

Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

<https://clegra.github.io/paracompl.html>

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Ph.D. Program Advanced Course

Gran Sasso Science Institute

L'Aquila, Italy

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Course overview

Monday, June 8 11.00 – 13.00	Tuesday, June 9 11.00 – 13.00	Wednesday, June 10 11.00 – 13.00	Thursday, June 11 11.00 – 13.00	Friday, June 13
<i>Algorithmic Techniques</i>			<i>Formal-Method & Algorithmic Techniques</i>	
Introduction & basic FPT results	Notions of bounded graph width	Example: Good Edge Labelings	Algorithmic Meta-Theorems	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	<i>Guest Lecture</i> Davi de Andrade Iacono	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
				motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Overview

- ▶ comparing parameterizations
- ▶ dynamical programming on trees, example:
 - ▶ WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- ▶ path-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ tree-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ fpt-results for other problems, obtained similarly
- ▶ other notions of width
 - ▶ clique-width
 - ▶ using f -width to define:
 - ▶ carving-width (and cut-width)
 - ▶ branch-width
 - ▶ rank-width
- ▶ comparing width-notions

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- ▷ $Q \subseteq \Sigma^*$ is the set of (*classical*) *problem instances*,
- ▷ $\kappa : \Sigma^* \rightarrow \mathbb{N}$ is a (general) function, *the parameterization*.

Parameterized problem $\langle Q, \Sigma, \kappa \rangle$

Instance: $x \in \Sigma^*$.

Parameter: $\kappa(x)$.

Problem: Is $x \in Q$?

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- ▷ $Q \subseteq \Sigma^*$ is the set of (*classical*) *problem instances*,
- ▷ $\kappa : \Sigma^* \rightarrow \mathbb{N}$ is a (general) function, *the parameterization*.

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in **FPT**) if:

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* [\mathbb{A} \text{ decides whether } x \in Q \text{ holds}$
 $\text{in time } \leq f(\kappa(x)) \cdot p(|x|)]$

†) Assumptions for a robust fpt-theory

$\kappa(x)$ is **polynomially computable**, or itself **fpt-computable**: for all $x \in \Sigma^*$ in time $\leq g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N}$ computable $\forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$.
- ▶ $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \text{FPT} \iff \langle Q, \kappa_2 \rangle \in \text{FPT}$$

$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \implies \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftrightarrow wd_1 \xrightarrow{g} wd_2$$

► FPT-results

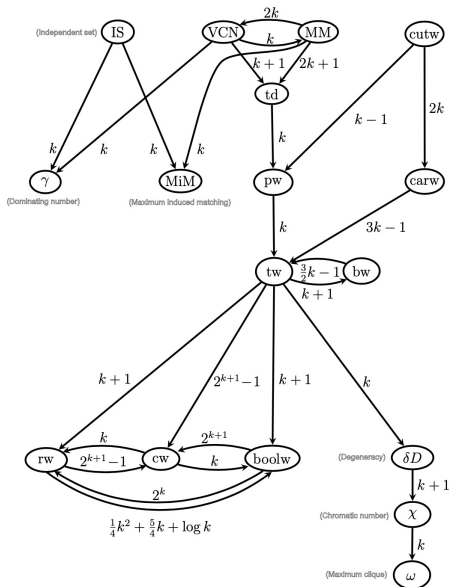
transfer upwards

(and conversely to \xrightarrow{g})

► (\notin FPT)-results

transfer downwards

(and along \xrightarrow{g})



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

(Ministero dell'Interno)

PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E \rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem: Is it possible that ℓ or more people can go out in the next hour so that everybody walks alone (with their children)?

corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is **independent set** in $\mathcal{G} : \iff \forall e = \{u, v\} \in E (\neg(u \in S \wedge v \in S))$
 $\iff \forall e = \{u, v\} \in E (u \notin S \vee v \notin S)$

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a **weight function** $w : V \rightarrow \mathbb{R}_0^+$.

Problem: What is the **max. weight of an independent set** of \mathcal{G} ?

S is a **vertex cover** of $\mathcal{G} : \iff \forall e = \{u, v\} \in E (u \in S \vee v \in S)$
 $\iff \forall e = \{u, v\} \in E (u \notin V \setminus S \vee v \notin V \setminus S)$
 $\iff V \setminus S$ is an independent set of \mathcal{G}

VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Problem: Does \mathcal{G} have a vertex cover of size at most ℓ ?

$S \subseteq V$ is **minimal** vertex cover $\iff V \setminus S$ is **maximal** independent set

Hence: solution of WEIGHTED-INDEPENDENT-SET

\implies solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^* -WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $w : V \rightarrow \mathbb{R}_0^+$.

Parameter: path-width / tree-width k .

Problem: What is the max. weight of an independent set of \mathcal{G} ?

p^* -VERTEX-COVER

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and $\ell \in \mathbb{N}$.

Parameter: path-width / tree-width k .

Problem: Does \mathcal{G} have a vertex cover of size at most ℓ ?

