On the parameterized complexity of computing good edge labelings

Topics in Algorithmic Graph Theory

J. Araújo

joint work with
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Motivation: RWA Problem

A major problem in Wavelength-Division Multiplexing (WDM) networks is the Routing and Wavelength Assignment Problem.

Input: A set of traffic requests and a (directed) network.
Problem: Find routes and their associated wavelength as well to satisfy them.
Constraint: Two routes using a same fiber must have different wavelengths.

Goal: Minimizing the number of used wavelengths.

This problem is NP-hard.

A classical approach

Split the problem in two.

1. Find the routes (or dipaths).
2. Find the wavelengths.

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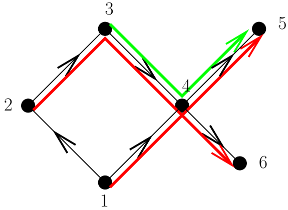
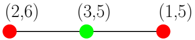
Step 2 = Coloring of a conflict graph.

Vertices = dipaths.   Wavelengths = colors.

Two paths sharing a fiber are linked by an edge.





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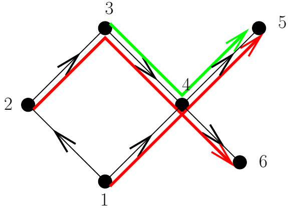
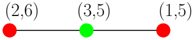
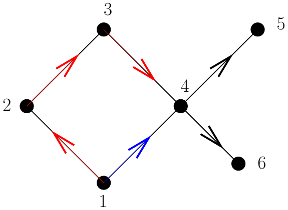




But graph coloring is NP-hard. [Karp, 1972]

UPP-DAG

UPP-DAG: acyclic digraphs in which there is at most one dipath from one vertex to another.



In an UPP-DAG the routing is forced!

Load: max. number of paths using a same fiber.
Hope: number of wavelengths is bounded by a function of the load.

Good edge-labeling

edge-labeling: function $\phi : E(G) \rightarrow \mathbb{R}$.

A path is increasing if the sequence of its edges labels is non-decreasing.

An edge-labeling of G is good if, for any two distinct vertices u, v, there is at most one increasing (u, v)-path.

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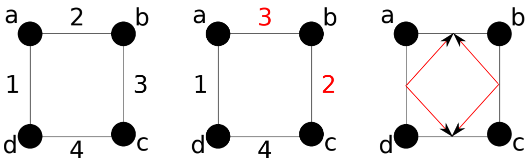

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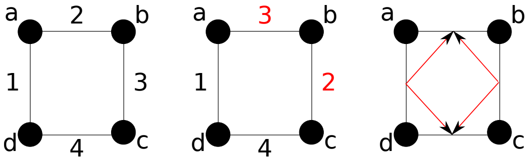

Problem [Bermond, Cosnard, and Pérennes [2013]]

Given a graph G, does G have a good edge-labeling?

Load vs number of wavelengths

Theorem (Bermond, Cosnard, and Pérennes [2013])

The conflict graph of a set of dipaths of load 2 in a UPP-DAG has a good edge-labeling.
Conversely, if G has a good edge-labeling, then it is the conflict graph of a set of dipaths of load 2 on a UPP-DAG.

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Corollary

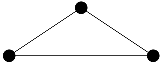
There exist sets of requests on UPP-DAGs s.t.

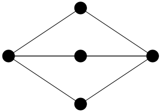
- *the load is 2,*
- *an arbitrarily large number of wavelengths are needed.*

Remarks

- G is good under ϕ if, and only if, G is good under injective integer-valued ϕ';
- G is good if, and only if, G does not have two disjoint (u, v)-increasing paths, for every $u, v \in V(G)$.

Bad graphs





C_3 and $K_{2,3}$ are bad.

Problem (J.-S. Sereni)

Are all $\{C_3, K_{2,3}\}$*-free graphs good?*

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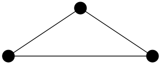
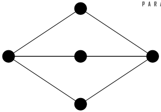




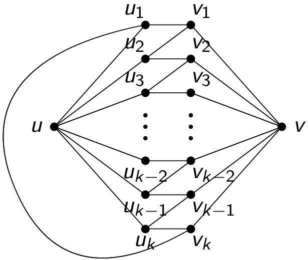
C_3 and $K_{2,3}$ are bad.

