

Lecture 5: Three More Models

Models of Computation

<https://clegra.github.io/moc.html>

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Ph.D. Program, Advanced Courses Period

Gran Sasso Science Institute

L'Aquila, Italy

July 10, 2026

Course overview

Monday, July 6 10.30 – 12.30	Tuesday, July 7 10.30 – 12.30	Wednesday, July 8 10.30 – 12.30	Thursday, July 9 10.30 – 12.30	Friday, July 10
<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ -calculus Herbrand–Gödel recursive functions partial-recursive/ μ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ζ -calculus evolutionary programming/genetic algorithms abstract state machines	<i>modern</i>
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

Overview

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- ▶ **Compare computational power of models of computation**
- ▶ **Fractran** (by **John Horton Conway**, 1987, [2])

Emil Post



Emil Leon Post (1897–1954)

Post's Correspondence Problem (PCP)

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Instance of PCP:

$$I = \{\langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle\}, \text{ where } k \geq 1, g_i, g'_i \in \Sigma^+ \text{ for } i \in \{1, \dots, k\}.$$

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Question: Is I solvable?

Do there exist $n \geq 1$, and $i_1, \dots, i_n \in \{1, \dots, k\}$ such that:

$$g_{i_1} g_{i_2} \dots g_{i_n} = g'_{i_1} g'_{i_2} \dots g'_{i_n} \quad ?$$

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Theorem

Codings of solvable instances of PCP:

$$\overbrace{\left\{ \left\{ \langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle \mid k \geq 1, g_i, g'_i \in \Sigma^+ \right\} \right\}}^{\text{PCP instance } I} \mid I \text{ is solvable}$$

form a set that is *recursively enumerable*, but *not recursive*.

Typical features of ‘computationally complete’ MoC’s

- ▶ storage (unbounded)
- ▶ control (finite, given)
- ▶ modification
 - ▶ of (immediately accessible) stored data
 - ▶ of control state
- ▶ conditionals
- ▶ loop (unbounded)
- ▶ stopping condition

Yves Lafont



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Interaction Nets

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Analogy with:

- ▶ electric circuits:
 - ▶ agents $\hat{=}$ gates,
 - ▶ edges $\hat{=}$ wires

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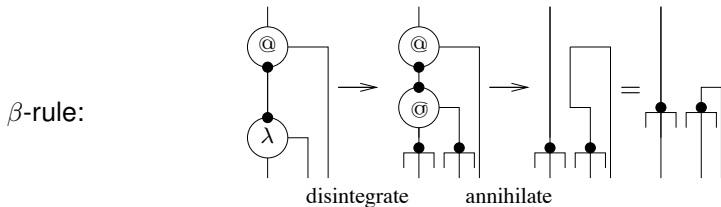
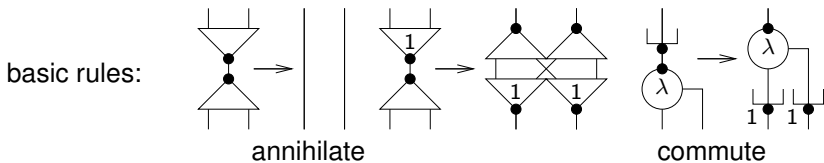
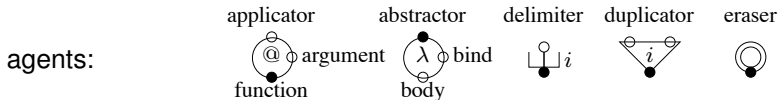
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Analogy with:

- ▶ electric circuits:
 - ▶ agents $\hat{=}$ gates,
 - ▶ edges $\hat{=}$ wires
- ▶ agents as computation entities:
 - ▶ interaction rules specify behavior

Lambdascope [Vincent van Oostrom, K.J. van de Looij, M. Zwitserlood, 2003]



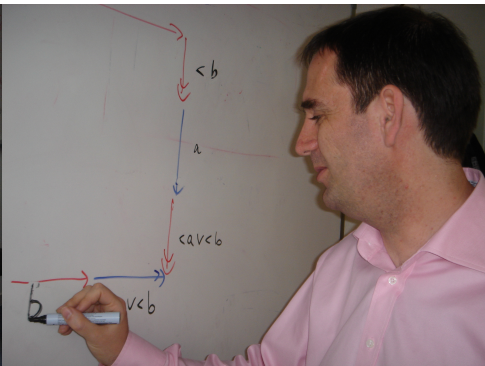
Gradations of optimal reduction

Optimal reduction in λ -calculus avoids:

- ▶ unnecessary work
- ▶ duplication of work

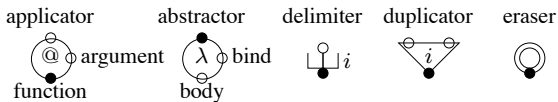
calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
λ -calculus (β -reduction \rightarrow_{β})	Lévy labelling '78	Lamping '89 Kathail '90 Abdadi/Gonthier/ /Levy '92 Asperti/Guerrini '93 VvO '03 (Terese)	context sharing
λ -calculus (weak- β red. $\rightarrow_{w\beta}$)	Blanc/Lévy/ Maranget '05/'07	Wadsworth '71 Shivers/Wand '04	extended-scope sharing
orthogonal TRS (induced rewrite relation \rightarrow)	VvO '03 (Terese)	Staples '80 VvO '03 (Terese)	subterm sharing

Vincent van Oostrom

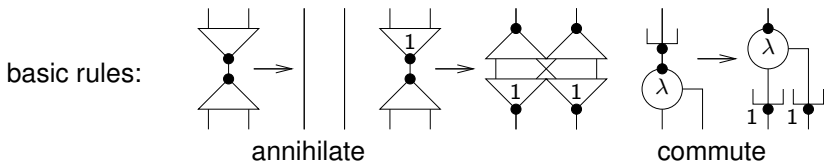
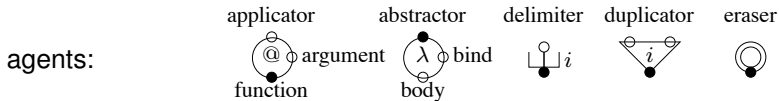


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agents:



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Graph rewriting tool by Jan Rochel



graph rewrite tool on Hackage:

<http://hackage.haskell.org/package/graph-rewriting-0.7.5>

Additional gradations of optimal reduction

calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
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	?	term/port graph implementation	scope sharing
	?	nested term graph implementation	extended scope sharing
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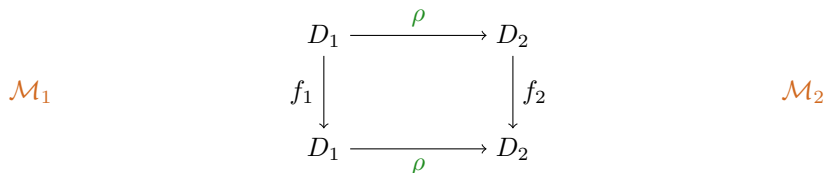
Comparing computational power via encodings

- ▶ Simulation of models of computation $\mathcal{M}_1 = \langle D_1, \mathcal{F}_1 \rangle$, $\mathcal{M}_2 = \langle D_2, \mathcal{F}_2 \rangle$:

Comparing computational power via encodings

- ▶ Simulation of functions:

function f_2 *simulates* function f_1 via *encoding* ρ if:

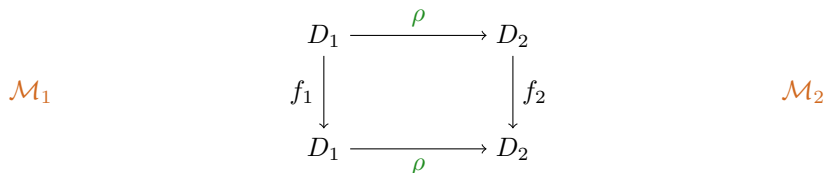


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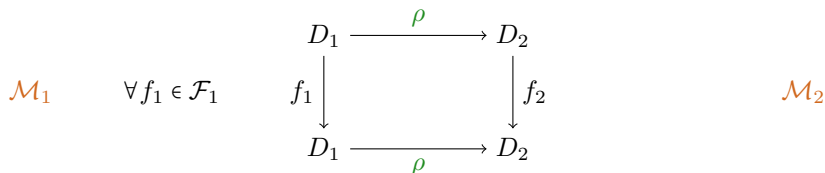


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$$\mathcal{M}_1 \quad \forall f_1 \in \mathcal{F}_1 \quad \begin{array}{ccc} D_1 & \xrightarrow{\rho} & D_2 \\ f_1 \downarrow & & \downarrow f_2 \\ D_1 & \xrightarrow{\rho} & D_2 \end{array} \quad \exists f_2 \in \mathcal{F}_2 \quad \mathcal{M}_2$$

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$$\forall f_1 \in \mathcal{F}_1 \exists f_2 \in \mathcal{F}_2 (f_2 \text{ simulates } f_1 \text{ via } \rho)$$

Weak requirements on encodings (Boker/Dershowitz)

Traditional requirements on encodings are:

- ▶ *informally computable/effective/mechanizable in principle*
- ▶ *computable* with respect to a specific model (Turing machine, ...)