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \rightarrow \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of \mathcal{T} ?

Obtain a directed tree $\mathcal{T} = \langle T, F, r \rangle$ (pick a root r , orient edges away).

- ▶ $A[v] :=$ max. weight of an independent set in subtree \mathcal{T}_v at v ,
- ▶ $B[v] :=$ max. weight of an ind. set in \mathcal{T}_v that does not contain v .

Computation of $A[v]$ and $B[v]$:

- ▶ in leafs: $B[v] = 0, \quad A[v] = w(v).$
- ▶ for inner vertices v with children v_1, \dots, v_q :

$$B[v] = \sum_{i=1}^q A[v_i], \quad A[v] = \max\left\{B[v], w(v) + \sum_{i=1}^q B[v_i]\right\}.$$

Solution: value of $A[r]$, can be computed bottom-up in linear time.

Dynamical programming on trees (example)

WEIGHTED-INDEPENDENT-SET

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and a weight function $w : T \rightarrow \mathbb{R}_0^+$.

Problem: What is the max. weight of an independent set of \mathcal{T} ?

Theorem

On trees with n nodes,

WEIGHTED-INDEPENDENT-SET \in DTIME($O(n)$).

VERTEX-COVER

Instance: A tree $\mathcal{T} = \langle T, F \rangle$, and $\ell \in \mathbb{N}$.

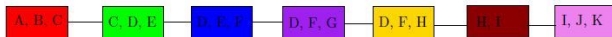
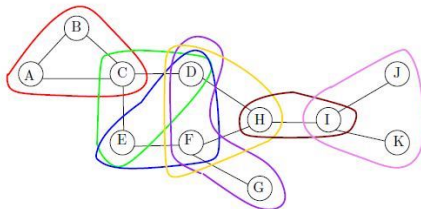
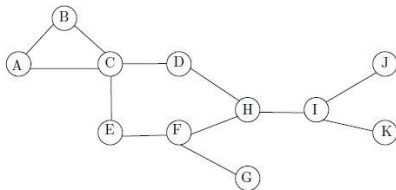
Problem: Does \mathcal{T} have a vertex cover of size at most ℓ ?

Corollary

On trees with n nodes,

VERTEX-COVER \in DTIME($O(n)$).

Path-decomposition (example)



Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \dots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^r B_i$ (every vertex of \mathcal{G} is in some bag).

(P2) $(\forall \{u, v\} \in E) (\exists i \in \{1, 2, \dots, r\}) [\{u, v\} \subseteq B_i]$
(every edge of \mathcal{G} is realized in some bag).

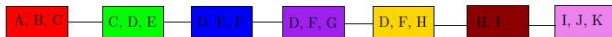
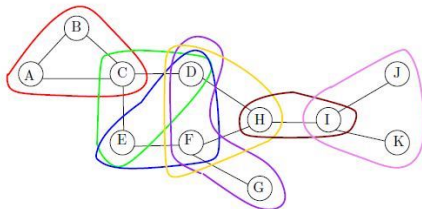
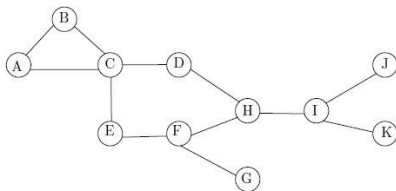
(P3) $(\forall v \in V) (\exists i, k \in \{1, \dots, r\}, i \leq k) [\{j \mid v \in B_j\} = [i, k]]$
(the list of bags that contains a vertex of \mathcal{G} is $\langle B_i, \dots, B_k \rangle$ for some interval $[i, k]$)

The *width* of path decomp. $\langle B_1, B_2, \dots, B_r \rangle$ is $\max \{|B_t| - 1 \mid 1 \leq t \leq r\}$.

The *path-width* $\text{pw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$\text{pw}(\mathcal{G}) :=$ minimal width of a path decomposition of \mathcal{G} .

Path-decomposition (example)



Path decomposition defines separations

Lemma

Let $\langle B_1, B_2, \dots, B_r \rangle$ be a path decomposition of a graph $\mathcal{G} = \langle V, E \rangle$. Then for all $i \in \{1, \dots, r-1\}$ it holds:

- ▶ $\langle \bigcup_{j=1}^i B_j, \bigcup_{j=i+1}^r B_j \rangle$ is a separation of \mathcal{G} with separator $B_i \cap B_{i+1}$.
- ▶ $\partial(\bigcup_{j=1}^i B_j) \subseteq B_i \cap B_{i+1}$.

- ▶ A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of \mathcal{G} if:

- ▶ $V = A \cup B$
- ▶ there is **no edge** between $A \setminus B$ and $B \setminus A$.

