Lecture 2: Machine Models, Basic Computability Theory

Models of Computation

https://clegra.github.io/moc.html

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Ph.D. Program, Advanced Courses Period
Gran Sasso Science Institute
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Course overview

Monday, July 6 10.30 – 12.30	Tuesday, July 7 10.30 – 12.30	Wednesday, July 8 10.30 – 12.30	Thursday, July 9 10.30 – 12.30	Friday, July 10
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran
				comparing computational power

Overview

- Post machine
- Turing machine
  - Turing's analysis of computations done by (human) computers

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  - formal definition
  - video
  - typical features of models of computation
  - Busy Beavers

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- Post machine
- Turing machine
  - Turing's analysis of computations done by (human) computers
  - formal definition
  - video
  - typical features of models of computation
  - Busy Beavers
- Elementary recursion theory
  - an unsolvable problem
  - Halting problem
  - recursively enumerable, and recursive sets
  - universal language
  - Chomsky hierarchy

(Comput.) Yes-or-no-questions / Decision problems

Example

Tautology Problem for the propositional calculus
Instance: A formula ϕ of propositional logic.
Question: Is ϕ a tautology?

Suppose $A \subseteq E$, where E a set of finitely describable objects.

A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

Decision problem for A in E: Find a decision method for A in E, or show that no such method can exist.

The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

(Comput.) What-questions / Computation Problems

Example

Computing the greatest common divisor
Instance: a pair $\langle a, b \rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.
Question: What is $\gcd(a, b)$, the greatest common divisor of a and b?

Suppose $F : A \to B$ is a mapping, where the elements of A, B are finitely describable objects.

A computation method for F is a method by which, given an element $a \in A$, we can obtain solution $F(a)$ in a finite number of steps.

Computation problem for F: Find a computation method for F, or show that no such method can exist.

A mapping F is calculable if there exists a computation method for F.

Representing function

Let $P(a_1, \ldots, a_n)$ be an n-ary number-theoretic predicate.

The representing function f of P:

$$f(a_1, \ldots, a_n) := \begin{cases} 1 & \ldots P(a_1, \ldots, a_n) \text{ is true} \\ 0 & \ldots P(a_1, \ldots, a_n) \text{ is false} \end{cases}$$

Hence:
A decision procedure can be handled as a computation procedure f by taking ‘0’ for ‘yes’, and ‘1’ for ‘no’.

Decision / Computation procedures (steps)

What is a computation method (procedure) more precisely, with respect to its steps?

– A mechanical, algorithmic computation procedure that:

- can be carried out by a machine M (ideal, not limited by resource problems, mechanical breakdown, etc.).
- for computing a function F on an argument a,
  - a is placed on the input device of the M,
  - which then produces $F(a)$ after finitely many steps.
- for computing a function F,
  - the machine M that is chosen for obtaining $F(a)$ may not be different for different arguments a

– Similar for a decision methods.

Solvability by an effective procedure

Q: Is the diophantine equation $15x + 9y + 12 = 0$ solvable?
(I.e. solvable for $x, y \in \mathbb{Z}$?)

From elementary number theory we know:

$$ax + by + c = 0 \text{ solvable in } \mathbb{Z} \iff \gcd(a, b) \mid c \qquad (\star)$$

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Using Euclid's algorithm we calculate $\gcd(15, 9)$:

15	:	9	=	1	rem	6
9	:	6	=	1	rem	3
6	:	3	=	2	rem	0

We find: $\gcd(15, 9) = 3$.
Due to $3 \mid 12$ and (**??**) we conclude:

A: Yes. (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin

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- methods that involve chance procedures: tossing a coin
- methods involving magic: asking a fortune teller

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin
- methods involving magic: asking a fortune teller
- methods that require (unformalised, unmechanised) insight

Effectively calculable?

Example

Hilbert's 10th Problem

Instance: An equation $p(x_1, \ldots, x_n) = 0$, where
p a polynomial with integer coefficients.

Question: Is the equation solvable for $x_1, \ldots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Effectively calculable? – No!

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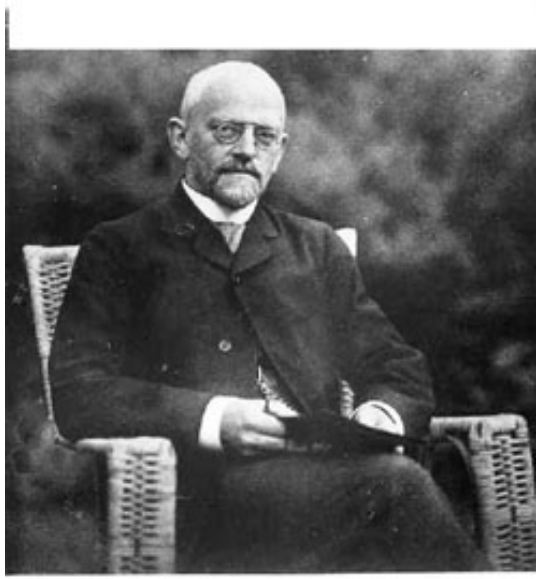
Theorem (Matijasevic, 1970)

Hilbert's 10th Problem is unsolvable.

David Hilbert (1862–1943)



Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

Timeline: From logic to computability

1900	Hilbert's 23 Problems in mathematics
1921	Schönfinkel: Combinatory logic
1928	Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called 'Entscheidungsproblem')
1929	Presburger: completeness/decidability of theory of addition on $\mathbb{Z}$
1930	Gödel: completeness theorem of predicate calculus
1931	Gödel: incompleteness theorems for first-order arithmetic
1932	Church: λ-calculus
1933/34	Herbrand/Gödel: general recursive functions
1936	Church/Kleene: λ-definable $\sim$ general recursive Church Thesis: 'effectively calculable' be defined as either Church shows: the 'Entscheidungsproblem' is unsolvable
1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Reading recommended

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [4].

2. Turing machine motivation:
   Turing's analysis of a human computer:
   Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [5].