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Definition (**power subsumption** pre-order [Boker/Dershowitz 2006 [1]])

- (i) $\mathcal{M}_1 \lesssim \mathcal{M}_2$ if: there is an **injective** ρ such that $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$
- (ii) $\mathcal{M}_1 \lesssim_{\text{bijective}} \mathcal{M}_2$ if: there is a **bijective** ρ such that $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$

Anomalies for decision models

However, we found anomalies of these definitions.

$\mathcal{M} = \langle D, \mathcal{F} \rangle$ is a *decision model* if $\{0, 1\} \subseteq D$, $\forall f \in \mathcal{F} (f[D] \subseteq \{0, 1\})$.

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Theorem (Endrullis/G/Hendriks, [3])

Let Σ and Γ with $\{0, 1\} \subseteq \Sigma, \Gamma$ be alphabets.

Then for every countable decision model $\mathcal{M} = \langle \Sigma^*, \mathcal{F} \rangle$, it holds:

$$\mathcal{M} \lesssim \text{DFA}(\Gamma) \quad \mathcal{M} \lesssim_{\text{bijective}} \text{DFA}(\Gamma)$$

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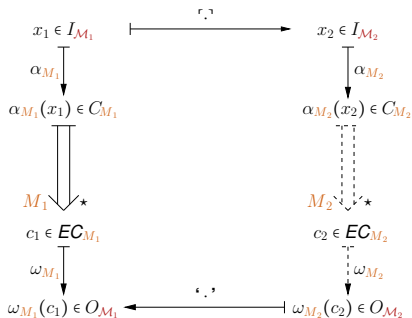
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- ▶ can be extended to **some** moc's with unbounded output domain
- ▶ but **do not extend** to **all** moc's

Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ **simulate each other** with respect to **computable** coding $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and decoding $\lrcorner \cdot \lrcorner : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ if:



Models of computation, viewed abstractly

A(n abstractly viewed) model of computation (MoC) is a class \mathcal{M} of machines/systems/. . . such that every $M \in \mathcal{M}$ it holds:

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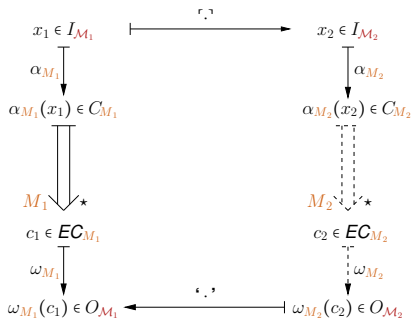
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- ▷ M defines a **one-step computation relation** \Rightarrow_M on the set C_M ; the transitive closure of \Rightarrow_M is designated by \Rightarrow_M^* ;
- ▷ M has a partial **output function** $\omega_M : EC_M \rightarrow O_{\mathcal{M}}$, which maps **some** end-configurations of M to output objects of M ; ω_M is **computable**, and membership of end-configurations in $dom(\omega_M)$ is **decidable**.

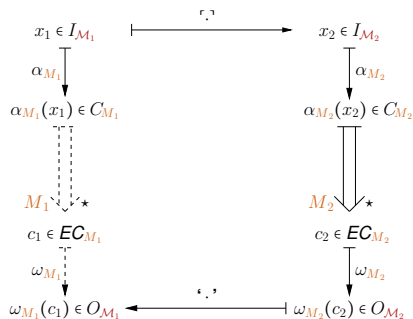
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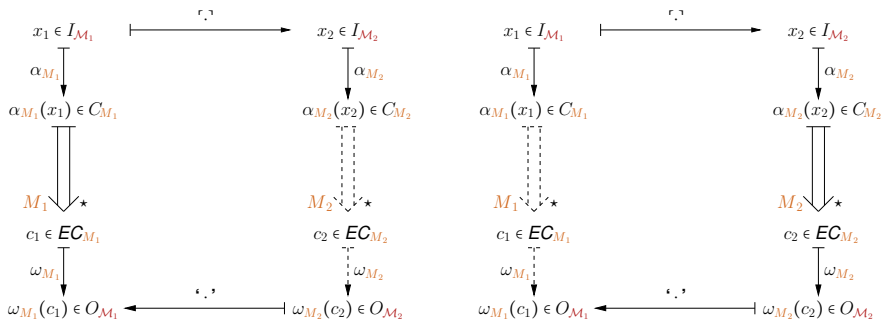
Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ **simulate each other** with respect to **computable** coding $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and decoding $\lrcorner \cdot \lrcorner : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ if:



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(defines a **Galois connection**)

Comparing Computational Power of MoC's

Definition

Let \mathcal{M}_1 and \mathcal{M}_2 be MoC's.