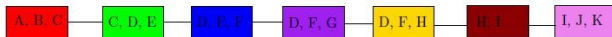
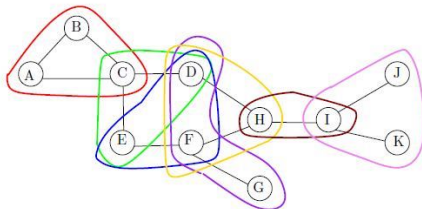
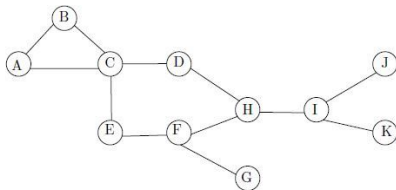
$A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.

- ▶ The *border (set of border vertices)* $\partial(A)$ of a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$.

Note that:

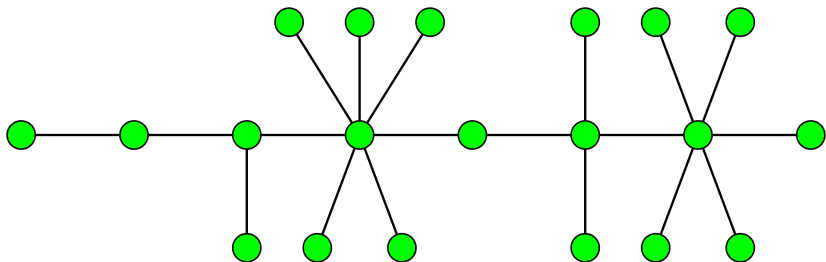
- ▶ $\langle A, (V \setminus A) \cup \partial(A) \rangle$ is a separation of \mathcal{G} , for all $A \subseteq V$.

Path-decomposition (example)



Caterpillar

Path-width?

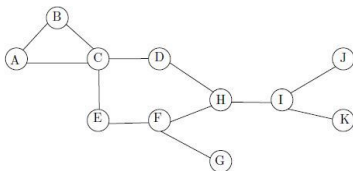


Nice path decomposition

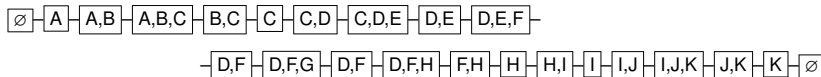
Definition

A *path decomposition* $\langle B_1, B_2, \dots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- ▶ $B_1 = B_r = \emptyset$
- ▶ Every index $i > 1$ is either of:
 - ▶ **introduce index**: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
 - ▶ **forget index**: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.



Nice path decomposition:



Nice path decomposition

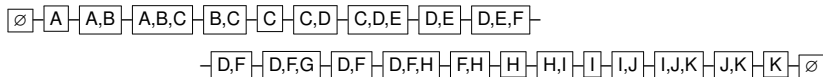
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 - ▶ **forget index**: there is $v \in V$ such that $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$.

Lemma

From every *path decomposition* $\langle B_1, B_2, \dots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a *nice path decomposition* $\langle B'_1, B'_2, \dots, B'_r \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.



Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is **independent set** in $\mathcal{G} : \iff \forall e = \{u, v\} (\neg(u \in S \wedge v \in S))$.

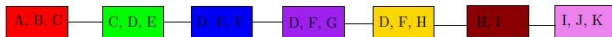
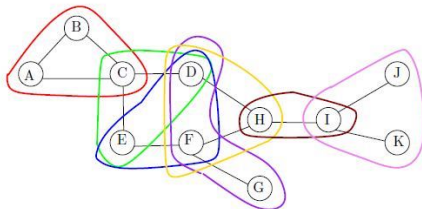
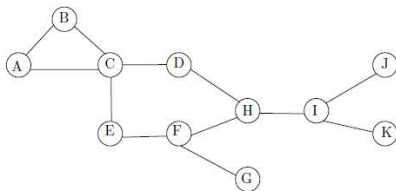
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a **weight function** $w : V \rightarrow \mathbb{R}_0^+$.

Parameter: **path-width** k .

Problem: What is the **max. weight of an independent set** of \mathcal{G} ?

Path-decomposition (example)



Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \dots, B_r \rangle$ be a **nice path decomposition** of $\mathcal{G} = \langle V, E \rangle$.

Then for every $i \in \{1, \dots, r\}$, and every $S \subseteq B_i$, we define:

$$c[i, S] := \begin{cases} -\infty & \text{if } S \text{ is not independent} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^i B_j \wedge \hat{S} \cap B_i = S & \\ \text{if } S \text{ is independent.} & \end{cases}$$

Recursive equations for computing $c[i, S]$ for **independent** S :

▶ **Case** $i = 1$: $c[1, \emptyset] = 0$

▶ **Case** $i + 1$:

▶ $i + 1$ **introduces** v : $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,

$$c[i + 1, S] = \begin{cases} c[i, S] & \text{if } v \notin S, \\ c[i, S \setminus \{v\}] + \mathbf{w}(v) & \text{if } v \in S; \end{cases}$$

▶ $i + 1$ **forgets** v : $B_{i+1} = B_i \setminus \{v\}$ and $v \in B_i$,

$$c[i + 1, S] = \max\{c[i, S], c[i, S \cup \{v\}]\}.$$

Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \dots, B_r \rangle$ be a nice path dec. of $\mathcal{G} = \langle V, E \rangle$ of width k .

