

On the parameterized complexity of computing good edge labelings

Topics in Algorithmic Graph Theory

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joint work with

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A major problem in **W**avelength-**D**ivision **M**ultiplexing (WDM) networks is the **R**outing and **W**avelength **A**ssignment Problem.

Input: A set of traffic requests and a (directed) network.

Problem: Find routes and their associated wavelength as well to satisfy them.

Constraint: Two routes using a same fiber must have different wavelengths.

Goal: **Minimizing** the number of used wavelengths.

This problem is **NP-hard**.

A classical approach

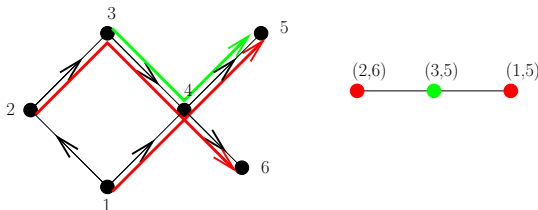
Split the problem in two.

1. Find the routes (or dipaths).
2. Find the wavelengths.

Step 2 = Coloring of a conflict graph.

Vertices = dipaths. Wavelengths = colors.

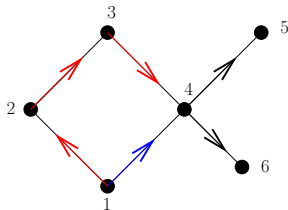
Two paths sharing a fiber are linked by an edge.



But graph coloring is **NP-hard**.

[Karp, 1972]

UPP-DAG: acyclic digraphs in which there is at most one dipath from one vertex to another.



In an UPP-DAG the routing is forced!

Load: max. number of paths using a same fiber.

Hope: number of wavelengths is bounded by a function of the load.

Theorem (Bermond, Cosnard, and Pérennes [2013])

The *conflict graph* of a set of dipaths of *load 2* in a *UPP-DAG* has a *good edge-labeling*.

Conversely, if G has a *good edge-labeling*, then it is the *conflict graph* of a set of dipaths of *load 2* on a *UPP-DAG*.

Theorem (Bermond, Cosnard, and Pérennes [2013])

There exist graphs with a good edge-labeling and large chromatic number.

Corollary

There exist sets of requests on UPP-DAGs s.t.

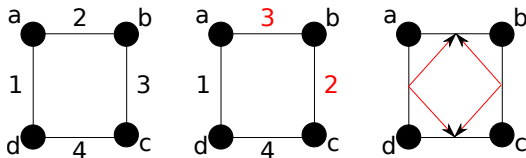
- ▶ the *load is 2*,
- ▶ an *arbitrarily large number of wavelengths* are needed.

edge-labeling: function $\phi : E(G) \rightarrow \mathbb{R}$.

A path is **increasing** if the sequence of its edges labels is non-decreasing.

An edge-labeling of G is **good** if, for any two distinct vertices u, v , there is at most one increasing (u, v) -path.

Example

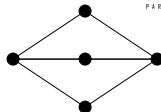
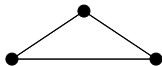


Problem [Bermond, Cosnard, and Pérennes [2013]]

Given a graph G , does G have a good edge-labeling?

- ▶ G is good under ϕ if, and only if, G is good under **injective integer-valued** ϕ' ;
- ▶ G is good if, and only if, G does not have two **disjoint** (u, v) -increasing paths, for every $u, v \in V(G)$.

Bad graphs



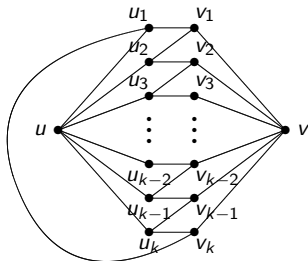
C_3 and $K_{2,3}$ are bad.

Problem (J.-S. Sereni)

Are all $\{C_3, K_{2,3}\}$ -free graphs good? **No!**

Proposition (Araújo, Cohen, Giroire, and Havet [2012])

The graph H_k has either an **increasing** (u, v) -path or an **increasing** (v, u) -path, for any edge-labeling.



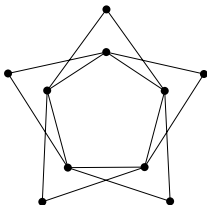


Figure: A bad graph: for any edge-labeling, in the central 5-cycle there are three adjacent edges uv , vw , wx forming an increasing path P_1 . But then, there are two other internally-disjoint (u, x) -paths P_2 and P_3 of length two. Given that a 2-path is either increasing or decreasing, two of P_1, P_2, P_3 are either increasing or decreasing paths.

