Lecture 4: Fixed-Parameter Intractability

(A Short Introduction to Parameterized Complexity)

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Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Overview

- Motivation for fixed-parameter intractability
- Fixed parameter reductions
- The classes para-NP and XP
- The class W[P]
- Logic preliminaries (continued)
- W-hierarchy
  - definitions
    - with Boolean circuits
    - as parameterized weighted Fagin definability problems
- A-hierarchy
  - definition as parameterized model-checking problems
- picture overview of these classes

Two classical problems

QUERIES

Instance: a relational database D, a conjunctive query α.

Compute: answer to query α from database D.

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Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ

Parameter: size $|\varphi|$ of formula φ

Question: Does $\mathcal{K} \models \varphi$ hold?

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- LTL-MODEL-CHECKING ∈ PSPACE-complete.

Comparing their parameterizations

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- QUERIES ∈ $O(n^k)$ for $n = \|D\|$, which does not give an FPT result.

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LTL-MODEL-CHECKING

Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ

Parameter: size $k = |\varphi|$ of formula φ

Question: Does $\mathcal{K} \vDash \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

Fixed-parameter intractability

'The purpose [. . .] is to give evidence that certain problems are not fixed-parameter tractable (just as the main purpose of the theory of NP*-completeness is to give evidence that certain problems are not polynomial time computable.)*

In classical theory, the notion of NP*-completeness is central to a nice, simple, and far-reaching theory for intractable problems.*

Unfortunately, the world of parameterized intractability is more complex: There is a big variety of seemingly different classes of intractable parameterized problems.'

(Flum, Grohe [2])

Fixed-Parameter tractable

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$$\exists f : \mathbb{N} \to \mathbb{N} \text{ computable } \exists p \in \mathbb{N}[X] \text{ polynomial}$$

$$\exists \mathbb{A} \text{ algorithm, takes inputs in } \Sigma^*$$

$$\forall x \in \Sigma^* \big[\mathbb{A} \text{ decides whether } x \in Q \text{ holds in time } \leq f(\kappa(x)) \cdot p(|x|) \big]$$

Slices of parameterized problems

The ℓ*-th slice*, for $\ell \in \mathbb{N}$, of a parameterized problem $\langle Q, \kappa \rangle$ is:

$$\langle Q, \kappa \rangle_{\ell} := \{ x \in Q \mid \kappa(x) = \ell \} \ .$$

Proposition (slices of FPT problems are in PTIME)

Let $\langle Q, \kappa \rangle$ be a parameterized problem, and $\ell \in \mathbb{N}$.
If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_{\ell} \in$ PTIME.

Proof

Let ℓ be fixed. Then for all $x \in \Sigma^*$:
Decide $x \in Q$, $\kappa(x) = \ell$ in time $\leq f(\kappa(x)) \cdot p(|x|) = f(\ell) \cdot p(|x|) \in$ PTIME.

A problem not in FPT

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Slices of FPT problems are in PTIME

If $\langle Q, \kappa \rangle \in$ FPT, then $\langle Q, \kappa \rangle_{\ell} \in$ PTIME.

p-COLORABILITY

Instance: A graph $\mathcal{G}$, and $\ell \in \mathbb{N}$.

Parameter: ℓ.

Problem: Decide whether $\mathcal{G}$ is ℓ-colorable.

Consequence: p-COLORABILITY $\notin$ FPT (unless P = NP).

It is well-known: 3-COLORABILITY $\in$ NP-complete. Now since 3-COLORABILITY is the third slice of p-COLORABILITY, the proposition entails p-COLORABILITY $\notin$ FPT unless P = NP.

Polynomial reductions / hardness / completeness

Definition

Let $\langle Q_1, \Sigma_1 \rangle$, $\langle Q_2, \Sigma_2 \rangle$ be classical problems.
An *polynomial-time reduction* from $\langle Q_1, \Sigma_1 \rangle$ to $\langle Q_2, \Sigma_2 \rangle$ is a mapping $R : \Sigma_1^* \to \Sigma_2^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a polynomial-time algorithm: there is a polynomial $p(X)$ such that R is computable in time $p(|x|)$.