Problem (J.-S. Sereni)

Are all $\{C_3, K_{2,3}\}$-free graphs good? No!

Proposition (Araújo, Cohen, Giroire, and Havet [2012])

The graph H_k has either an increasing (u, v)-path or an increasing (v, u)-path, for any edge-labeling.



A newer bad example

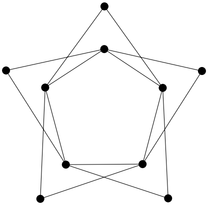


Figure: A bad graph: for any edge-labeling, in the central 5-cycle there are three adjacent edges uv, vw, wx forming an increasing path P_1. But then, there are two other internally-disjoint (u, x)-paths P_2 and P_3 of length two. Given that a 2-path is either increasing or decreasing, two of P_1, P_2, P_3 are either increasing or decreasing paths.

[de Andrade, Araújo, Morelle, Sau, and Silva, 2024]

An even newer bad example

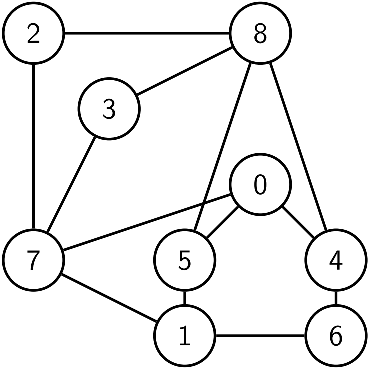


Figure: A bad example on 9 vertices.

Other results

Theorem (Araújo, Cohen, Giroire, and Havet [2012])

Deciding if a given bipartite graph G is good is NP*-complete.*

Classes of good graphs: [Araújo, Cohen, Giroire, and Havet, 2012]

- C_3-free outerplanar;
- $\{C_3, K_{2,3}\}$ subcubic;
- $|E(G)| < \frac{3}{2}(|V(G)| - 1)$.

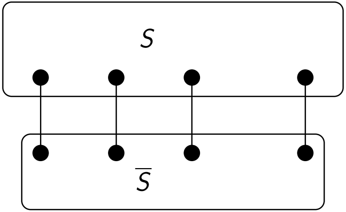
Key idea: Matching cuts!

Good graphs

G is good if, and only if, it does not contain a bad graph H as subgraph.

Critical graph

A graph G is critical if it is bad and, for every proper $H \subset G$, H is good.

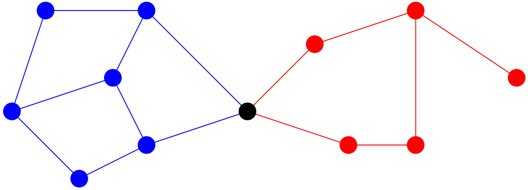


Lemma (Araújo, Cohen, Giroire, and Havet [2012])

A critical graph does not contain any matching-cut.

Cut-vertex

cut-vertex: vertex x s.t. $G - x$ is not connected.



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Other works in the literature

Theorem (Mehrabian [2012])

A good graph G on n vertices such that its maximum degree is within a constant factor of its average degree has at most $n^{1+o(1)}$ edges.

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Theorem (Mehrabian [2012])

For any Δ, there is a g such that any graph with maximum degree at most Δ and girth at least g is good.

Other works in the literature

Theorem (Mehrabian, Mitsche, and Pralat [2013])

Any good graph on n vertices has at most $n \log_2(n)/2$ edges and this bound is tight for infinitely many values of n.

Unpublished work by Bode, Farzad, and Theis [2011].

Temporal Graphs

Edge-labeled graph $\rightarrow$ temporal graph.

[Kempe, Kleinberg, and Kumar, 2000, Michail, 2015]

Increasing path $\rightarrow$ temporal path.

Good edge-labeling $\rightarrow$ temporization with low tmp. connectivity.

A natural variant: c-GEL

- For $c \in \mathbb{Z}_+^*$, an edge-labeling λ of G is a c-edge-labeling if λ takes at most c distinct values.
- A good c-edge-labeling $\iff$ c-gel.
- A graph admitting a c-gel is c-good, otherwise it is c-bad.

GOOD c-EDGE-LABELING (c-GEL for short)
Input: A graph G.
Question: Does G admit a c-gel?