Emil Post

Emil Leon Post (1897–1954)

Post about . . .

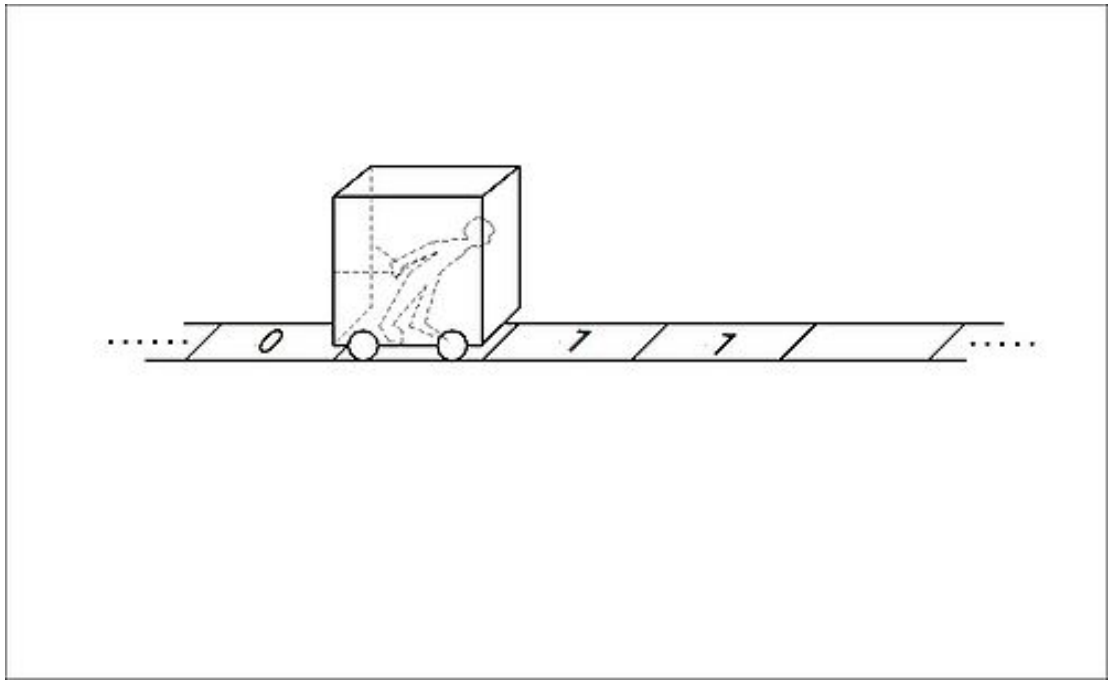
. . . a result of his from 1921 similar to the Incompleteness Theorem:

Theorem (Gödel, 1931 (paraphrased here))

Every axiomatisable, consistent first-order-logic system of number theory is incomplete: it contains true, but unprovable formulas.

> *"For full generality a complete analysis would have to be given of all possible ways in which the human mind could set up finite processes for generating sequences."*

Post machine (1936)



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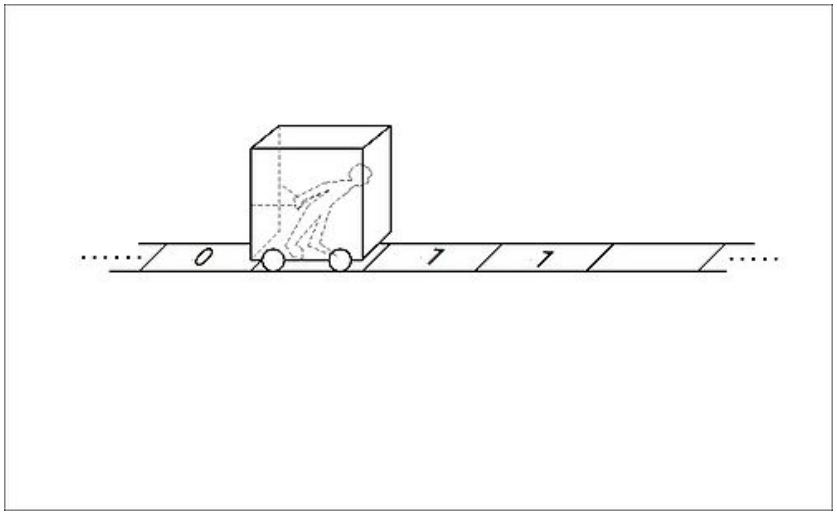

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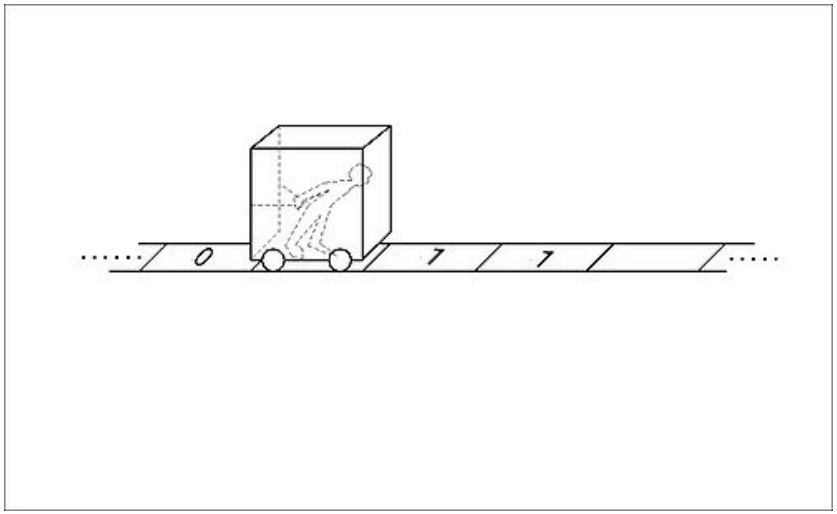


"The worker is assumed to be capable of performing the following primitive acts:

(a) Marking the box he is in (assumed empty),

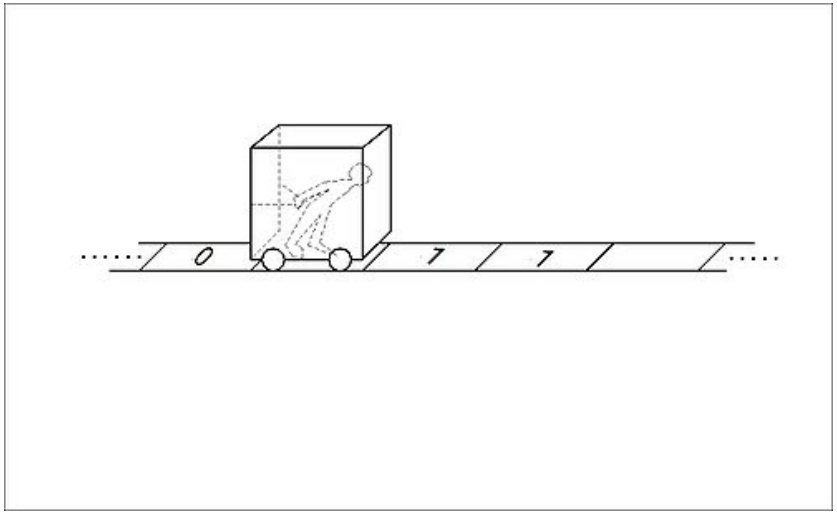
(b) Erasing the mark in the box he is in (assumed marked),

(c) Moving to the box on his right,

(d) Moving to the box on his left,

(e) Determining whether the box he is in, is or is not marked."

Post machine (1936)



‘Directions’ (= list of instructions):

- Start at the starting point and follow direction 1.
- Then a finite number of directions numbered 1, 2, 3, . . . , n, where the i-th has one of the following forms:
  - Perform operation $O_i \in \{(a), (b), (c), (d)\}$, then follow direction j_i.
  - Perform operation (e) and according as the answer is yes or no correspondingly follow direction j_i' or j_i''.
  - Stop.

Exercise

Exercise

Construct a Post machine that adds one to a natural number in unary representation.

Typical features of 'computationally complete' MoC's

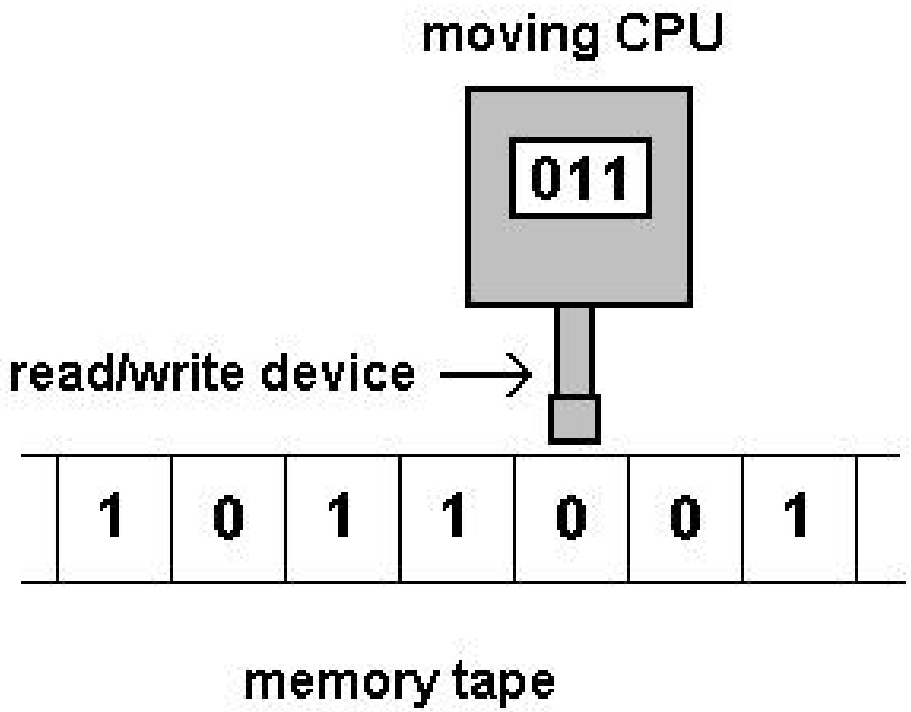
- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

(Credits due to: Vincent van Oostrom)

Turing computability

Alan Turing (1912 –1954)

Turing machine



Turing machine: formal definition

Definition

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- $\delta : (Q \setminus F) \times \Gamma \rightharpoonup Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is a partial function, called the transition function;
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Turing's analysis of a human 'computer'

Section 9 in Turing's 1937 paper 'On computable numbers, with an application to the Entscheidungsproblem' [5].

A direct appeal to intuition in analysing human computation:

- paper is divided into squares
- one-dimensional paper ('tape' divided into squares)
- number of symbols is finite
- behaviour of computer at any time is determined by:
  - observed symbols
  - her/his 'state of mind'
- bound B on the number of symbols/squares
  that the computer can observe at any moment
- number of 'states of mind' of the computer is finite

Turing's analysis of a human 'computer'

- modification of tape symbols

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  - other directly observable squares? – T. argues: not necessary

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- modification of 'state of mind'

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  - (b) change of one of the observed squares to another square at most L squares away

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  - A change (b) of observed square,
    together with a possible change of state of mind.