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$\langle Q_1, \Sigma_1 \rangle \leq_{\text{pol}} \langle Q_2, \Sigma_2 \rangle :=$
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$$\langle Q_1, \Sigma_1\rangle \in \mathrm{P} \Longleftarrow \langle Q_2, \Sigma_2\rangle \in \mathrm{P}.$$
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Let C be class of classical problems.

- $\langle Q, \Sigma \rangle$ is C-hard: if, for all $\langle Q', \Sigma' \rangle \in$ C, $\langle Q', \Sigma' \rangle \leq_{\text{pol}} \langle Q, \Sigma \rangle$.

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- $\langle Q, \Sigma \rangle$ is C-hard: if, for all $\langle Q', \Sigma' \rangle \in$ C, $\langle Q', \Sigma' \rangle \leq_{\text{pol}} \langle Q, \Sigma \rangle$.
- $\langle Q, \Sigma \rangle$ is C-complete: if $\langle Q, \Sigma \rangle$ is C-hard, and $\langle Q, \Sigma \rangle \in$ C.

Fixed-parameter tractable reductions

Definition

Let $\langle Q_1, \Sigma_1, \kappa \rangle$, $\langle Q_2, \Sigma_2, \kappa_2 \rangle$ be parameterized problems.
An *fpt-reduction* from $\langle Q_1, \kappa_1 \rangle$ to $\langle Q_2, \kappa_2 \rangle$ is a mapping $R : \Sigma_1^* \to (\Sigma_2)^*$:

R1. $(x \in Q_1 \iff R(x) \in Q_2)$ for all $x \in \Sigma_1^*$.

R2. R is computable by a fpt-algorithm (with respect to κ): there are f computable and $p(X)$ polynomial such that R is computable in time $f(\kappa_1(x)) \cdot p(|x|)$.

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R3. $\kappa_2(R(x)) \le g(\kappa_1(x))$ for all $x \in \Sigma_1^*$, for some computable function $g : \mathbb{N} \to \mathbb{N}$.

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Proposition

If $\langle Q_1, \kappa_1\rangle \le_{\text{fpt}} \langle Q_2, \kappa_2\rangle$, then:

$$\begin{aligned} \langle Q_1, \kappa_1\rangle \in \text{FPT} &\Longleftarrow \langle Q_2, \kappa_2\rangle \in \text{FPT}. \\ \langle Q_1, \kappa_1\rangle \notin \text{FPT} &\Longrightarrow \langle Q_2, \kappa_2\rangle \notin \text{FPT}. \end{aligned}$$

Comparing parameterizations (revisited)

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
- $\kappa_1 \succ \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \neg\big(\kappa_2 \succeq \kappa_1\big)$.

Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $\kappa_1 \succeq \kappa_2$:

$$\begin{aligned}
\langle Q, \kappa_1 \rangle \in \mathrm{FPT} \quad &\Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \mathrm{FPT}\,, \\
\langle Q, \kappa_1 \rangle \notin \mathrm{FPT} \quad &\Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \mathrm{FPT}\,.
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Proposition

For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with $Q \subseteq \Sigma^*$:

$$\kappa_1 \succeq \kappa_2 \quad\Longleftrightarrow\quad \langle Q, \kappa_1 \rangle \leq_{\mathsf{fpt}} \langle Q, \kappa_2 \rangle \text{ via } R : \Sigma^* \to \Sigma^*,\ x \mapsto x.$$

Fixed-parameter tractable reductions

Examples

- p-CLIQUE $\equiv_{\text{fpt}}$ p-INDEPENDENT-SET.
- p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.

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Non-Example

- For graphs $\mathcal{G} = \langle V, E \rangle$, and sets $X \subseteq V$:

  $$X \text{ is independent set of } \mathcal{G} \iff V \setminus X \text{ is a vertex cover of } \mathcal{G}$$

  yields a polynomial reduction between p-INDEPENDENT-SET and p-VERTEX-COVER, but does not yield an fpt-reduction.

Fpt-reduction closure / hardness / reducibility

Let C be a class of parameterized problems.

