

# Lecture 5: Three More Models

## Models of Computation

<https://clegra.github.io/moc.html>

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# Course overview

Monday, July 6 10.30 – 12.30	Tuesday, July 7 10.30 – 12.30	Wednesday, July 8 10.30 – 12.30	Thursday, July 9 10.30 – 12.30	Friday, July 10
<i>intro</i>	<i>classic models</i>			<i>additional models</i>
<b>Introduction to Computability</b>	<b>Machine Models</b>	<b>Recursive Functions</b>	<b>Lambda Calculus</b>	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	$\lambda$ -terms, $\beta$ -reduction, $\lambda$ -definable functions, partial recursive = $\lambda$ -definable = Turing computable	
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	
				14.30 – 16.30
				<b>Three more Models of Computation</b>
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

# Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic $\lambda$ -calculus Herbrand–Gödel recursive functions partial-recursive/ $\mu$ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra $\zeta$ -calculus evolutionary programming/genetic algorithms abstract state machines	<i>modern</i>
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

# Overview

- ▶ **Post's Correspondence Problem** (by **Emil Post**, 1946, [6])
- ▶ **Interaction Nets** (by **Yves Lafont**, 1990, [4])
  - ▶ **Lambdascope** (Vincent van Oostrom, 2003, [5])
  - ▶ **Lambdascope animation tool** (Jan Rochel, 2010, [7])
- ▶ **Compare computational power of models of computation**
- ▶ **Fractran** (by **John Horton Conway**, 1987, [2])

# Emil Post



Emil Leon Post (1897–1954)

# Post's Correspondence Problem (PCP)

Emil Leon Post:

- ▶ "A Variant of a Recursively Unsolvable Problem"  
Bulletin of the American Mathematical Society, 1946.

Instance of PCP:

$I = \{\langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle\}$ , where  $k \geq 1$ ,  $g_i, g'_i \in \Sigma^+$  for  $i \in \{1, \dots, k\}$ .

Question: Is  $I$  solvable?

Do there exist  $n \geq 1$ , and  $i_1, \dots, i_n \in \{1, \dots, k\}$  such that:

$$g_{i_1} g_{i_2} \dots g_{i_n} = g'_{i_1} g'_{i_2} \dots g'_{i_n} \quad ?$$

Theorem

Codings of solvable instances of PCP:

$$\overbrace{\left\{ \left\{ \langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle \mid k \geq 1, g_i, g'_i \in \Sigma^+ \right\} \right\}}^{\text{PCP instance } I} \mid I \text{ is solvable}$$

form a set that is *recursively enumerable*, but *not recursive*.

# Typical features of ‘computationally complete’ MoC’s

- ▶ storage (unbounded)
- ▶ control (finite, given)
- ▶ modification
  - ▶ of (immediately accessible) stored data
  - ▶ of control state
- ▶ conditionals
- ▶ loop (unbounded)
- ▶ stopping condition

# Yves Lafont



Yves Lafont

# Interaction Nets

Yves Lafont (1990) [4] ([link pdf](#)) proposed:

- ▶ a programming language with a simple graph rewriting semantics

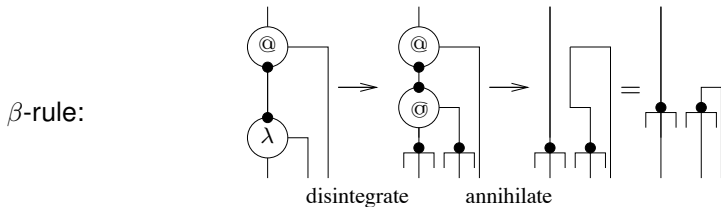
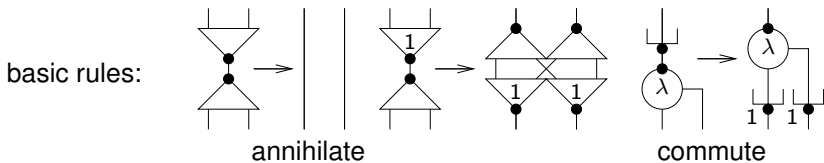
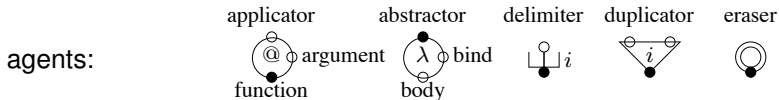
An interaction net is specified by:

- ▶ a set of agents
- ▶ a set of interaction rules

Analogy with:

- ▶ electric circuits:
  - ▶ agents  $\hat{=}$  gates,
  - ▶ edges  $\hat{=}$  wires
- ▶ agents as computation entities:
  - ▶ interaction rules specify behavior

# Lambdascope [Vincent van Oostrom, K.J. van de Looij, M. Zwitserlood, 2003]





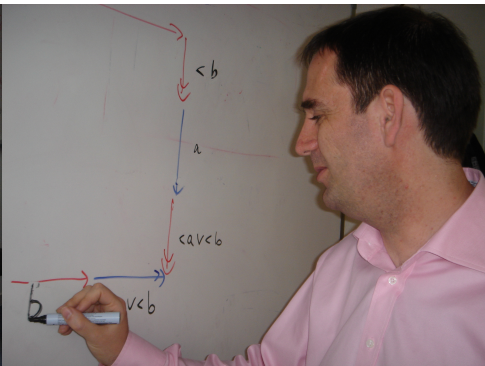
# Gradations of optimal reduction

Optimal reduction in  $\lambda$ -calculus avoids:

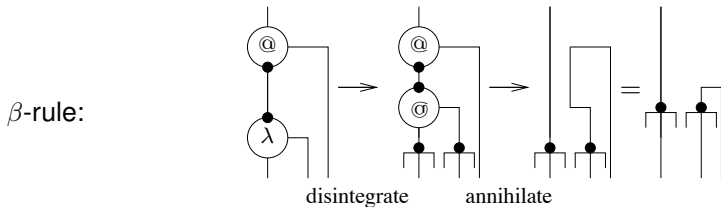
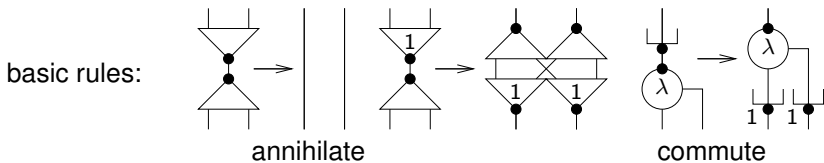
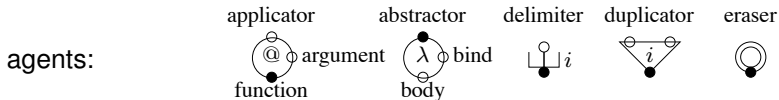
- ▶ unnecessary work
- ▶ duplication of work

calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
$\lambda$ -calculus ( $\beta$ -reduction $\rightarrow_{\beta}$ )	Lévy labelling '78	Lamping '89 Kathail '90 Abdadi/Gonthier/ /Levy '92 Asperti/Guerrini '93 VvO '03 (Terese)	context sharing
$\lambda$ -calculus (weak- $\beta$ red. $\rightarrow_{w\beta}$ )	Blanc/Lévy/ Maranget '05/'07	Wadsworth '71 Shivers/Wand '04	extended-scope sharing
orthogonal TRS (induced rewrite relation $\rightarrow$ )	VvO '03 (Terese)	Staples '80 VvO '03 (Terese)	subterm sharing

# Vincent van Oostrom



# Lambdascope [Vincent van Oostrom, K.J. van de Looij, M. Zwitserlood, 2003]



# Graph rewriting tool by Jan Rochel



graph rewrite tool on Hackage:

<http://hackage.haskell.org/package/graph-rewriting-0.7.5>

# Additional gradations of optimal reduction

calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
$\lambda$ -calculus ( $\beta$ -reduction $\rightarrow_{\beta}$ )	Lévy labelling '78	Lamping '89 Kathail '90 Abdadi/Gonthier/ /Levy '92 Asperti/Guerrini '93 VvO '03 (Terese)	context sharing
	?	term/port graph implementation	scope sharing
	?	nested term graph implementation	extended scope sharing
$\lambda$ -calculus (weak- $\beta$ red. $\rightarrow_{w\beta}$ )	Blanc/Lévy/ Maranget '05/'07	Wadsworth '71 Shivers/Wand '04	extended-scope sharing
orthogonal TRS (induced rewrite relation $\rightarrow$ )	VvO '03 (Terese)	Staples '80 VvO '03 (Terese)	subterm sharing



# Weak requirements on encodings (Boker/Dershowitz)

Traditional requirements on encodings are:

- ▶ *informally computable/effective/mechanizable in principle*
- ▶ *computable* with respect to a specific model (Turing machine, ...)

Boker & Dershowitz [1]: want a ‘**robust definition that does not itself depend on the notion of computability**’, and therefore suggest as encodings:

- (i) *injective* functions
- (ii) *bijective* functions

Definition (**power subsumption** pre-order [Boker/Dershowitz 2006 [1]])

- (i)  $\mathcal{M}_1 \lesssim \mathcal{M}_2$  if: there is an **injective**  $\rho$  such that  $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$
- (ii)  $\mathcal{M}_1 \lesssim_{\text{bijective}} \mathcal{M}_2$  if: there is a **bijective**  $\rho$  such that  $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$

# Anomalies for decision models

However, we found anomalies of these definitions.

$\mathcal{M} = \langle D, \mathcal{F} \rangle$  is a *decision model* if  $\{0, 1\} \subseteq D$ ,  $\forall f \in \mathcal{F} (f[D] \subseteq \{0, 1\})$ .

Theorem (Endrullis/G/Hendriks, [3])

Let  $\Sigma$  and  $\Gamma$  with  $\{0, 1\} \subseteq \Sigma, \Gamma$  be alphabets.

Then for every countable decision model  $\mathcal{M} = \langle \Sigma^*, \mathcal{F} \rangle$ , it holds:

$$\mathcal{M} \lesssim \text{DFA}(\Gamma) \quad \mathcal{M} \lesssim_{\text{bijective}} \text{DFA}(\Gamma)$$

$\text{TMD}(\Sigma)$ : class of Turing machine deciders with input alphabet  $\Sigma$

Anomaly (example)

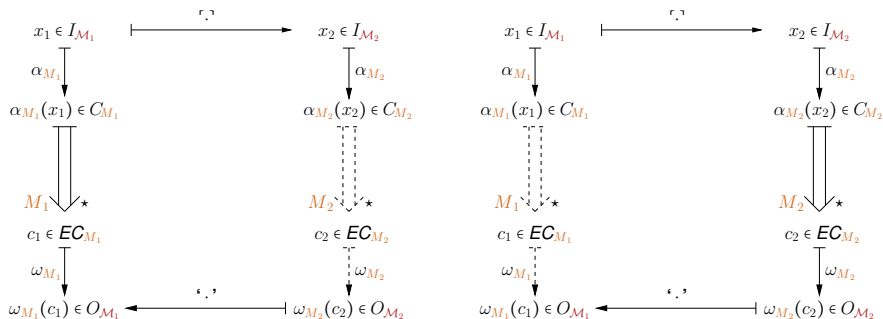
$$\text{TMD}(\Sigma) \lesssim_{\text{bijective}} \text{DFA}(\Gamma)$$

These anomalies for **decision models** and **bijective encodings**:

- ▶ depend on **uncomputable encodings**
- ▶ can be extended to **some** moc's with unbounded output domain
- ▶ but **do not extend** to **all** moc's

# Simulations between models of computation

models  $M_1 \in \mathcal{M}_1$  and  $M_2 \in \mathcal{M}_2$  simulate each other with respect to computable coding  $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$  and decoding  $\lrcorner \cdot \lrcorner : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$  if:



(defines a **Galois connection**)