[de Andrade, Araújo, Morelle, Sau, and Silva, 2024]

An even newer bad example

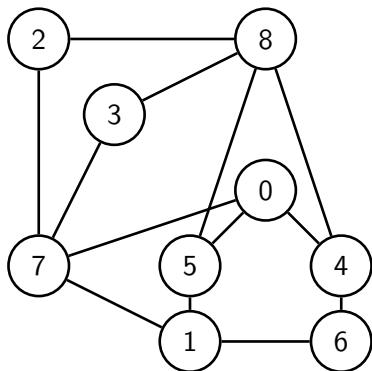


Figure: A bad example on 9 vertices.

Theorem (Araújo, Cohen, Giroire, and Havet [2012])

Deciding if a given bipartite graph G is good is NP-complete.

Classes of good graphs:

[Araújo, Cohen, Giroire, and Havet, 2012]

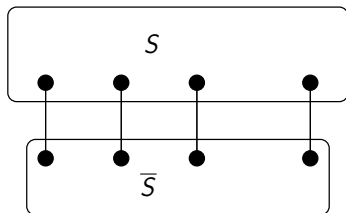
- ▶ C_3 -free outerplanar;
- ▶ $\{C_3, K_{2,3}\}$ subcubic;
- ▶ $|E(G)| < \frac{3}{2}(|V(G)| - 1)$.

Key idea: Matching cuts!

G is good if, and only if, it does not contain a bad graph H as subgraph.

Critical graph

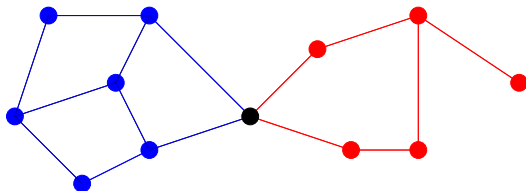
A graph G is critical if it is **bad** and, for every proper $H \subset G$, H is **good**.



Lemma (Araújo, Cohen, Giroire, and Havet [2012])

A critical graph does not contain any matching-cut.

cut-vertex: vertex x s.t. $G - x$ is not connected.



Lemma (Araújo, Cohen, Giroire, and Havet [2012])

A critical graph does not contain any cut-vertex.

Theorem (Mehrabian [2012])

A good graph G on n vertices such that its maximum degree is within a constant factor of its average degree has at most $n^{1+o(1)}$ edges.

Corollary (Mehrabian [2012])

There are bad graphs with arbitrarily large girth.

Theorem (Mehrabian [2012])

For any Δ , there is a g such that any graph with maximum degree at most Δ and girth at least g is good.

Theorem (Mehrabian, Mitsche, and Pralat [2013])

Any good graph on n vertices has at most $n \log_2(n)/2$ edges and this bound is tight for infinitely many values of n .

Unpublished work by Bode, Farzad, and Theis [2011].

Temporal Graphs

Edge-labeled graph \rightarrow temporal graph.

[Kempe, Kleinberg, and Kumar, 2000, Michail, 2015]

Increasing path \rightarrow temporal path.

Good edge-labeling \rightarrow temporization with low tmp. connectivity.

- ▶ For $c \in \mathbb{Z}_+^*$, an edge-labeling λ of G is a **c -edge-labeling** if λ takes at most c distinct values.
- ▶ A good c -edge-labeling \iff **c -gel**.
- ▶ A graph admitting a c -gel is **c -good**, otherwise it is **c -bad**.

GOOD c -EDGE-LABELING (c -GEL for short)

Input: A graph G .

Question: Does G admit a c -gel?

An example about c -GEL

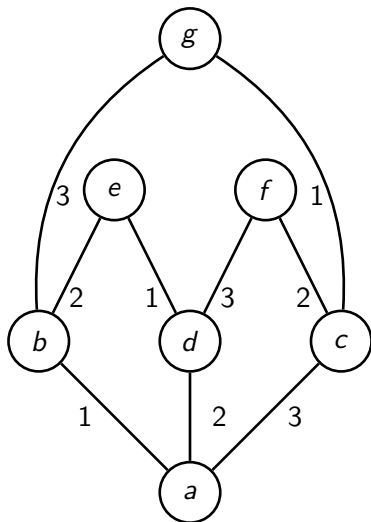


Figure: A 3-GEL exists

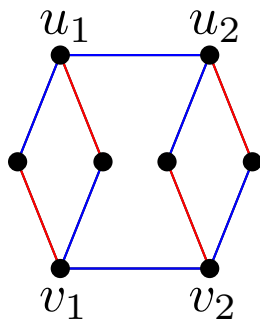


Figure: Edges u_1u_2 and v_1v_2 must have the same label in a 2-GEL

[de Andrade, Araújo, Morelle, Sau, and Silva, 2024]

- ▶ c -GEL is in NP, for every $c \geq 2$.
- ▶ c -GEL is NP-hard, for every $c \geq 2$, even for bipartite bounded-degree graphs having $(c + 1)$ -GEL.