We define for all parameterized problems $\langle Q, \kappa \rangle$:

- $\langle Q, \kappa \rangle$ is C*-hard under fpt-reductions* if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$

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  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
- $\langle Q, \kappa \rangle$ is C*-complete under fpt-reductions*
  if $\langle Q, \kappa \rangle \in$ C and $\langle Q, \kappa \rangle$ is C-hard under fpt-reductions,

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- $\big[\langle Q, \kappa \rangle\big]^{\text{fpt}} := \big\{\langle Q', \kappa' \rangle \mid \langle Q', \kappa' \rangle \leq_{\text{fpt}} \langle Q, \kappa \rangle\big\}$.
- $\big[\mathsf{C}\big]^{\text{fpt}} := \bigcup_{\langle Q, \kappa \rangle \in \mathsf{C}} \big[\langle Q, \kappa \rangle\big]^{\text{fpt}}$
  is the *closure of* C *under fpt-reductions.*
- $\langle Q, \kappa \rangle$ is C*-hard under fpt-reductions*
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
- $\langle Q, \kappa \rangle$ is C*-complete under fpt-reductions*
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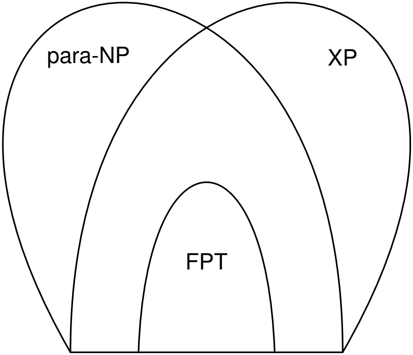
  is the *closure of* C *under fpt-reductions.*
- $\langle Q, \kappa \rangle$ is C*-hard under fpt-reductions*
  if every problem in C is fpt-reducible to $\langle Q, \kappa \rangle$
  that is: $\mathsf{C} \subseteq \big[\langle Q, \kappa \rangle\big]^{\text{fpt}}$, and hence $\big[\mathsf{C}\big]^{\text{fpt}} \subseteq \big[\langle Q, \kappa \rangle\big]^{\text{fpt}}$.
- $\langle Q, \kappa \rangle$ is C*-complete under fpt-reductions*
  if $\langle Q, \kappa \rangle \in \mathsf{C}$ and $\langle Q, \kappa \rangle$ is C-hard under fpt-reductions,
  and then: $\big[\mathsf{C}\big]^{\text{fpt}} = \big[\langle Q, \kappa \rangle\big]^{\text{fpt}}$.

para-NP and XP



para-NP

Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is in para-NP if there is a computable function $f : \mathbb{N} \to \mathbb{N}$, and a polynomial $p \in \mathbb{N}[X]$ such that there is a non-deterministic algorithm $\mathbb{A}$ such that:

- $\mathbb{A}$ decides, for all $x \in \Sigma^*$, whether $x \in Q$ in $\leq f(\kappa(x)) \cdot p(|x|)$ steps.

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- NP $\subseteq$ para-NP.

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Example

- p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, p-HITTING-SET, p-COLORABILITY $\in$ para-NP.

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Example

- p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, p-HITTING-SET, p-COLORABILITY ∈ para-NP.

- FPT = para-NP if and only if PTIME = NP.

para-NP

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- para-NP is closed under fpt-reductions.
- NP ⊆ para-NP.

Example

- p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, p-HITTING-SET, p-COLORABILITY ∈ para-NP.

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  - p-COLORABILITY $\in$ para-NP-complete.

XP (slicewise polynomial problems)

Recall: slices of FPT-problems are in PTIME. This suggests a class:

$\mathrm{XP}_{\mathrm{nu}}$, *non-uniform* XP : the class of parameterized problems $\langle Q, \kappa \rangle$, whose slices $\langle Q, \kappa \rangle_k$ are all in PTIME.

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A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is in XP if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that there is an algorithm $\mathbb{A}$ such that:

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equivalently, if in addition to computable $f : \mathbb{N} \to \mathbb{N}$ there are polynomials $p_k \in \mathbb{N}[X]$ for all $k \in \mathbb{N}$ such that:

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XP (slicewise polynomial problems)

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- p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, p-HITTING-SET $\in$ XP.
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FPT $\subsetneqq$ XP.

Model checking

The *model checking problem* for a class Φ of first-order formulas:

MC(Φ)

Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Problem: Decide whether $\mathcal{A} \vDash \varphi$ (that is, $\varphi(\mathcal{A}) \neq \varnothing$).