For every $i \in \{1, \dots, r\}$, and every **independent** $S \subseteq B_i$, we define:

$$c[i, S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^i B_j \wedge \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[i, S]$, the **maximum possible weight of an independent set $S \subseteq V$** can be computed as:

$$= \max_{S \subseteq B_r} c[r, S] = c[r, \emptyset]$$

Then for all $i \in \{1, \dots, n\}$:

- ▶ $|B_i| \leq k + 1$,
- ▶ \Rightarrow number of values $c[i, S]$ at index i : $2^{|B_i|} = 2^{k+1}$,
- ▶ \Rightarrow **adjacency/independence check** for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- ▶ time for comp. a value at i , using map of values at $i - 1$: $\sim O(k)$
- ▶ time for comp. all values at i , using values at $i - 1$: $2^{k+1} \cdot O(k^2)$

\Rightarrow the time for computing all values at r :

$$(2^{k+1} \cdot O(k^2)) \cdot r + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } r = 2n.$$

Dynamical programming with path width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and *path-width* $\text{pw}(\mathcal{G}) = k$,
 p^* -WEIGHTED-INDEPENDENT-SET $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$.

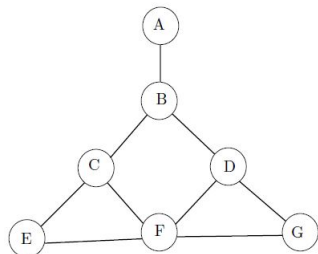
S is a *minimal* vertex cover

$\iff V \setminus S$ is a *maximal* independent set.

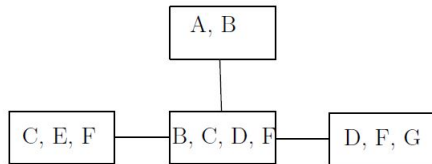
Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and *path-width* $\text{pw}(\mathcal{G}) = k$,
 p^* -VERTEX-COVER $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$.

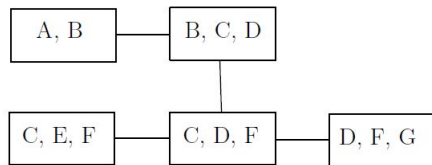
Tree decomposition (example)



The original graph G



A tree-decomposition of width 3



A tree-decomposition of width 2

Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

(T1) $V = \bigcup_{t \in T} B_t$ (every vertex of \mathcal{G} is in some bag).

(T2) $(\forall \{u, v\} \in E) (\exists t \in T) [\{u, v\} \subseteq B_t]$
 (the vertices of every edge of \mathcal{G} are realized in some bag).

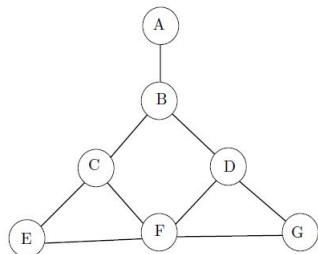
(T3) $(\forall v \in V) [\text{subgraph of } \mathcal{T} \text{ defd. by } \{t \in T \mid v \in B_t\} \text{ is connected}]$
 (the tree vertices (in \mathcal{T}) whose bags contain some vertex of \mathcal{G} induce a subgraph of \mathcal{T} that is connected).

The *width* of a tree decomposition $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ is $\max \{|B_t| - 1 \mid t \in T\}$.

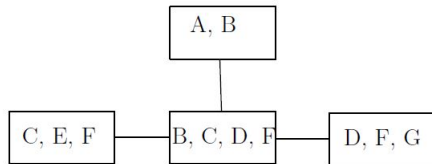
The *tree-width* $tw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$tw(\mathcal{G}) :=$ minimal width of a tree decomposition of \mathcal{G} .

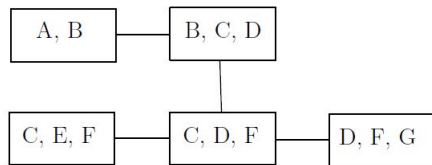
Tree decomposition (example)



The original graph G



A tree-decomposition of width 3



A tree-decomposition of width 2

Tree decomposition defines separations

Lemma

Let $\langle \mathcal{T}, \{B_t\}_{t \in \mathcal{T}} \rangle$ be a tree decomposition of a graph $\mathcal{G} = \langle V, E \rangle$. Let $e = \langle a, b \rangle$ be an edge of \mathcal{T} . The $\mathcal{T} \setminus e$ is the union of a tree \mathcal{T}_a containing a , and a tree \mathcal{T}_b containing b .

Then for $A := \bigcup_{t \in V(\mathcal{T}_a)} B_t$ and $B := \bigcup_{t \in V(\mathcal{T}_b)} B_t$ it holds:

- ▶ $\langle A, B \rangle$ is a separation of \mathcal{G} with separator $B_a \cap B_b$.
- ▶ $\partial(A), \partial(B) \subseteq B_a \cap B_b$.