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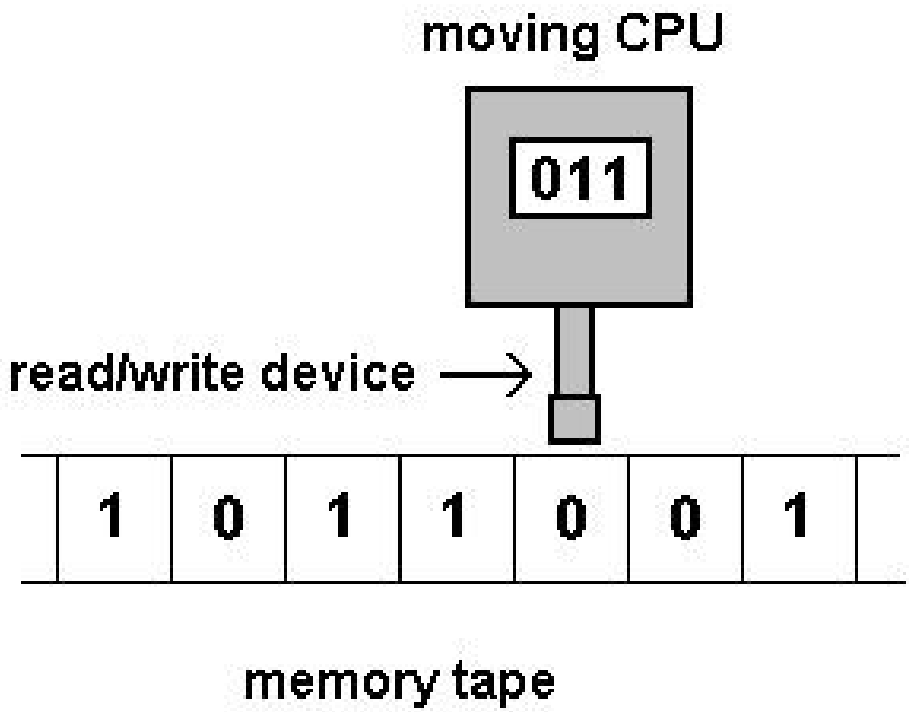
"It is my contention that these operations include all those which are used in the computation of a number."

Turing machine



Church–Turing Thesis

Thesis (Church–Turing, 1937)

Every effectively calculable function is computable by a Turing-machine.

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Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \textbb, F \rangle$ be a Turing machine.

A configuration of M is an element $w_1 q w_2 \in \Gamma^* \times Q \times \Gamma^*$ such that the first letter in w_1 and the last letter in w_2 are different from $\textbb$.

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$L(M) := \{w \in \Sigma^* \mid M \text{ accepts } w\}$ is the language accepted by M.

Recursively enumerable, and recursive languages

Definition

Let $L \subseteq \Sigma^*$ be a language.

L is called recursively enumerable if

- $L = L(M)$ for some Turing machine M with input symbols Σ.

L is called recursive if

- there is a Turing machine M with input symbols Σ such that
  - $L = L(M)$
  - M halts on all of its inputs.

Turing-computable (total) functions

Definition

A total function $f : \mathbb{N}^k \to \mathbb{N}$ is Turing-computable if there exists a Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \text{␣}, F \rangle$ and a calculable coding function $\langle \cdot \rangle : \mathbb{N} \to \Sigma^*$ such that:

- for all $n_1, \ldots, n_k \in \mathbb{N}$ there exists $q \in F$ such that:

$$q_0 \langle n_1 \rangle \text{␣} \langle n_2 \rangle \text{␣} \ldots \text{␣} \langle n_k \rangle \vdash_M^* q \langle f(n_1, \ldots, n_k) \rangle$$

Exercises

Exercise

Construct a Turing machine that adds one to a natural number in binary representation.

(In the film this Turing machine is executed five times consecutively.)

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Exercise

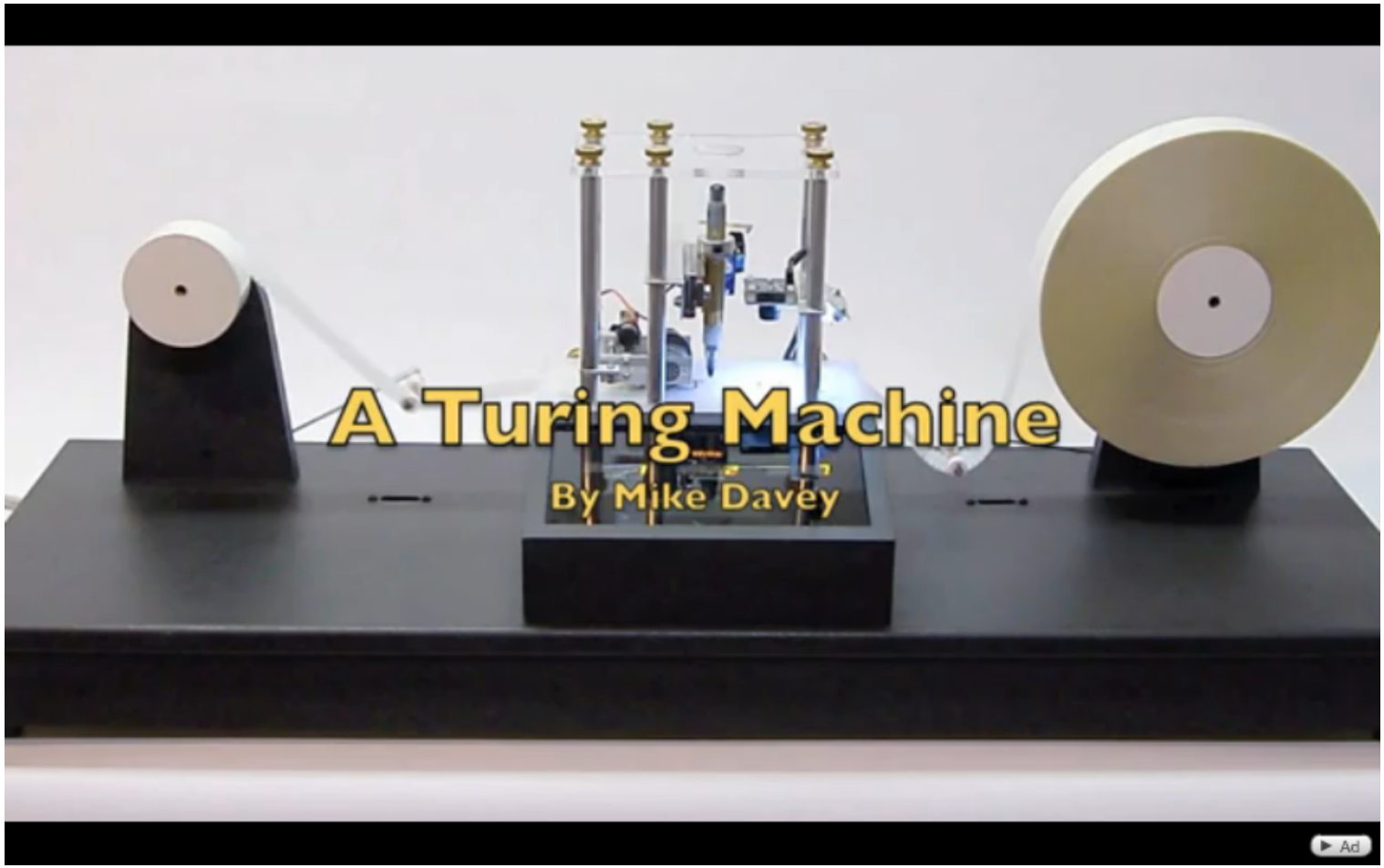
Construct a Turing machine that, if started on the empty tape, writes the sequence