Theorem

MC(FO) *can be solved in time* $O(|\varphi| \cdot |A|^{w} \cdot w)$, *where* w *is the width of the input formula* φ *(max. no. of free variables in a subformula of* φ*).*

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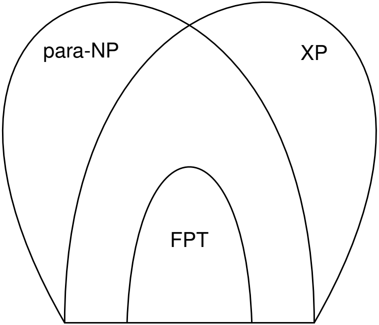
The *parameterized model checking problem* for a class Φ of formulas:

p-MC(Φ).
  Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
  Parameter: $|\varphi|$.
  Problem: Decide whether $\mathcal{A} \models \varphi$.

Theorem

p-MC(Φ) $\in$ XP*.*

FPT versus para-NP and XP

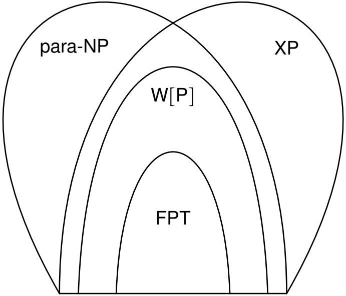


Proposition

- FPT $\subseteq$ para-NP, and:
  FPT $=$ para-NP if and only if PTIME $=$ NP.
- para-NP $\nsubseteq$ XP if PTIME $\neq$ NP.
- FPT $\subsetneqq$ XP.

W[P]

> *‘There is no definite single class that can be viewed as “the parameterized NP”. Rather, there is a whole hierarchy of classes playing this role.*
>
> *The class W[P] can be placed on top of this hierarchy. It is one of the most important parameterized complexity classes.’*
>
> (Flum, Grohe [2])



W[P] and limited non-determinism

$\langle Q, \Sigma \rangle \in$ NP$[f]$ means:

$\iff \exists p(X)$ polynomial $\exists \mathbb{M}$ non-deterministic Turingmachine
$\big(\forall x \in \Sigma^{*}\big(\,(x \in Q \iff \mathbb{M}$ accepts $x)$
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Fact

$$\mathsf{NP}[\log n] = \mathsf{P}, \qquad \mathsf{NP}[n^{O(1)}] = \mathsf{NP}.$$

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Definition

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  - at most $f(\kappa(x)) \cdot p(|x|)$ steps,
  - at most $h(\kappa(x)) \cdot \log |x|$ of them being nondeterministic,
- W[P] contains all problems $\langle Q, \kappa \rangle$ that can be decided by a κ-restricted nondeterministic Turing machine.

W[P] (properties)

Theorems

T1. FPT ⊆ W[P] ⊆ XP ∩ para-NP

T2. W[P] is closed under fpt-reductions.

T3. p-CLIQUE, p-INDEPENDENT-SET, p-DOMINATING-SET, and p-HITTING-SET are in W[P].

The W-hierarchy – Boolean circuits

A *(Boolean) circuit* is a DAG in which nodes are labeled:

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W[P] complete problems

p-WSAT(CIRC)

Instance: A circuit $\mathcal{C}$ and $k \in \mathbb{N}$

Parameter: k.

Problem: Decide whether $\mathcal{C}$ is k-satisfiable.

Theorem

p-WSAT(CIRC) *is* W[P]*-complete under fpt-reductions.*

Definition

The depth of the circuit is the max. length of a path from an input node to the output node. Small nodes have indegree at most 2 while large nodes have indegree > 2. The weft of a circuit is the max. number of large nodes on a path from an input node to the output node. We denote by $\text{CIRC}_{t,d}$ the class of circuits with weft $\leq t$ and depth $\leq d$.

Application

p-DOMINATING-SET $\in$ W[P], since it reduces to p-WSAT($\text{CIRC}_{2,3}$).