Recall, for a graph $\mathcal{G} = \langle V, E \rangle$:

- ▶ A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of \mathcal{G} if:
 - ▶ $V = A \cup B$
 - ▶ there is **no edge** between $A \setminus B$ and $B \setminus A$.

$A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.

- ▶ The *border (vertices)* $\partial(A)$ of a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$.

Computing tree-width

TREE-WIDTH

Instance: A graph \mathcal{G} and $k \in \mathbb{N}$.

Problem: Decide whether $tw(\mathcal{G}) = k$.

Theorem

TREE-WIDTH is NP-complete.

p -TREE-WIDTH

Instance: A graph $\mathcal{G} = \langle V, E \rangle$ and $k \in \mathbb{N}$.

Parameter: k .

Problem: Decide whether $tw(\mathcal{G}) = k$.

Theorem

p -TREE-WIDTH is fixed-parameter tractable,
in time $2^{p(k)} \cdot n$ where $n := |V|$.

Nice tree decomposition

Definition

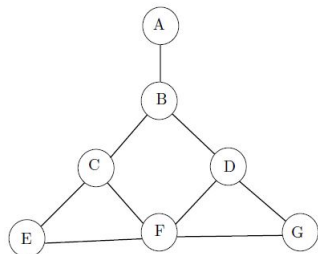
A *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t \in \mathcal{T}} \rangle$ of graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- ▶ $B_r = \emptyset$, and $B_\ell = \emptyset$ for every leaf $\ell \in \mathcal{T}$.
- ▶ Every non-leaf node $t \in \mathcal{T}$ is of one of three types:
 - ▶ **introduce node**: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is **introduced** at t .
 - ▶ **forget node**: t has exactly one child t' such that $B_t = B_{t'} \setminus \{w\}$ for some $w \in B_{t'}$; we say w is **forgotten** at t .
 - ▶ **join node**: a node t with two children t_1, t_2 such that $B_t = B_{t_1} = B_{t_2}$.

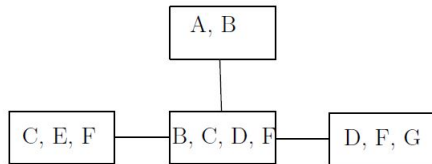
Lemma

From every *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t \in \mathcal{T}} \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of *width* k a *nice tree decomposition* $\langle \mathcal{T}', \{B'_t\}_{t \in \mathcal{T}'} \rangle$ of *width* k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

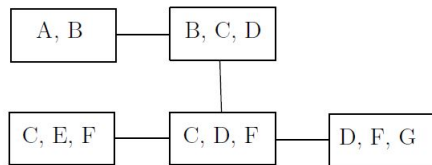
Tree decomposition (example)



The original graph G



A tree-decomposition of width 3



A tree-decomposition of width 2

Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is **independent set** in $\mathcal{G} : \iff \forall e = \{u, v\} (\neg(u \in S \wedge v \in S))$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a **weight function** $w : V \rightarrow \mathbb{R}_0^+$.

Parameter: **tree-width** k .

Problem: What is the **max. weight of an independent set** of \mathcal{G} ?

Dynamical programming using tree-width (example)

For every node t of a **nice tree decomposition**, and every $S \subseteq B_t$, we define:

$$c[t, S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent } \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S & \\ \text{if } S \text{ is independent.} \end{cases}$$

Recursive equations for computing $c[t, S]$ for **independent** S :

- ▶ leaf node t : $c[t, \emptyset] = 0$
- ▶ introduction node t of vertex v with child t' :

$$c[t, S] = \begin{cases} c[t', S] & \text{if } v \notin S \\ c[t', S \setminus \{v\}] + w(v) & \text{otherwise} \end{cases}$$

- ▶ forget node t of vertex v with child t' :

$$c[t, S] = \max\{c[t', S], c[t', S \cup \{v\}]\}$$

- ▶ join node t with children t_1 and t_2 :

$$c[t, S] = c[t_1, S] + c[t_2, S] - w(S)$$

Dyn. programming using tree-width (Weigh. Ind. Set)

Let $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ be a **nice tree decomposition** of $\mathcal{G} = \langle V, E \rangle$ of width k . For every $t \in T$, and every **independent** $S \subseteq B_t$:

$$c[t, S] := \begin{cases} \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_t = \bigcup_{s \in T_t} B_s \wedge \hat{S} \cap B_i = S \end{cases}$$

Time Complexity: Based on the values of $c[t, S]$, the **maximum possible weight of an independent set** $S \subseteq V$ can be computed as:

$$= \max_{S \subseteq B_r} c[r, S] = c[r, \emptyset]$$

Then for all $t \in T$:

- ▶ $|B_t| \leq k + 1$,
- ▶ \Rightarrow number of values $c[t, S]$ at index t : $2^{|B_t|} = 2^{k+1}$,
- ▶ \Rightarrow **adjacency/independence check** for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- ▶ time for comp. a value at t , using map of values at $t - 1$: $O(k)$
- ▶ time for comp. all values at t , using values at $t - 1$: $2^{k+1} \cdot O(k^2)$

\Rightarrow the time for computing all values at the root r :

$$(2^{k+1} \cdot O(k^2)) \cdot |T| + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } |T| \in O(k \cdot n).$$

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and *tree-width* $tw(\mathcal{G}) = k$,
 p^* -WEIGHTED-INDEPENDENT-SET $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$.