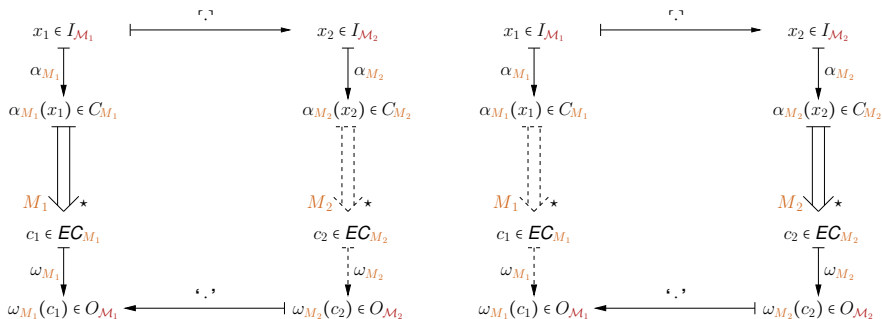
# Models of computation, viewed abstractly

A(n abstractly viewed) **model of computation (MoC)** is a class  $\mathcal{M}$  of machines/systems/. . . such that every  $M \in \mathcal{M}$  it holds:

- ▷  $M$  has a countable set  $I_{\mathcal{M}}$  of **input objects**, and a countable set  $O_{\mathcal{M}}$  of **output objects** that are specific to the MoC  $\mathcal{M}$ ;
- ▷  $M$  has a set  $C_M$  of **configurations** of  $M$ , which contains the subset  $EC_M \subseteq C_M$  of **end-configurations** of  $M$ ;
- ▷  $M$  has an injective **input function**  $\alpha_M : I_{\mathcal{M}} \rightarrow C_M$ , which maps input objects of  $M$  to configurations of  $M$ ;  $\alpha_M$  is **computable**;
- ▷  $M$  defines a **one-step computation relation**  $\Rightarrow_M$  on the set  $C_M$ ; the transitive closure of  $\Rightarrow_M$  is designated by  $\Rightarrow_M^*$ ;
- ▷  $M$  has a partial **output function**  $\omega_M : EC_M \rightarrow O_{\mathcal{M}}$ , which maps **some** end-configurations of  $M$  to output objects of  $M$ ;  $\omega_M$  is **computable**, and membership of end-configurations in  $dom(\omega_M)$  is **decidable**.

# Simulations between models of computation

models  $M_1 \in \mathcal{M}_1$  and  $M_2 \in \mathcal{M}_2$  simulate each other with respect to computable coding  $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$  and decoding  $\lrcorner \cdot \lrcorner : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$  if:



(defines a [Galois connection](#))

# Comparing Computational Power of MoC's

## Definition

Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be MoC's.

- 1 The computational power of  $\mathcal{M}_1$  is **subsumed** by that of  $\mathcal{M}_2$ , denoted symbolically by  $\mathcal{M}_1 \leq \mathcal{M}_2$ , if:

( $\exists$  a pair  $\langle \ulcorner \cdot \urcorner, \lrcorner \cdot \lrcorner \rangle$  of **computable** encoding and decoding functions  $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$  and  $\lrcorner \cdot \lrcorner : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$

$(\forall M_1 \in \mathcal{M}_1) (\exists M_2 \in \mathcal{M}_2)$

$[M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \ulcorner \cdot \urcorner, \lrcorner \cdot \lrcorner \rangle]$ .

- 2 The computational power of  $\mathcal{M}_1$  is **equivalent** to that of  $\mathcal{M}_2$ , denoted by  $\mathcal{M}_1 \sim \mathcal{M}_2$ , if both  $\mathcal{M}_1 \leq \mathcal{M}_2$  and  $\mathcal{M}_2 \leq \mathcal{M}_1$  hold.

# Comparing Computational Power of MoC's

## Theorem

For all models  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , and encoding and decoding functions  $\lceil \cdot \rceil : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$  and  $\lfloor \cdot \rfloor : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$  it holds:

$$\mathcal{M}_1 \leq_{(\lceil \cdot \rceil, \lfloor \cdot \rfloor)} \mathcal{M}_2 \implies \mathcal{F}(\mathcal{M}_1) \subseteq \{ \lfloor \cdot \rfloor \circ f \circ \lceil \cdot \rceil \mid f \in \mathcal{F}(\mathcal{M}_2) \}.$$

# Turing completeness and equivalence

By  $\mathcal{TM}(\Sigma)$  we mean the model of Turing machines over input alphabet  $\Sigma$ .