Limited non-determinism (classically)

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Fact

$$\text{NP}[\log n] = \text{P}, \qquad \text{NP}[n^{O(1)}] = \text{NP}.$$

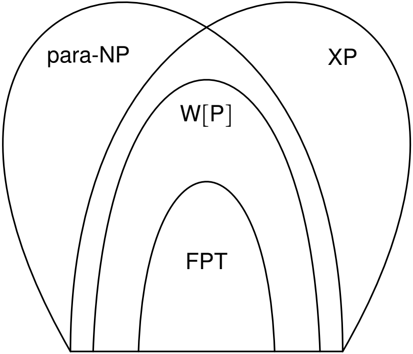
Theorem (Cai, Chen, 1997)

The following are equivalent:

(i) FPT = W[P].

(ii) *There is a computable, nondecreasing, unbounded function* $\iota : \mathbb{N} \to \mathbb{N}$ *such that* P = NP$[\iota(n) \cdot \log n]$.

FPT and W[P] versus para-NP and XP



Proposition

$\mathrm{FPT} \subseteq \mathrm{W[P]} \subseteq \mathrm{XP} \cap \text{para-NP}$.

Why is the theory of W[P]/W/A-hardness important?

- Prevents from wasting hours tackling a problem which is fundamentally difficult;
- Finding results on a problem is always a ping-pong game between trying to design a hardness/FPT result;
- There is a hierarchy on parameters and it is worth knowing which is the smallest one such that the problem remains FPT;
- There is a hierarchy on complexity classes and it is worth noting to which extent a problem is hard.

Logic preliminaries (continued)

- *atomic formulas/atoms*: a formula $x = y$ or $Rx_1 \dots x_n$
- *literal*: an atom or a negated atom
- *quantifier-free formula*: a formula without quantifiers
- formula in *negation-normal form*:
  negations only occur in front of atoms
- formula in *prenex normal form*: formula of the form
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  and $Q_1, \dots, Q_k \in \{\exists, \forall\}$
- Σ_0 and Π_0 : the class of quantifier-free formulas
- Σ_{t+1} : class of all formulas $\exists x_1 \dots \exists x_k \varphi$ where $\varphi \in \Pi_t$
- Π_{t+1} : class of all formulas $\forall x_1 \dots \forall x_k \varphi$ where $\varphi \in \Sigma_t$

Weighted Fagin definability

Let $\varphi(X)$ be a f-o formula with a free relation variable X with arity s.
Let τ be a vocabulary for φ, plus a relation symbol R of arity s.

A *solution for φ in a τ-structure $\mathcal{A}$* is a relation $S \subseteq A^s$ such that $\mathcal{A} \vDash \varphi(\overline{S})$.

The *weighted Fagin definability problem* for $\varphi(X)$ is:

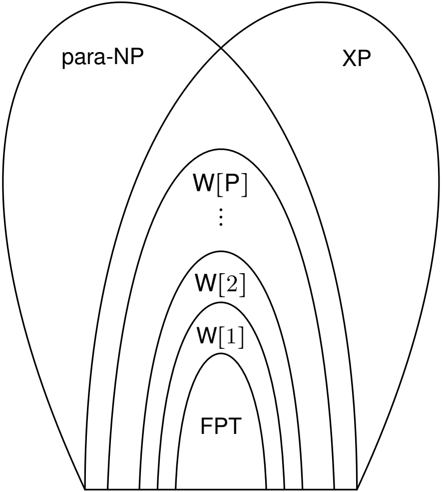
WD$_\varphi$

Instance: A structure $\mathcal{A}$ and $k \in \mathbb{N}$.

Problem: Decide whether there is a solution $S \subseteq A^s$ for φ of cardinality $|S| = k$.

WD$_\Phi$: the class of all problems WD$_\varphi$ with $\varphi \in \Phi$, where Φ is a class of first-order formulas with free relation variable X.

W-Hierarchy



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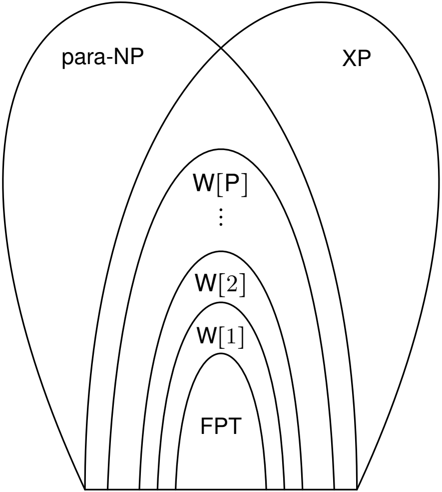

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p-WD$_\varphi$ (φ a fo-formula with free relation variable X of arity s)
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Definition (Downey–Fellows, 1995)

$W[t] := \left[p\text{-WD-}\Pi_t\right]^{\text{fpt}}$, for $t \geq 1$, form the *W-hierarchy*.