$$0101101110111101111 0 \ldots$$

on the tape, but does not halt.

(Compare your machine with Turing's machine for this purpose.)

Mike Davey's Turing machine (link)



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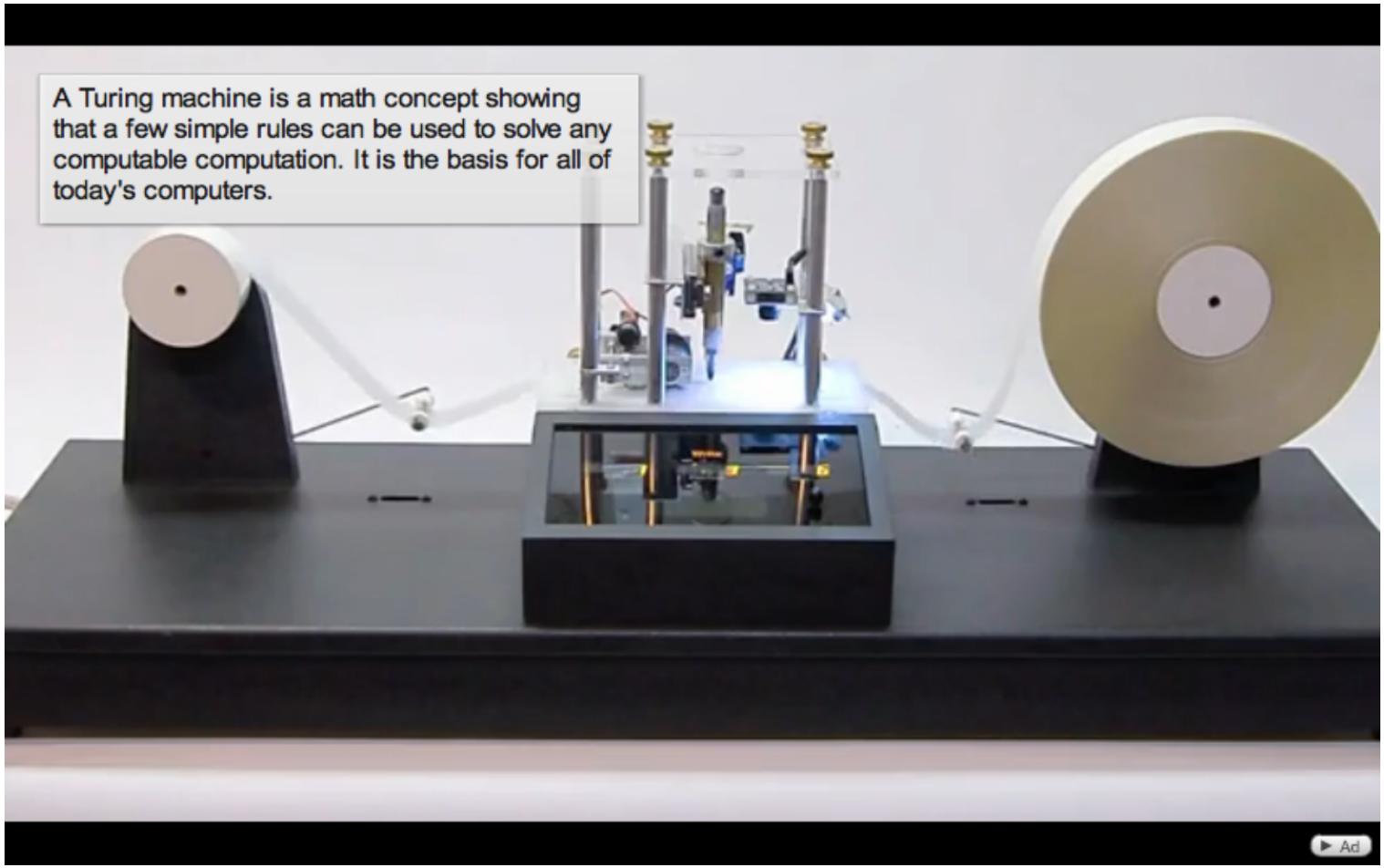