S is a *minimal* vertex cover

$\iff V \setminus S$ is a *maximal* independent set.

Corollary

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and *tree-width* $tw(\mathcal{G}) = k$,
 p^* -VERTEX-COVER $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$.

Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E \rangle$ of size n and **nice tree decompositions** $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ of tree width k .

- ▶ **Formulate** a family of properties that can be restricted to subtrees of \mathcal{T} such that
 - ▶ a solution of P can be obtained from the properties at the root of \mathcal{T} .
- ▶ **Find** recursion equations for bottom-up evaluation on \mathcal{T} .
- ▶ **Prove** correctness of these recursion equations by showing two inequalities for each type of node:
 - ▶ one relating an optimum solution for the node to some solutions for its children,
 - ▶ one relating optimum solutions for a node's children to a solution for the node.
- ▶ **Obtain** an estimate of the time needed to compute the properties in a node t depending on n and k .
- ▶ **Sum up** the time needed to compute the solution(s) at root r of \mathcal{T} .
- ▶ **Add** time needed to obtain the solution of P from properties at r .

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and $tw(\mathcal{G}) = k$,

- ▶ p^* -VERTEX-COVER, INDEPENDENT-SET $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -DOMINATING-SET $\in \text{DTIME}(4^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -ODD CYCLE TRAVERSAL $\in \text{DTIME}(3^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* -MAXCUT $\in \text{DTIME}(2^k \cdot k^{O(1)} \cdot n)$,
- ▶ p^* - q -COLORABILITY $\in \text{DTIME}(q^k \cdot k^{O(1)} \cdot n)$.

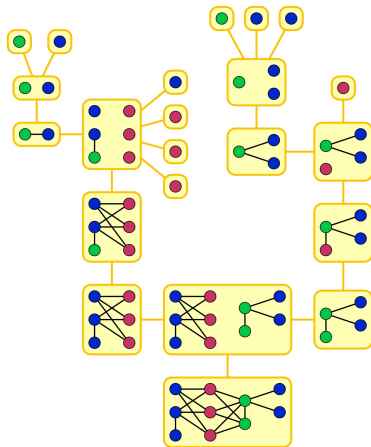
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and $tw(\mathcal{G}) = k$, the following problems are in $\text{DTIME}(k^{O(k)} \cdot n)$:

- ▶ p^* -STEINER-TREE,
- ▶ p^* -FEEDBACK-VERTEX-SET,
- ▶ p^* -HAMILTONIAN-PATH and p^* -LONGEST-PATH,
- ▶ p^* -HAMILTONIAN-CYCLE and p^* -LONGEST-CYCLE,
- ▶ p^* -CHROMATIC-NUMBER,
- ▶ p^* -CYCLE-PACKING,
- ▶ p^* -CONNECTED-VERTEX-COVER,
- ▶ p^* -CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



Clique-Width

For $k \in \mathbb{N}$, the k -expressions are defined by:

$$\varphi, \varphi_1, \varphi_2 ::= i \mid \text{edge}_{i-j}(\varphi) \mid \text{recolor}_{i \rightarrow j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

for $i, j \in [k]$ with $i \neq j$. k -expressions φ *generate* graphs $\mathcal{G}(\varphi)$:

- ▷ $\mathcal{G}(i)$ is the graph with a single vertex of color i .
- ▷ $\mathcal{G}(\text{edge}_{i-j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by adding edges between every vertex of color i and every vertex of color j .
- ▷ $\mathcal{G}(\text{recolor}_{i \rightarrow j}(\varphi))$ results from $\mathcal{G}(\varphi)$ by recoloring every vertex of color i by color j .
- ▷ $\mathcal{G}(\varphi_1 \oplus \varphi_2)$ is the disjoint union of $\mathcal{G}(\varphi_1)$ and $\mathcal{G}(\varphi_2)$.