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Examples

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- p-DOMINATING-SET $\in$ W$[2]$.
- p-HITTING-SET $\in$ W$[2]$.

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- p-Clique $\in$ W$[1]$.

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Problem: Is there a relation $S \subseteq A^s$ of cardinality $|S| = k$ with $\mathcal{A} \models \varphi(S)$.

p-WD-Φ: the class of all problems p-WD-φ with $\varphi \in \Phi$, Φ is a class of first-order formulas.

Definition (Downey–Fellows, 1995)

$W[t] := \left[p\text{-WD-}\Pi_t\right]^{\text{fpt}}$, for $t \geq 1$, form the *W-hierarchy*.

Examples

- p-DOMINATING-SET $\in$ W$[2]$.

W-Hierarchy

p-WD$_\varphi$ (φ a fo-formula with free relation variable X of arity s)

Instance: A structure $\mathcal{A}$ and $k \in \mathbb{N}$.

Parameter: k.

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- p-HITTING-SET $\in$ W$[2]$.

W-hierarchy

Definition

(W-hierarchy) For $t \geq 1$, a parameterized problem $\langle Q, \kappa \rangle$ belongs to the class W$[t]$ if there is a parameterized reduction from $\langle Q, \kappa \rangle$ to p-WSAT(CIRC$_{t,d}$) (with parameter t) for some $d \geq 1$.

FPT $\subseteq$ W$[1] \subseteq$ W$[2] \ldots$

- p-CLIQUE, p-INDEPENDENT-SET are W$[1]$-Complete.
- p-DOMINATING-SET, p-HITTING-SET are W$[2]$-Complete.

Hypothesis: W$[1] \neq$ FPT

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Proposition

This definition of the W-hierarchy is equivalent to the one here before. That is, it holds, for all $t \geq 1$:

$$\mathrm{W}[t] = \left[\{p\text{-WSAT}(\mathrm{CIRC}_{t,d}) \mid d \geq 1\}\right]^{\mathrm{fpt}}.$$

W-Hierarchy (properties)

Immediate from definition follows: $\left[p\text{-WD-FO}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty} \text{W}[i]$.

Theorems

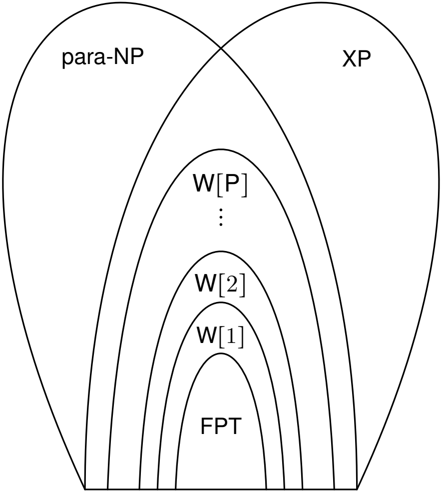
T1. p-WD-FO $\subseteq$ W[P], and hence $\text{W}[t] \subseteq \text{W[P]}$ for all $t \geq 1$.

T2. $p\text{-WD-}\Sigma_1 \subseteq \text{FPT}$.

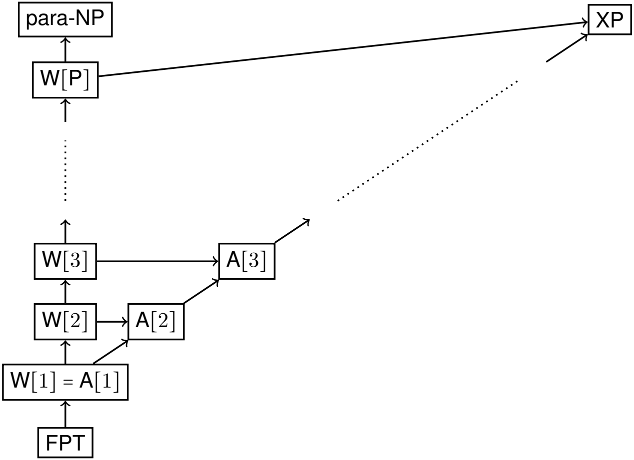
T3. $p\text{-WD-}\Sigma_{t+1} \subseteq p\text{-WD-}\Pi_t$, for all $t \geq 1$.