An example about c-GEL

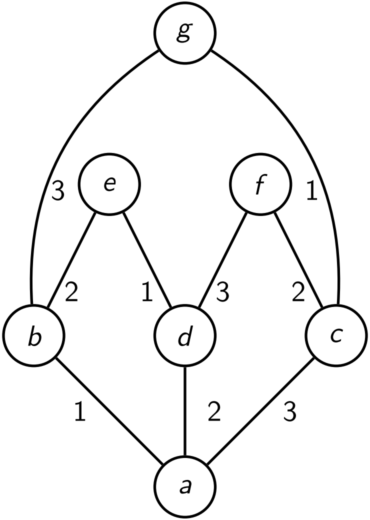

Figure: A 3-GEL exists

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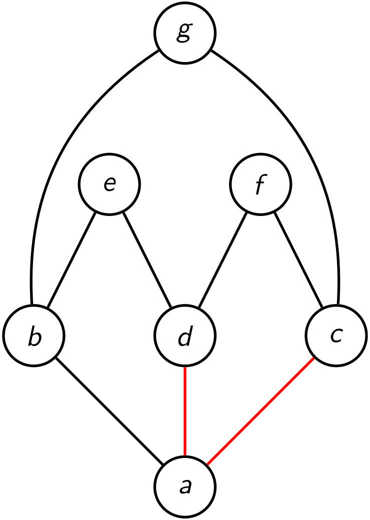

Figure: An example where no 2-GEL exists

Another useful example

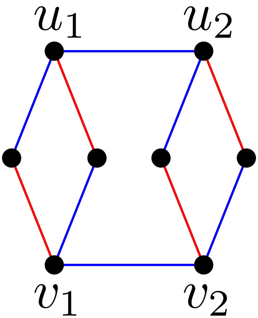

Figure: Edges $u_1 u_2$ and $v_1 v_2$ must have the same label in a 2-GEL

Our Contributions

[de Andrade, Araújo, Morelle, Sau, and Silva, 2024]

- c-GEL is in NP, for every $c \geq 2$.

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Parameterized Complexity:

- Linear kernel for neighborhood diversity.

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  - Via Courcelle's Theorem.
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- *GEL* is FPT parameterized by $\mathsf{tw} + \Delta$:
  - GEL can be solved in time $2^{\mathcal{O}(\mathsf{tw}\Delta^2 + \mathsf{tw}^2 \log \Delta)} \cdot n$ on n-vertex graphs, minimizing number of used labels.
- Deciding whether G has an UPP orientation is NP-complete.

Not-All-Equal 3-SAT

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Input: 3-CNF boolean formula φ, each clause with exactly three literals.

Question: Does there exist a truth assignment such that, for each clause C, its literals are not all set to `true` (nor to `false`)?

Example

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$$

$x_1 = x_2 = x_3 =$ `true` is NOT a satisfying assignment!

NP-complete problem. [Darmann and Döcker, 2020]

NP-hardness for 2-GEL

Reduction from NAE 3-SAT:

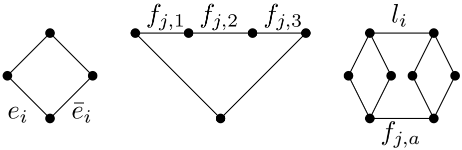

Figure: Gadgets for the reduction to 2-GEL: from left to right, X_i, C_j, and $P_{i,j,a}$.

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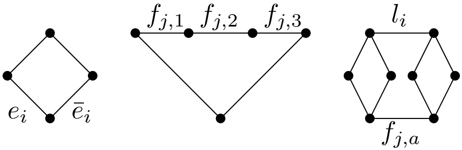

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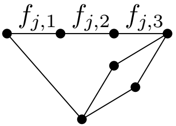
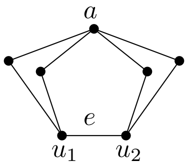


Figure: Bipartite variation of the clause gadget.

NP-hardness for c-GEL



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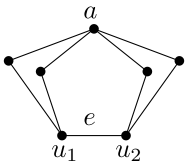

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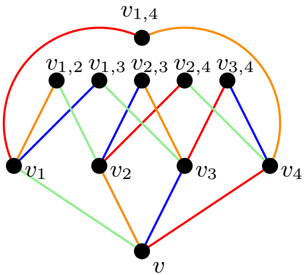

Figure: The c-color gadget D_c for $c = 4$ with a c-gel where each color represents a label.

