Lecture 6: Three More Models

Models of Computation

https://clegra.github.io/moc/moc.html

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University of Novi Sad
Novi Sad, Serbia

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Course overview

intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	**Three more Models of Computation**
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	imperative programming	*algebraic programming*	*functional programming*	

Some Models of Computation

<table>
<tr><th>machine model</th><th>mathematical model</th><th>sort</th></tr>
<tr><td>Turing machine
Post machine
register machine</td><td>Combinatory Logic
λ-calculus
Herbrand–Gödel recursive functions
partial-recursive/μ-recursive functions
Post canonical system (tag system)
Post's Correspondence Problem
Markov algorithms
Lindenmayer systems</td><td>classical</td></tr>
<tr><td></td><td>Fractran</td><td>less well known</td></tr>
<tr><td>cellular automata
neural networks</td><td>term rewrite systems
interaction nets
logic-based models of computation
concurrency and process algebra
ς-calculus
evolutionary programming/genetic algorithms</td><td rowspan="2">modern</td></tr>
<tr><td colspan="2">abstract state machines</td></tr>
<tr><td colspan="2">hypercomputation</td><td>speculative</td></tr>
<tr><td colspan="2">quantum computing
bio-computing
reversible computing</td><td>physics-/biology-inspired</td></tr>
</table>

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Emil Post

Emil Leon Post (1897–1954)

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Instance of PCP:

$I = \{\langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle\}$, where $k \geq 1$, $g_i, g'_i \in \Sigma^+$ for $i \in \{1, \dots, k\}$.

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Question: Is I solvable?

Do there exist $n \geq 1$, and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ such that:

$$g_{i_1} g_{i_2} \ldots g_{i_n} = g'_{i_1} g'_{i_2} \ldots g'_{i_n} \quad ?$$

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Theorem

Codings of solvable instances of PCP:

$$\Big\{\Big\langle \overbrace{\{\langle g_1, g'_1\rangle, \ldots, \langle g_k, g'_k\rangle \mid k \geq 1, g_i, g'_i \in \Sigma^+\}}^{\text{PCP instance } I} \Big\rangle \mid I \text{ is solvable}\Big\}$$

form a set that is recursively enumerable, but not recursive.

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)
- control (finite, given)
- modification
  - of (immediately accessible) stored data
  - of control state
- conditionals
- loop (unbounded)
- stopping condition

Yves Lafont

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- electric circuits:
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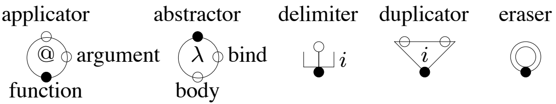
Analogy with:

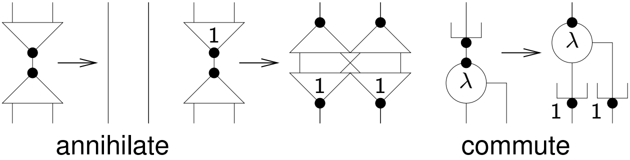
- electric circuits:
  - agents $\triangleq$ gates,
  - edges $\triangleq$ wires
- agents as computation entities:
  - interaction rules specify behavior

Lambdascope [Vincent van Oostrom, K.J. van de Looij, M. Zwitserlood, 2003]

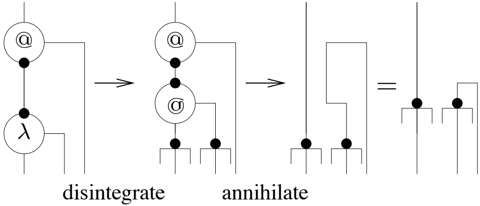
agents:



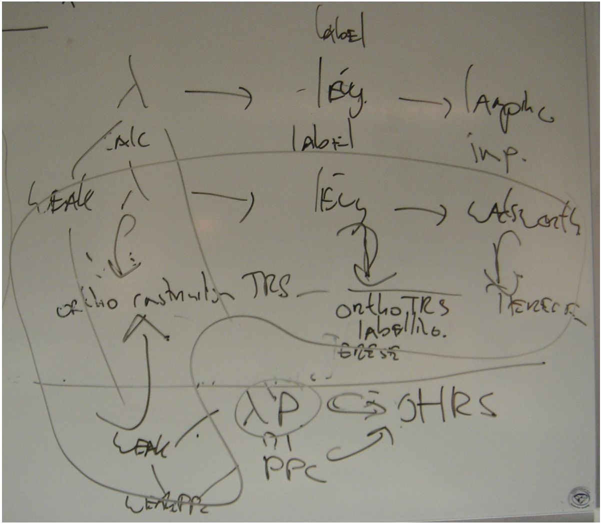
basic rules:



β-rule:



Gradations of optimal reduction [VvO, plaatje 2008/10]



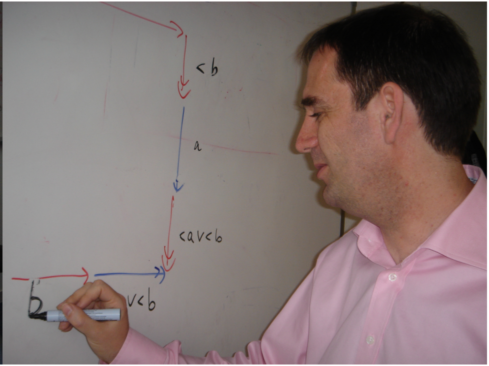
Gradations of optimal reduction

Optimal reduction in λ-calculus avoids:

- unnecessary work
- duplication of work

calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
λ-calculus (β-reduction $\rightarrow_{\beta}$)	Lévy labelling '78	Lamping '89 Kathail '90 Abdadi/Gonthier/ /Levy '92 Asperti/Guerrini '93 VvO '03 (Terese)	context sharing
λ-calculus (weak-β red. $\rightarrow_{w\beta}$)	Blanc/Lévy/ Maranget '05/'07	Wadsworth '71 Shivers/Wand '04	extended-scope sharing
orthogonal TRS (induced rewrite relation $\rightarrow$)	VvO '03 (Terese)	Staples '80 VvO '03 (Terese)	subterm sharing

Vincent van Oostrom



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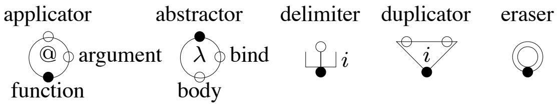

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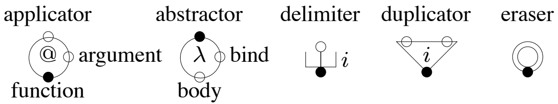

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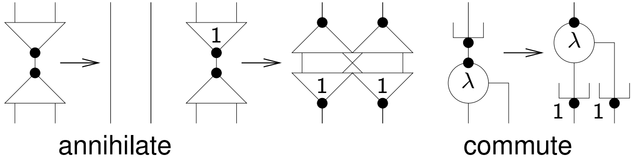

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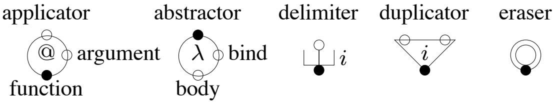

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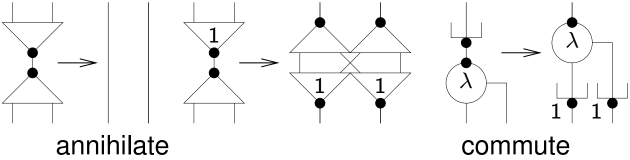