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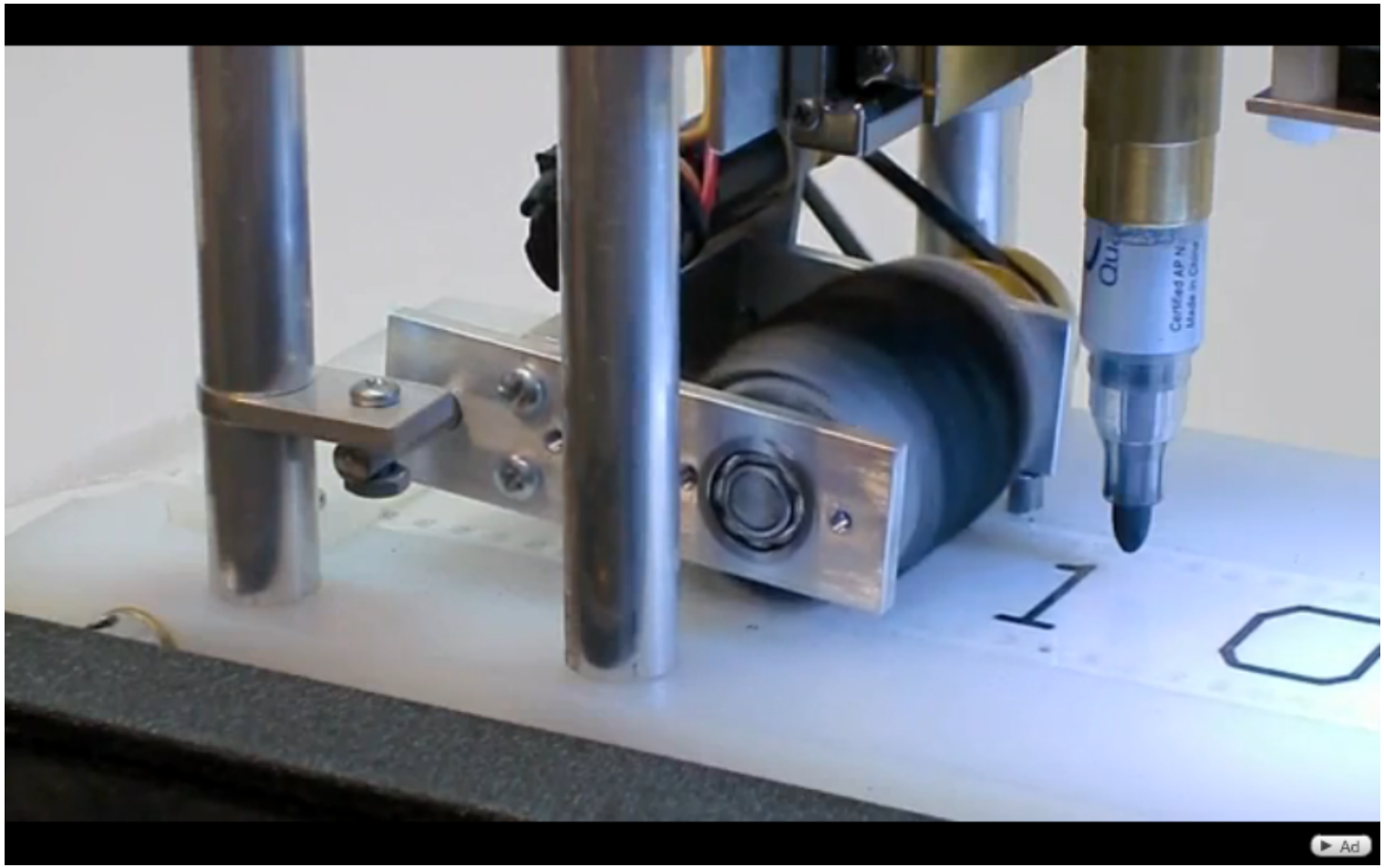

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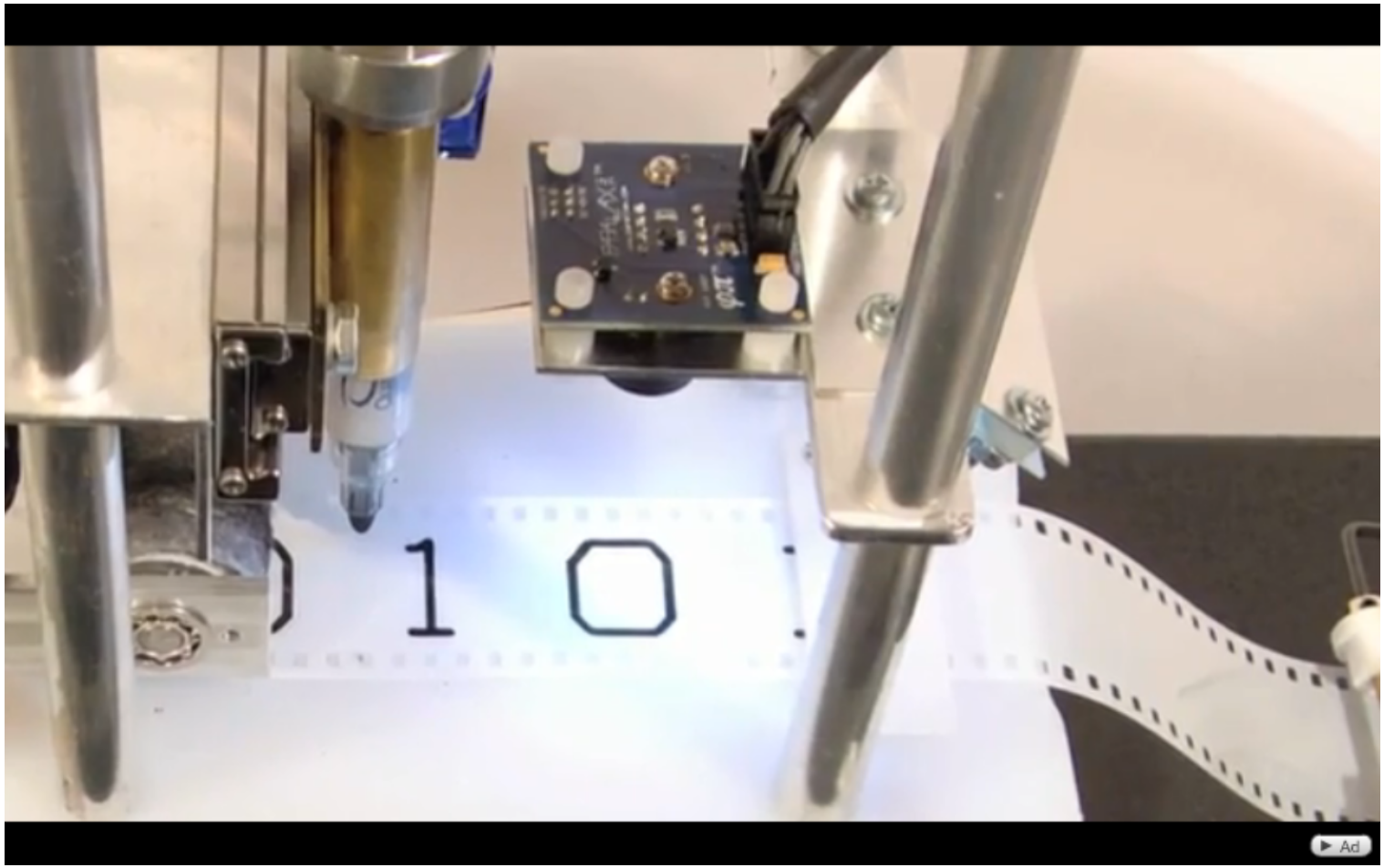