T4. $\text{W}[t] = \left[p\text{-WD-}\Sigma_{t+1}\right]^{\text{fpt}}$ for all $t \geq 1$.

W-Hierarchy versus para-NP and XP



A-Hierarchy



A-Hierarchy (definition and examples 1,2)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

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Examples

- p-CLIQUE $\in$ A$[1]$.
- p-DOMINATING-SET $\in$ A$[2]$.

A-Hierarchy (definition and examples 3,4)

The parameterized model checking problem for a class Φ of formulas:

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The parameterized model checking problem for a class Φ of formulas:

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Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-HITTING-SET $\in$ A$[2]$.
- p-SUBGRAPH-ISOMORPHISM $\in$ A$[1]$.

A-Hierarchy (example 5)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \emptyset$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-SUBGRAPH-ISOMORPHISM $\in$ A$[1]$.

p-SUBGRAPH-ISOMORPHISM
Instance: Graphs $\mathcal{G}$ and $\mathcal{H}$.
Parameter: The number of vertices of $\mathcal{H}$.
Problem: Does $\mathcal{G}$ have a subgraph isomorphic to $\mathcal{H}$.

A-Hierarchy (example 6)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-VERTEX-DELETION $\in$ A$[2]$.

p-VERTEX-DELETION
Instance: Graphs $\mathcal{G}$ and $\mathcal{H}$, and $k \in \mathbb{N}$.
Parameter: $k + \ell$, where ℓ the number of vertices of $\mathcal{H}$.
Problem: Is it possible to delete at most k vertices from $\mathcal{G}$ such that the resulting graph has no subgraph isomorphic to $\mathcal{H}$?

A-Hierarchy (example 7)

The parameterized model checking problem for a class Φ of formulas:

p-MC(Φ)
Instance: A structure $\mathcal{A}$ and a formula $\varphi \in \Phi$.
Parameter: $|\varphi|$.
Problem: Decide whether $\varphi(\mathcal{A}) \neq \varnothing$.

Definition (Flum, Grohe, 2001)

$A[t] := \left[p\text{-MC}(\Sigma_t)\right]^{\text{fpt}}$, for $t \geq 1$, form the *A-hierarchy*.

Examples

- p-CLIQUE-DOMINATING-SET $\in$ A$[2]$.

p-CLIQUE-DOMINATING-SET
Instance: Graphs $\mathcal{G}$, and $k, \ell \in \mathbb{N}$.
Parameter: $k + \ell$, where ℓ the number of vertices of $\mathcal{H}$.
Problem: Decide whether $\mathcal{G}$ contains a set of k vertices from $\mathcal{G}$ that dominates every clique of ℓ elements.

A-Hierarchy (properties)

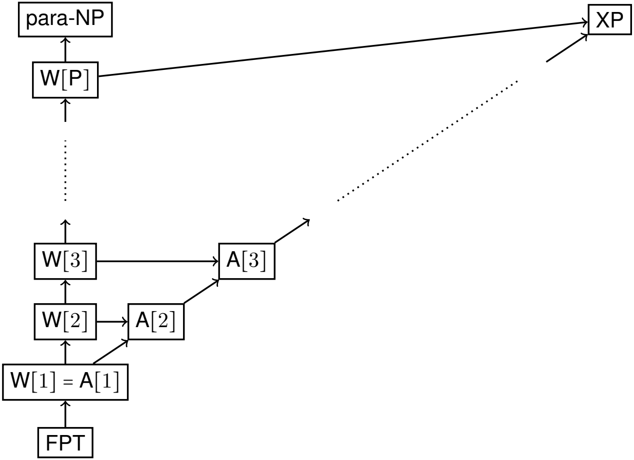
Theorems

T1. A$[1] \subseteq$ W[P].