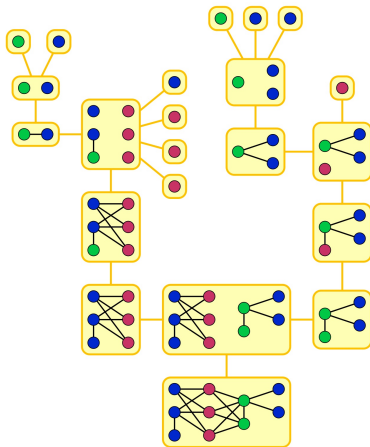
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width* $clw(\mathcal{G})$ of $\mathcal{G} = \langle V, E \rangle$ is defined by:

$$clw(\mathcal{G}) := \text{the least } k \in \mathbb{N} \text{ such that, for some } k\text{-expression } \varphi, \\ \mathcal{G} = \mathcal{G}(\varphi) \text{ (when removing colors)}$$

Clique width (example)

Building a graph \mathcal{G} of clique-width $c/w(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

Example

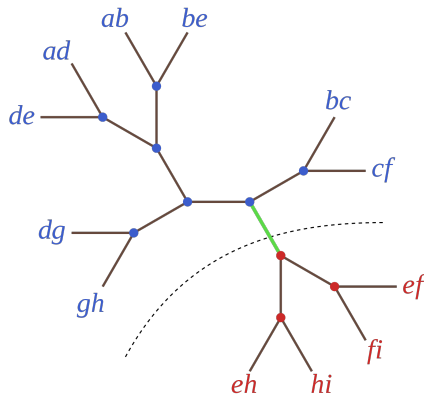
- ▶ The class of cliques has clique-width 2.
 - ▶ The class of stars has clique-width 2.
 - ▶ The class of trees has clique-width 3.
 - ▶ The class of $n \times n$ grids has clique-width $\Theta(n)$.
- ▶ subgraphs/induced subgraphs:
 - ▶ clique-width is preserved under taking induced subgraphs,
 - ▶ clique-width is **not preserved** under taking subgraphs (e.g. minors).
 - ▶ $clw < tw$:
 - ▶ $clw \leq tw$: $clw(\mathcal{G}) \leq 3 \cdot 2^{tw(\mathcal{G})-1}$
 - ▶ $\neg(tw \leq clw)$: for example, $clw(K_n) = 2$, and $tw(K_n) = n - 1$.
 - ▶ Deciding whether $clw(\mathcal{G}) \leq k$ is **NP-hard**. With parameter k it is in XP (slice-wise polynomial), but unknown to be in FPT.
 - ▶ Every graph property expressible in **MSO (monadic second-order logic)** can be decided in linear time w.r.t. the graph's clique-width.

f -Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \rightarrow \mathbb{R}_0^+$ such that:

f is *symmetric*: $\iff \forall X \subseteq U [f(X) = f(U \setminus X)]$;

f is *fair*: $\iff f(\emptyset) = f(U) = 0$.



Branch-Width

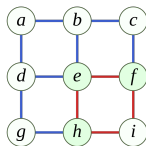
Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \left\{ v \in V \mid \exists e_1 \in X \exists e_2 \in E \setminus X \right. \\ \left. [v \text{ is incident to } e_1 \text{ and } e_2] \right\}$$

The *branch-width* $bw(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is defined as

$$bw(\mathcal{G}) := w_f(E) \quad \text{for } f : 2^E \rightarrow \mathbb{R}_0^+, X \mapsto |\partial(X)|$$



Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.

For $X \subseteq V$ we define the $GF(2)$ -matrix:

$$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \setminus X}, \text{ where, for all } x \in X, y \in V \setminus X:$$

$$b_{x,y} = 1 \iff \{x, y\} \in E.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by \mathcal{G} between X and $V \setminus X$.)

The *rank-width* $rw(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$rw(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \quad \text{for } \rho_{\mathcal{G}} : 2^V \rightarrow \mathbb{N}_0, X \mapsto \text{rank of } B_{\mathcal{G}}(X)$$

Properties

- ▶ $rw(\mathcal{G}) \leq tw(\mathcal{G})$.
- ▶ tree-width cannot be bounded functionally by rank-width:
 $rw(K_n) = 1$, but $tw(K_n) = n - 1$.

Carving-Width and Cut-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.

For $X \subseteq V$ the *edge-cut* of X is:

$$\text{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} .$$

The *carving-width* $\text{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\text{carw}(\mathcal{G}) := w_{\text{cut}}(E) \quad \text{for} \quad \text{cut} : 2^V \rightarrow \mathbb{N}_0, X \mapsto |\text{cut}_{\mathcal{G}}(X)| .$$

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.

For a permutation $\pi : \{1, \dots, n\} \rightarrow V$ on V we define:

$$\text{width}(\pi) := \max_{1 \leq i \leq n} \text{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \leq j \leq i\}) .$$

The *cut-width* $\text{cutw}(\mathcal{G})$ of \mathcal{G} is:

$$\text{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \text{width}(\pi) .$$

Coverage in Multi-Interface Networks



$CMI(p)$ (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \rightarrow 2^{\{1, \dots, a\}}$ available-interface allocation, $c : \{1, \dots, a\} \rightarrow \mathbb{R}^+$ interface cost function.

Solution: An allocation $W_A : V \rightarrow 2^{\{1, \dots, a\}}$ of active interfaces **covering** \mathcal{G} such that $W_A(v) \subseteq W(v)$, and $|W_A(v)| \leq p$ for all $v \in V$, if possible; otherwise, a negative answer.