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Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

Variants of Turing machines

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Busy Beavers

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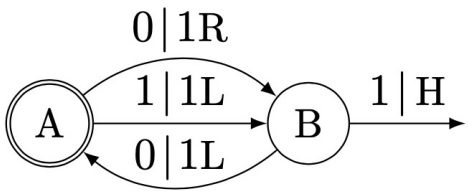
2-State Busy Beaver



initial state	···	0	0	0	0A	0	0	···
step 1	···	0	0	0	1	0B	0	···
step 2	···	0	0	0	1A	1	0	···
step 3	···	0	0	0B	1	1	0	···
step 4	···	0	0A	1	1	1	0	···
step 5	···	0	1	1B	1	1	0	···
step 6 (halt)	···	0	1	1H	1	1	0	···

Figure 1: A 2-state Busy Beaver and its execution on an initially all-0 tape. Starting in state A, the machine finds a 0 on its tape and therefore follows the arrow labeled $0|1R$, which causes it to replace the 0 by a 1, move one square to the right, and transition into state B, and so on until the machine reaches the Halt arrow on its 6^{th} step.

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- Scott Aaronson: “The Busy Beaver Frontier”, [1].
- —“Why Philosophers Should Care About Computational Complexity?” [2]

Elementary Recursion Theory

An unsolvable problem

The diagonalisation language:

$$L_d := \{w \mid w = \langle M \rangle, w \notin L(M)\} = \{\langle M \rangle \mid \langle M \rangle \notin L(M)\}$$

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Theorem

There exist unsolvable decision problems.

Exercise: Halting Problem

Exercise

Try to adapt the diagonalisation argument to show that for the Halting Problem

$$H = \{ \langle \langle M \rangle, w \rangle \mid M \text{ halts on input } w \}$$

it holds:

- H is not recursive

and show that:

- H is recursively enumerable

Properties of r.e./recursive sets (I)

For $L \subseteq \Sigma^*$, $\bar{L} := \Sigma^* \setminus L$ is called the complement of L.

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- the accepting states of M are made non-accepting in $\bar{M}$.
- $\bar{M}$ has a new accepting state r.
- for each $q \in Q$ and tape symbol $s \in \Gamma$ such that $\delta_M(q, s)$ is undefined, add the transition $\delta_{\bar{M}}(q, s) = \langle r, s, R \rangle$.

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It follows that $\bar{L} = L(\bar{M})$, and that $\bar{M}$ halts on all inputs. □

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Let M_1 and M_2 be Tm's such that $L = L(M_1)$ and $\bar{L} = L(M_2)$.

To decide, for a given $w \in \Sigma^*$, whether $w \in L$, build a Tm M that executes M_1 and M_2 on w in parallel, and such that:

- if M_1 accepts w, then also M accepts w.
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Hence M accepts w iff $w \in L(M_1) = L$. Thus $L(M) = L$.

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Since for all w, either $w \in L$ or $w \in \bar{L}$, it follows that either M_1 or M_2 halts on w, and hence M halts on all inputs.

Hence $L = L(M)$ is recursive.

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- L_u is r.e.: $L_u = L(M_u)$ for an universal machine M_u.
- L_u is not recursive:

  Suppose that L_u is recursive. Then $\bar{L_u}$ is recursive, and hence there exists a Tm. M such that $\bar{L_u} = L(M)$.

  M can be used to build a Tm. M' that accepts the diagonalisation language L_d, entailing $L_u = L(M')$.

  [picture of M' to be given]

  But then L_u would actually be r.e., in contradiction with what we proved before.

Busy Beaver is not computable

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Proposition

The busy beaver function $BB : \mathbb{N} \to \mathbb{N},\ n \mapsto BB(n)$ *is not (Turing-)computable.*

More generally: one can solve the Halting Problem, given oracle access to any function $b : \mathbb{N} \to \mathbb{N}$ *such that* $b(n) \geq BB(n)$ *for all* $n \in \mathbb{N}$.