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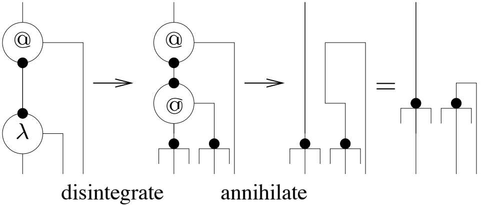
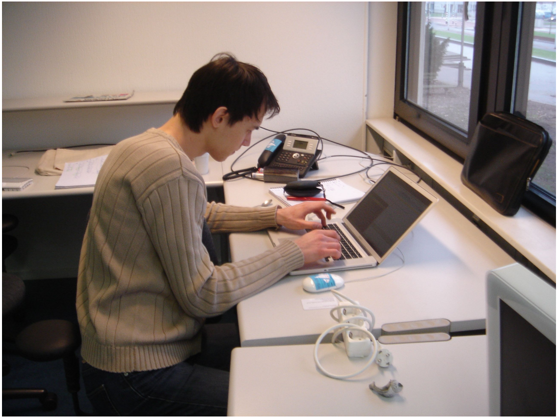


Graph rewriting tool by Jan Rochel



graph rewrite tool on Hackage:

http://hackage.haskell.org/package/graph-rewriting-0.7.5

Additional gradations of optimal reduction

calculus (rewrite relation)	labelling	graph rewriting implementation	sharing notion
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	?	term/port graph implementation	scope sharing
	?	nested term graph implementation	extended scope sharing
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Comparing computational power via encodings

- Simulation of models of computation $\mathcal{M}_1 = \langle D_1, \mathcal{F}_1 \rangle$, $\mathcal{M}_2 = \langle D_2, \mathcal{F}_2 \rangle$:

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- Simulation of functions:
  function f_2 *simulates* function f_1 via *encoding* ρ if:

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Weak requirements on encodings (Boker/Dershowitz)

Traditional requirements on encodings are:

- *informally computable*/*effective*/*mechanizable in principle*
- *computable* with respect to a specific model (Turing machine, . . .)

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Definition (power subsumption pre-order [Boker/Dershowitz 2006 [1]])

(i) $\mathcal{M}_1 \lesssim \mathcal{M}_2$ if: there is an injective ρ such that $\mathcal{M}_1 \lesssim_\rho \mathcal{M}_2$

(ii) $\mathcal{M}_1 \lesssim_{\text{bijective}} \mathcal{M}_2$ if: there is a bijective ρ such that $\mathcal{M}_1 \lesssim_\rho \mathcal{M}_2$

Anomalies for decision models

However, we found anomalies of these definitions.

$\mathcal{M} = \langle D, \mathcal{F} \rangle$ is a *decision model* if $\{0,1\} \subseteq D, \ \ \forall f \in \mathcal{F} \, (f[D] \subseteq \{0,1\})$.

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Let Σ and Γ with $\{0,1\} \subseteq \Sigma, \Gamma$ be alphabets.

Then for every countable decision model $\mathcal{M} = \langle \Sigma^*, \mathcal{F} \rangle$, it holds:

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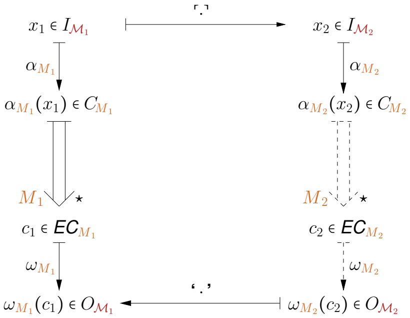
These anomalies for decision models and bijective encodings:

- depend on uncomputable encodings
- can be extended to some moc's with unbounded output domain
- but do not extend to all moc's

Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ simulate each other with respect to computable coding $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and decoding '$\cdot$' $: O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$ if:



Models of computation, viewed abstractly

A(n abstractly viewed) model of computation (MoC) is a class $\mathcal{M}$ of machines/systems/… such that every $M \in \mathcal{M}$ it holds:

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- M has an injective input function $\alpha_M : I_{\mathcal{M}} \to C_M$, which maps input objects of M to configurations of M; α_M is computable;
- M defines a one-step computation relation $\Rrightarrow_M$ on the set C_M; the transitive closure of $\Rrightarrow_M$ is designated by $\Rrightarrow_M^*$;