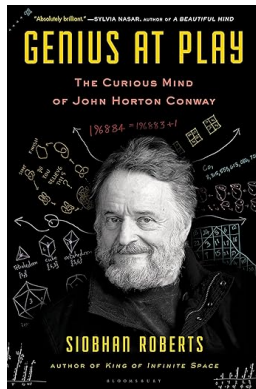
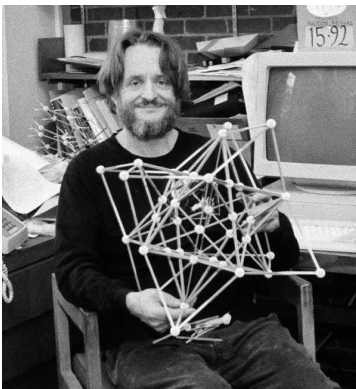
## Definition

Let  $\mathcal{M}$  a model of computation.

$\mathcal{M}$  is **Turing-complete** if  $\mathcal{TM}(\Sigma) \leq \mathcal{M}$  for some alphabet  $\Sigma$  with  $\Sigma \neq \emptyset$ .

$\mathcal{M}$  is **Turing-equivalent** if  $\mathcal{M} \sim \mathcal{TM}(\Sigma)$  for some alphabet  $\Sigma \neq \emptyset$ .

# John Horton Conway



John Horton Conway (1937–2020)

# Fractran

## John Horton Conway:

- ▶ article:
  - ▶ **FRACTRAN:**  
A Simple Universal Programming Language for Arithmetic
- ▶ talk video:
  - ▶ "Fractran: A Ridiculous Logical Language"

# Summary

- ▶ [Post's Correspondence Problem](#) (by [Emil Post](#), 1946, [6])
- ▶ [Interaction Nets](#) (by [Yves Lafont](#), 1990, [4])
- ▶ Compare computational power of models of computation
- ▶ [Fractran](#) (by [John Horton Conway](#), 1987, [2])

# Some Models of Computation

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cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra $\zeta$ -calculus evolutionary programming/genetic algorithms abstract state machines	<i>modern</i>
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

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computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = Turing-computable, Church's Thesis	$\lambda$ -terms, $\beta$ -reduction, $\lambda$ -definable functions, partial recursive = $\lambda$ -definable = Turing computable	
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				comparing computational power

# References I



Udi Boker and Nachum Dershowitz.

Comparing computational power.

*Logic Journal of the IGPL*, 14(5):633–647, 10 2006.



John Horton Conway.

FRACTRAN: A Simple Universal Programming Language for Arithmetic.

58(2):345–363, April 1936.



Jörg Endrullis, Clemens Grabmayer, and Dimitri Hendriks.

Regularity-Preserving but not Reflecting Encodings.

In *Proceedings of the 30th Annual ACM/IEEE Symposium on Logic in Computer Science 2015 (Kyoto, Japan, July 6–10, 2015)*, pages 535–546, July 2015.



Yves Lafont.

Interaction Nets.

*Proceedings of POPL'90*, pages 95–108, 1990.

## References II



Vincent van Oostrom, Kees-Jan van de Looij, and Marijn Zwieterlood.

Lambdascope.

Extended Abstract, Workshop ALPS, Kyoto, April 10th 2004, 2004.

<http://www.phil.uu.nl/~oostrom/publication/pdf/lambdascope.pdf>.



Emil Leon Post.

A Variant of a Recursively Unsolvable Problem.

*Bulletin of the American Mathematical Society*, 52:264–268, 1946.

## References III



Jan Rochel.

graph-rewriting-lambdascope: Lambdascope, an optimal evaluator of the lambda calculus.

Haskell package on Hackage, <https://hackage.haskell.org/package/graph-rewriting-lambdascope>, 2010.

Lambdascope interaction-net animation tool.