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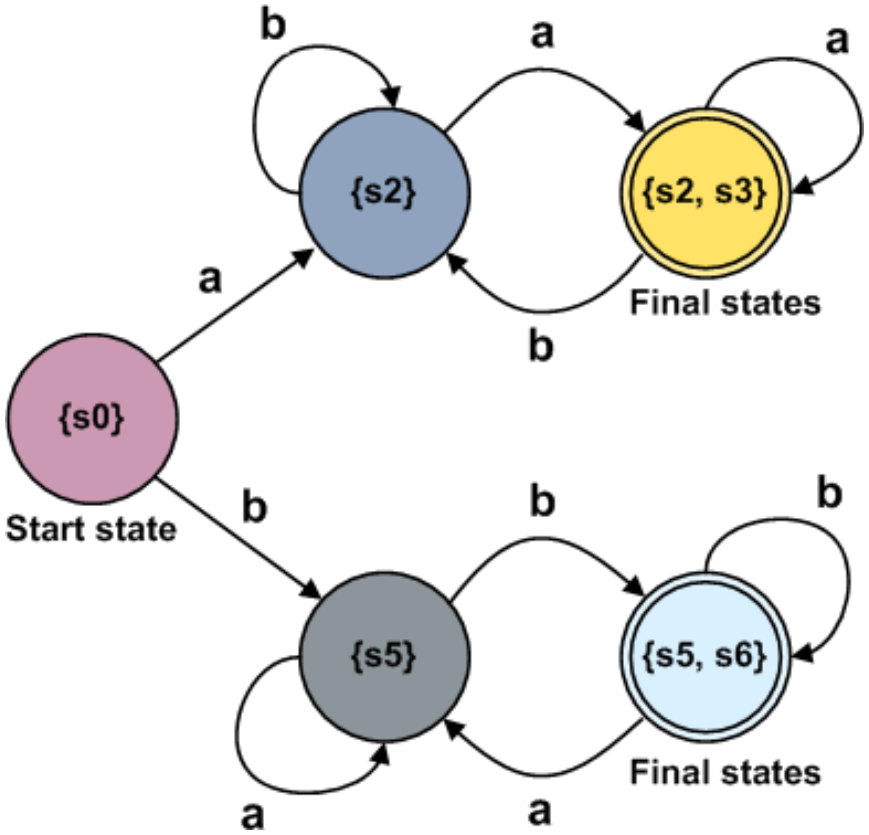
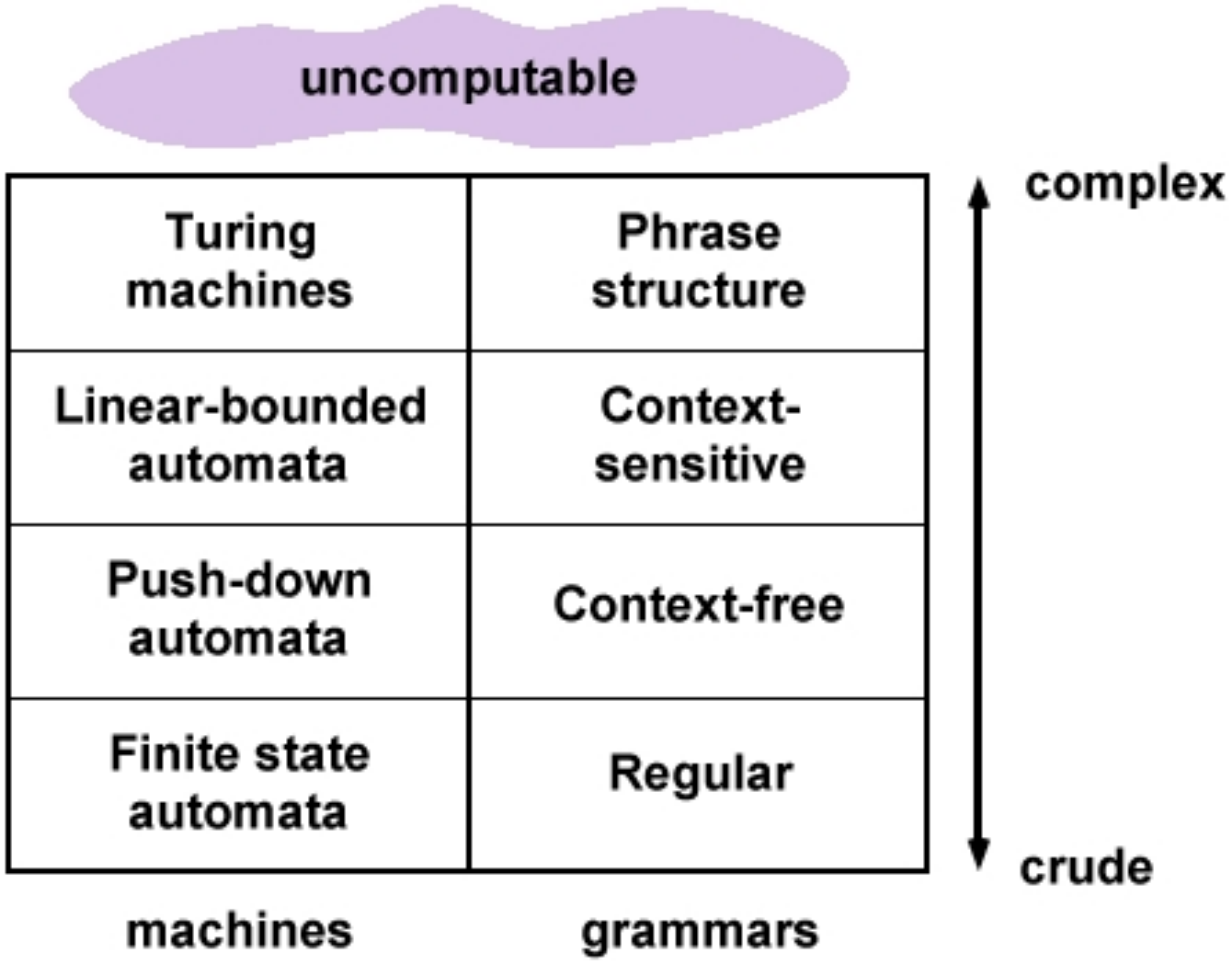
Proposition

Let $f : \mathbb{N} \to \mathbb{N}$ *be a Turing-computable function.*
Then there exists an $n_f \in \mathbb{N}$ *such that* $BB(n) > f(n)$ *for all* $n \geq n_f$.

Finite-state automaton



Formal-languages Chomsky hierarchy



Summary

- Post machine
- Turing machine
  - Turing's analysis of computations done by (human) computers
  - formal definition
  - video
  - typical features of models of computation
  - Busy Beavers
- Elementary recursion theory
  - an unsolvable problem
  - Halting problem
  - recursively enumerable, and recursive sets
  - universal language
  - Chomsky hierarchy

Course overview

Monday, July 6 10.30 – 12.30	Tuesday, July 7 10.30 – 12.30	Wednesday, July 8 10.30 – 12.30	Thursday, July 9 10.30 – 12.30	Friday, July 10
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

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