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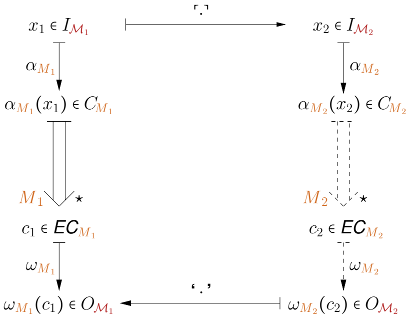
A(n abstractly viewed) model of computation (MoC) is a class $\mathcal{M}$ of machines/systems/... such that every $M \in \mathcal{M}$ it holds:

- M has a countable set $I_{\mathcal{M}}$ of input objects, and a countable set $O_{\mathcal{M}}$ of output objects that are specific to the MoC $\mathcal{M}$;
- M has a set C_M of configurations of M, which contains the subset $EC_M \subseteq C_M$ of end-configurations of M;
- M has an injective input function $\alpha_M : I_{\mathcal{M}} \to C_M$, which maps input objects of M to configurations of M; α_M is computable;
- M defines a one-step computation relation $\Rightarrow_M$ on the set C_M; the transitive closure of $\Rightarrow_M$ is designated by $\Rightarrow_M^*$;
- M has a partial output function $\omega_M : EC_M \rightharpoonup O_{\mathcal{M}}$, which maps some end-configurations of M to output objects of M; ω_M is computable, and membership of end-configurations in $dom(\omega_M)$ is decidable.

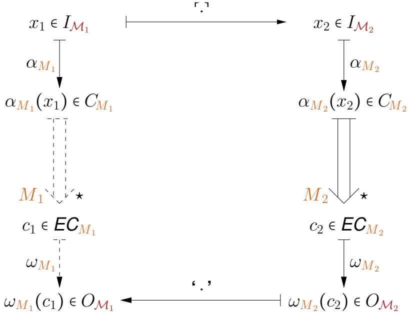
Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ simulate each other with respect to computable coding $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and decoding ‘ · ’ $: O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$ if:



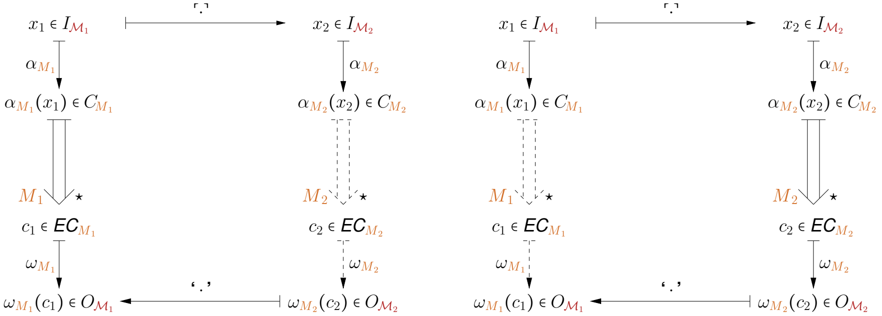
Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ simulate each other with respect to computable coding $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and decoding ‘ · ’ $: O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$ if:



Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ simulate each other with respect to computable coding $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and decoding $`\cdot' : O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$ if:



(defines a Galois connection)

Comparing Computational Power of MoC's

Definition

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be MoC's.

1. The computational power of $\mathcal{M}_1$ is subsumed by that of $\mathcal{M}_2$, denoted symbolically by $\mathcal{M}_1 \le \mathcal{M}_2$, if:

   ($\exists$ a pair $\langle \ulcorner\cdot\urcorner, `\cdot' \rangle$ of computable encoding and decoding functions $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and $`\cdot' : O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$

   $$(\forall M_1 \in \mathcal{M}_1)\,(\exists M_2 \in \mathcal{M}_2)$$

   $$\Big[\, M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \ulcorner\cdot\urcorner, `\cdot' \rangle \,\Big].$$

Comparing Computational Power of MoC’s

Definition

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be MoC’s.