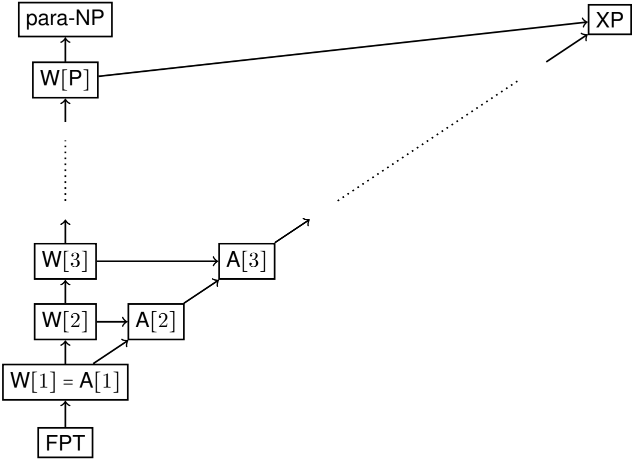
T2. W$[t] \subseteq$ A$[t]$, for all $t \in \mathbb{N}$.

- Unlikely: A$[t] \subseteq$ W$[t]$, for $t > 1$.
  Reason:
  - the A-hierarchy are parameterizations of problems that are complete for the levels of the polynomial hierarchy
  - the W-hierarchy is a refinement of NP in parameterized complexity
- Unlikely: $\left[p\text{-MC(FO)}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty}$ A$[i]$,
  contrasting with: $\left[p\text{-WD-FO}\right]^{\text{fpt}} = \bigcup_{i=1}^{\infty}$ W$[i]$.

W-Hierarchy and A-Hierarchy versus para-NP and XP



W-Hierarchy and A-Hierarchy versus para-NP and XP



Revisiting the two problems at start today

QUERIES
Instance: a relational database D, a conjunctive query α.
Parameter: size $k = |\alpha|$ of query α
Compute: answer to query α from database D.

- QUERIES $\in$ NP-complete.
- QUERIES $\in O(n^k)$ for $n = \|D\|$, which does not give an FPT result.

LTL-MODEL-CHECKING
Instance: a Kripke structure (state space) $\mathcal{K}$, an LTL formula φ
Parameter: size $k = |\varphi|$ of formula φ
Question: Does $\mathcal{K} \vDash \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
- LTL-MODEL-CHECKING $\in O(k \cdot 2^{2^k} \cdot n) \in$ FPT for $n = \|\mathcal{K}\|$.

Revisiting the two problems at start today

QUERIES
Instance: a relational database D, a conjunctive query α.
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Compute: answer to query α from database D.

- QUERIES $\in$ NP-complete.
- QUERIES $\in O(n^k)$ for $n = \|D\|$, which does not give an FPT result.
- QUERIES $\in$ W[1] (= strong evidence for it likely not to be in FPT).

LTL-MODEL-CHECKING
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Question: Does $\mathcal{K} \vDash \varphi$ hold?

- LTL-MODEL-CHECKING $\in$ PSPACE-complete,
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Summary

- Motivation for fixed-parameter intractability
- Fixed parameter reductions
- The classes para-NP and XP
- The class W[P]
- Logic preliminaries (continued)
- W-hierarchy
    - definitions
        - with Boolean circuits
        - as parameterized weighted Fagin definability problems
- A-hierarchy
    - definition as parameterized model-checking problems
- picture overview of these classes

Course overview

Monday, July 14 10.30 – 12.30	Tuesday, July 15 10.30 – 12.30	Wednesday, July 16 10.30 – 12.30	Thursday, July 17 10.30 – 12.30	Friday, July 18
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
Introduction & basic FPT results	**Notions of bounded graph width**	**Algorithmic Meta-Theorems**	**FPT-Intractability Classes & Hierarchies**	
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. width	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies	
				14.30 – 16.30
				examples, question hour

Example suggestions

Examples

1. FPT results transfer backwards over fpt-reductions:
   If $\langle Q_1, \kappa_1 \rangle \leq_{\text{fpt}} \langle Q_2, \kappa_2 \rangle$, then $Q_2 \in$ FPT implies $Q_1 \in$ FPT.
2. Find the idea for:
   p-DOMINATING-SET $\equiv_{\text{fpt}}$ p-HITTING-SET.
3.

References

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