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- The function $g(k)$ is called the size of the kernel, and a kernel is polynomial (resp. linear, quadratic) if $g(k)$ is a polynomial (resp. linear, quadratic) function.

[Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, and Saurabh, 2015, Fomin, Lokshtanov, Saurabh, and Zehavi, 2019]

Reduction Rules

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Rule (For GEL only)

If G has a matching cut S, consider each connected component of $G - S$ separately.

Linear Kernel by nd(G)

- The neighborhood diversity of G, nd(G), is the minimum w s.t. $V(G)$ can be partitioned into w sets of vertices having the same type.
- u and v have the same type if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$.
- nd(G) can be computed in cubic time.

[Lampis, 2012]

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Idea:

- Find partition.
- Apply Reduction rules.
- Check that each part is small.

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Idea:

- Use 2-approx. alg. to get vertex cover S with $|S| \leq 2 \cdot \text{vc}(G)$.
- Apply reduction rules.
- No vertex of degree 1 remains.
- At most $2 \cdot \binom{2k}{2}$ vertices of $\overline{S}$ with degree at least 2.

Further Research

$$\mathrm{tw}(G) - 1 \leq \mathrm{fvs}(G) \leq \mathrm{sfm}(G) \leq \mathrm{vc}(G)$$

$\mathrm{nd}(G) \leq 2^{\mathrm{vc}(G)} + \mathrm{vc}(G)$, but incomparable to tw, fvs, or sfm.

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Question

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any graph G, if G is good then G is $f(\mathsf{tw})$-good, where tw is the treewidth of G?

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Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any graph G, if G is good then G is $f(\mathrm{tw})$-good, where tw *is the treewidth of G?*

Question

Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any graph G, if G is good then G is $f(\Delta)$-good, where Δ is the maximum degree of G?

Thank you for your attention!

References I

J. Araújo, N. Cohen, F. Giroire, and F. Havet. Good edge-labelling of graphs. *Discrete Applied Mathematics*, 160(18):2502–2513, 2012. doi: 10.1016/J.DAM.2011.07.021.

J. Bermond, M. Cosnard, and S. Pérennes. Directed acyclic graphs with the unique dipath property. *Theoretical Computer Science*, 504:5–11, 2013. doi: 10.1016/J.TCS.2012.06.015.

M. Bode, B. Farzad, and D. O. Theis. Good edge-labelings and graphs with girth at least five. *CoRR*, abs/1109.1125, 2011.

M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015. ISBN 978-3-319-21274-6. doi: 10.1007/978-3-319-21275-3.

A. Darmann and J. Döcker. On a simple hard variant of Not-All-Equal 3-Sat. *Theoretical Computer Science*, 815: 147–152, 2020. doi: https://doi.org/10.1016/j.tcs.2020.02.010.

References II

D. de Andrade, J. Araújo, L. Morelle, I. Sau, and A. Silva. On the parameterized complexity of computing good edge-labelings, 2024. URL https://arxiv.org/abs/2408.15181.

F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi. *Kernelization: Theory of Parameterized Preprocessing*. Cambridge University Press, 2019. doi: 10.1017/9781107415157.

R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972. ISBN 0-306-30707-3.

D. Kempe, J. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks. In *Proc. of the 32nd annual ACM Symposium on Theory of Computing (STOC)*, pages 504–513, 2000. doi: 10.1145/335305.335364.

References III

M. Lampis. Algorithmic meta-theorems for restrictions of treewidth. *Algorithmica*, 64(1):19–37, 2012. doi: 10.1007/S00453-011-9554-X.

A. Mehrabian. On the density of nearly regular graphs with a good edge-labeling. *SIAM Journal on Discrete Mathematics*, 26(3): 1265–1268, 2012. doi: 10.1137/11085414X.

A. Mehrabian, D. Mitsche, and P. Pralat. On the maximum density of graphs with unique-path labelings. *SIAM Journal on Discrete Mathematics*, 27(3):1228–1233, 2013. doi: 10.1137/120898528.

O. Michail. *An Introduction to Temporal Graphs: An Algorithmic Perspective*, pages 308–343. Springer International Publishing, Cham, 2015. ISBN 978-3-319-24024-4. doi: 10.1007/978-3-319-24024-4_18. URL https://doi.org/10.1007/978-3-319-24024-4_18.