1. The computational power of $\mathcal{M}_1$ is subsumed by that of $\mathcal{M}_2$, denoted symbolically by $\mathcal{M}_1 \le \mathcal{M}_2$, if:

   ($\exists$ a pair $\langle \ulcorner\cdot\urcorner, `\cdot' \rangle$ of computable encoding and decoding functions $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and $`\cdot' : O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$

   $$(\forall M_1 \in \mathcal{M}_1)\,(\exists M_2 \in \mathcal{M}_2)$$

   $$\big[\, M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \ulcorner\cdot\urcorner, `\cdot' \rangle \,\big].$$

2. The computational power of $\mathcal{M}_1$ is equivalent to that of $\mathcal{M}_2$, denoted by $\mathcal{M}_1 \sim \mathcal{M}_2$, if both $\mathcal{M}_1 \le \mathcal{M}_2$ and $\mathcal{M}_2 \le \mathcal{M}_1$ hold.

Comparing Computational Power of MoC's

Theorem

For all models $\mathcal{M}_1$ and $\mathcal{M}_2$, and encoding and decoding functions $\ulcorner\cdot\urcorner : I_{\mathcal{M}_1} \to I_{\mathcal{M}_2}$ and $`\cdot' : O_{\mathcal{M}_2} \to O_{\mathcal{M}_1}$ it holds:

$$\mathcal{M}_1 \leq_{\langle \ulcorner\cdot\urcorner, `\cdot' \rangle} \mathcal{M}_2 \implies \mathcal{F}(\mathcal{M}_1) \subseteq \left\{ `\cdot' \circ f \circ \ulcorner\cdot\urcorner \,\middle|\, f \in \mathcal{F}(\mathcal{M}_2) \right\}.$$

Turing completeness and equivalence

By $\mathcal{TM}(\Sigma)$ we mean the model of Turing machines over input alphabet Σ.

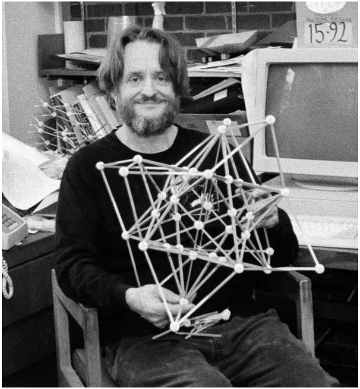
Definition

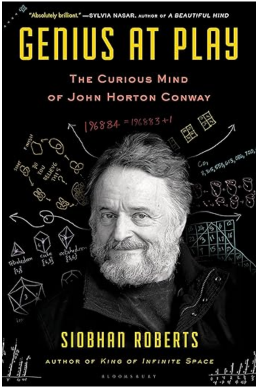
Let $\mathcal{M}$ a model of computation.

$\mathcal{M}$ is Turing-complete if $\mathcal{TM}(\Sigma) \leq \mathcal{M}$ for some alphabet Σ with $\Sigma \neq \varnothing$.

$\mathcal{M}$ is Turing-equivalent if $\mathcal{M} \sim \mathcal{TM}(\Sigma)$ for some alphabet $\Sigma \neq \varnothing$.

John Horton Conway





John Horton Conway (1937–2020)

Fractran

John Horton Conway:

- article:
  - FRACTRAN:
    A Simple Universal Programming Language for Arithmetic
- talk video:
  - ”Fractran: A Ridiculous Logical Language”

Summary

- Post's Correspondence Problem (by Emil Post, 1946, [6])
- Interaction Nets (by Yves Lafont, 1990, [4])
- Compare computational power of models of computation
- Fractran (by John Horton Conway, 1987, [2])

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ-calculus Herbrand–Gödel recursive functions partial-recursive/μ-recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	*classical*
	Fractran	*less well known*
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ς-calculus evolutionary programming/genetic algorithms	*modern*
abstract state machines		*modern*
hypercomputation		*speculative*
quantum computing bio-computing reversible computing		*physics-/biology-inspired*

Course overview

intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	**Three more Models of Computation**
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	imperative programming	*algebraic programming*	*functional programming*	

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Lambdascope interaction-net animation tool.