Problem: Obtain, if possible, a **minimal solution** with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks (parameterized)

Theorem

$CMI(2) \in \text{NP-complete}$, also for graphs with max. node degree ≥ 4 .

p^* -CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \rightarrow 2^{\{1, \dots, a\}}$ available-interface allocation, $c : \{1, \dots, a\} \rightarrow \mathbb{R}^+$ interface cost function.

Parameter: path-width / carving-width k

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is,

$$c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i).$$

Theorem (Aloisio, Navarra, 2020, [1])

- ▶ For path-width $pw(\mathcal{G}) = k$,

$$p^*\text{-CMI}(2) \in \text{DTIME}(n \cdot (a + \binom{a}{2})^{k+1}).$$
- ▶ For carving-width $carw(\mathcal{G}) = k$, $p^*\text{-CMI}(2) \in \text{DTIME}(n \cdot a^{4k}).$

Coverage in Multi-Interface Networks (parameterized)

Theorem (Aloisio, Navarra, 2020, [1])

- ▶ For *path-width* $pw(\mathcal{G}) = k$,
 p^* -CMI(2) \in DTIME($n \cdot (a + \binom{a}{2})^{k+1}$).
- ▶ For *carving-width* $carw(\mathcal{G}) = k$, p^* -CMI(2) \in DTIME($n \cdot a^{4k}$).

$(p^*)'$ -CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \rightarrow 2^{\{1, \dots, a\}}$ available-interface allocation, $c : \{1, \dots, a\} \rightarrow \mathbb{R}^+$ interface cost function.

Parameter: $a +$ (path-width / carving-width k)

Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is,
 $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Corollary

$(p^*)'$ -CMI(p) \in FPT.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ parameterizations.

- ▶ $\kappa_1 \geq \kappa_2 : \iff \exists g : \mathbb{N} \rightarrow \mathbb{N}$ computable $\forall x \in \Sigma^* [g(\kappa_1(x)) \geq \kappa_2(x)]$.
- ▶ $\kappa_1 \approx \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \kappa_2 \geq \kappa_1$.
- ▶ $\kappa_1 > \kappa_2 : \iff \kappa_1 \geq \kappa_2 \wedge \neg(\kappa_2 \geq \kappa_1)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \rightarrow \mathbb{N}$ with $\kappa_1 \geq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \text{FPT} \iff \langle Q, \kappa_2 \rangle \in \text{FPT}$$

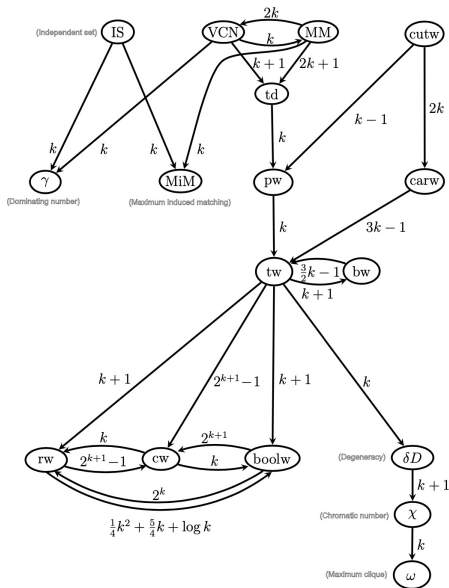
$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \implies \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 : \Leftarrow wd_1 \xrightarrow{g(k)} wd_2$$

- ▶ FPT-results
transfer upwards
(and conversely to \xrightarrow{g})
- ▶ (\notin FPT)-results
transfer downwards
(and along \xrightarrow{g})



Summary

- ▶ comparing parameterizations
- ▶ dynamical programming on trees, example:
 - ▶ WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- ▶ path-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ tree-width
 - ▶ example: fpt-algorithm for bounded path-width
- ▶ fpt-results for other problems, obtained similarly
- ▶ other notions of width
 - ▶ clique-width
 - ▶ using f -width to define:
 - ▶ carving-width (and cut-width)
 - ▶ branch-width
 - ▶ rank-width
- ▶ example problem: coverage in multi-interface networks
- ▶ comparing width-notions

Course overview

Monday, June 8 11.00 – 13.00	Tuesday, June 9 11.00 – 13.00	Wednesday, June 10 11.00 – 13.00	Thursday, June 11 11.00 – 13.00	Friday, June 13
<i>Algorithmic Techniques</i>			<i>Formal-Method & Algorithmic Techniques</i>	
Introduction & basic FPT results	Notions of bounded graph width	Example: Good Edge Labelings	Algorithmic Meta-Theorems	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	<i>Guest Lecture</i> Davi de Andrade Iacono	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	
				14.30 – 16.30
				FPT-Intractability Classes & Hierarchies
				motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Thursday

- ▶ recalling notions from logic:
 - ▶ propositional, and first-order logic
 - ▶ monadic second-order logic (MSO)
- ▶ **Courcelle's Theorem**: obtaining FPT-results by
 - ▶ model-checking of MSO-properties on graphs and structures of **bounded tree-/clique-width**

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