- ① The computational power of \mathcal{M}_1 is **subsumed** by that of \mathcal{M}_2 , denoted symbolically by $\mathcal{M}_1 \leq \mathcal{M}_2$, if:

(\exists a pair $\langle \lceil \cdot \rceil, \lfloor \cdot \rfloor \rangle$ of **computable** encoding and decoding functions $\lceil \cdot \rceil : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and $\lfloor \cdot \rfloor : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$

$(\forall M_1 \in \mathcal{M}_1) (\exists M_2 \in \mathcal{M}_2)$

$[M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \lceil \cdot \rceil, \lfloor \cdot \rfloor \rangle]$.

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Let \mathcal{M}_1 and \mathcal{M}_2 be MoC's.

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$(\forall M_1 \in \mathcal{M}_1) (\exists M_2 \in \mathcal{M}_2)$

$[M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \ulcorner \cdot \urcorner, \lrcorner \cdot \lrcorner \rangle]$.

- 2 The computational power of \mathcal{M}_1 is **equivalent** to that of \mathcal{M}_2 , denoted by $\mathcal{M}_1 \sim \mathcal{M}_2$, if both $\mathcal{M}_1 \leq \mathcal{M}_2$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ hold.

Comparing Computational Power of MoC's

Theorem

For all models \mathcal{M}_1 and \mathcal{M}_2 , and encoding and decoding functions $\lceil \cdot \rceil : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and $\lfloor \cdot \rfloor : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ it holds:

$$\mathcal{M}_1 \leq_{(\lceil \cdot \rceil, \lfloor \cdot \rfloor)} \mathcal{M}_2 \implies \mathcal{F}(\mathcal{M}_1) \subseteq \{ \lfloor \cdot \rfloor \circ f \circ \lceil \cdot \rceil \mid f \in \mathcal{F}(\mathcal{M}_2) \}.$$

Turing completeness and equivalence

By $\mathcal{TM}(\Sigma)$ we mean the model of Turing machines over input alphabet Σ .

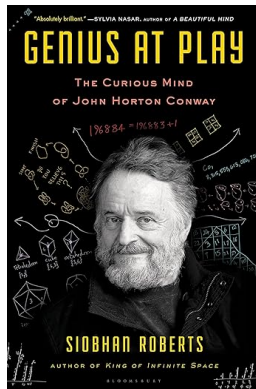
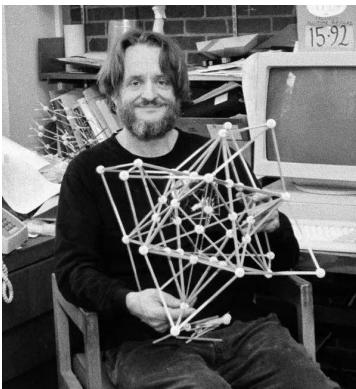
Definition

Let \mathcal{M} a model of computation.

\mathcal{M} is **Turing-complete** if $\mathcal{TM}(\Sigma) \leq \mathcal{M}$ for some alphabet Σ with $\Sigma \neq \emptyset$.

\mathcal{M} is **Turing-equivalent** if $\mathcal{M} \sim \mathcal{TM}(\Sigma)$ for some alphabet $\Sigma \neq \emptyset$.

John Horton Conway



John Horton Conway (1937–2020)

Fractran

John Horton Conway:

- ▶ article:
 - ▶ **FRACTRAN:**
A Simple Universal Programming Language for Arithmetic
- ▶ talk video:
 - ▶ "Fractran: A Ridiculous Logical Language"

Summary

- ▶ [Post's Correspondence Problem](#) (by [Emil Post](#), 1946, [6])
- ▶ [Interaction Nets](#) (by [Yves Lafont](#), 1990, [4])
- ▶ Compare computational power of models of computation
- ▶ [Fractran](#) (by [John Horton Conway](#), 1987, [2])

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ -calculus Herbrand–Gödel recursive functions partial-recursive/ μ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ζ -calculus evolutionary programming/genetic algorithms abstract state machines	<i>modern</i>
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

Course overview

Monday, July 6 10.30 – 12.30	Tuesday, July 7 10.30 – 12.30	Wednesday, July 8 10.30 – 12.30	Thursday, July 9 10.30 – 12.30	Friday, July 10
<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

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Lambdascope interaction-net animation tool.