Parameterized Complexity:

- ▶ Linear kernel for neighborhood diversity.
- ▶ Quadratic kernel for vertex cover.
- ▶ GEL is FPT parameterized by star-forest modulator.
- ▶ c -GEL is FPT parameterized by $tw + c$:
 - ▶ Via Courcelle's Theorem.
 - ▶ Explicit dynamic programming algorithm running in time $c^{\mathcal{O}(tw^2)} \cdot n$ on n -vertex graphs.
- ▶ GEL is FPT parameterized by $tw + \Delta$:
 - ▶ GEL can be solved in time $2^{\mathcal{O}(tw\Delta^2 + tw^2 \log \Delta)} \cdot n$ on n -vertex graphs, minimizing number of used labels.
- ▶ Deciding whether G has an UPP orientation is NP-complete.

Not-All-Equal 3-SAT

Input: 3-CNF boolean formula φ , each clause with **exactly** three literals.

Question: Does there exist a truth assignment such that, for each clause C , **its literals are not all set to true** (nor to false)?

Example

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$$

$x_1 = x_2 = x_3 = \text{true}$ is **NOT** a satisfying assignment!

NP-complete problem.

[Darmann and Döcker, 2020]

Reduction from NAE 3-SAT:

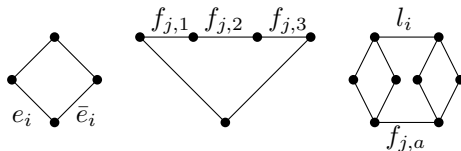


Figure: Gadgets for the reduction to 2-GEL: from left to right, X_i , C_j , and $P_{i,j,a}$.

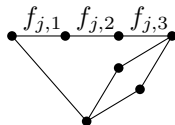


Figure: Bipartite variation of the clause gadget.

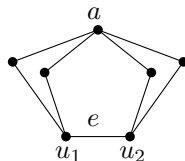


Figure: The extremal gadget X .

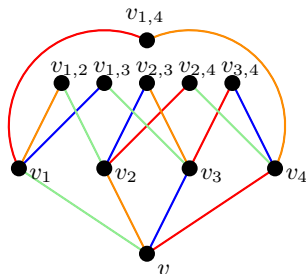


Figure: The c -color gadget D_c for $c = 4$ with a c -gel where each color represents a label.

- ▶ A **parameterized problem** is a language $L \subseteq \Sigma^* \times \mathbb{N}$, for some finite alphabet Σ . k is the **parameter**.
- ▶ For a computable function $g: \mathbb{N} \rightarrow \mathbb{N}$, a **kernelization algorithm** (or simply a **kernel**) for a parameterized problem L of **size** g is an algorithm A that given any instance (x, k) of L , runs in polynomial time and returns an instance (x', k') such that $(x, k) \in L \Leftrightarrow (x', k') \in L$ with $|x'|, k' \leq g(k)$.
- ▶ The function $g(k)$ is called the **size** of the kernel, and a kernel is **polynomial** (resp. **linear**, **quadratic**) if $g(k)$ is a polynomial (resp. linear, quadratic) function.

[Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, and Saurabh, 2015, Fomin, Lokshtanov, Saurabh, and Zehavi, 2019]

Rule

If G contains K_3 or $K_{2,3}$ as a subgraph, report a no-instance.

Rule

If G is disconnected, consider each connected component separately.

Rule

Let v be a cut-vertex in G and let C be the vertex set of a connected component of $G \setminus v$. If $G[C \cup \{v\}]$ is good (or c -good if we deal with c -GEL), delete C from G .

Rule (For GEL only)

If G has a matching cut S , consider each connected component of $G - S$ separately.

- ▶ The **neighborhood diversity** of G , $\text{nd}(G)$, is the minimum w s.t. $V(G)$ can be partitioned into w sets of vertices having **the same type**.
- ▶ u and v have **the same type** if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$.
- ▶ $\text{nd}(G)$ can be computed in cubic time.

[Lampis, 2012]

Idea:

- ▶ Find partition.
- ▶ Apply Reduction rules.
- ▶ Check that each part is small.

Note that $nd(G) \leq 2^{vc(G)} + vc(G)$.

Idea:

- ▶ Use 2-approx. alg. to get vertex cover S with $|S| \leq 2 \cdot vc(G)$.
- ▶ Apply reduction rules.
- ▶ No vertex of degree 1 remains.
- ▶ At most $2 \cdot \binom{2k}{2}$ vertices of \bar{S} with degree at least 2.

$$\text{tw}(G) - 1 \leq \text{fvs}(G) \leq \text{sfm}(G) \leq \text{vc}(G)$$

$\text{nd}(G) \leq 2^{\text{vc}(G)} + \text{vc}(G)$, but incomparable to tw , fvs , or sfm .

Is GEL FPT parameterized by fvs or by tw only? Is it even in XP?

Question

Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any graph G , if G is good then G is $f(\text{tw})$ -good, where tw is the treewidth of G ?

Question

Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any graph G , if G is good then G is $f(\Delta)$ -good, where Δ is the maximum degree of G ?

Thank you for your attention!

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