

Lecture 5: Three More Models

Models of Computation

<https://clegra.github.io/moc/moc.html>

Clemens Grabmayer

Teaching Mobility Program (PNRR-TNE DESK)

University of Novi Sad

Novi Sad, Serbia

March ., 2026

Course overview

<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	Three more Models of Computation
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ -calculus Herbrand–Gödel recursive functions partial-recursive/ μ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ζ -calculus evolutionary programming/genetic algorithms	<i>modern</i>
	abstract state machines	
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

Overview

- ▶ Post's Correspondence Problem (by Emil Post, 1946, [6])
- ▶ Interaction Nets (by Yves Lafont, 1990, [4])
 - ▶ Lambdascope (Vincent van Oostrom, 2003, [5])
 - ▶ Lambdascope animation tool (Jan Rochel, 2010, [7])
- ▶ Compare computational power of models of computation
- ▶ Fractran (by John Horton Conway, 1987, [2])

Post's Correspondence Problem (PCP)

Emil Leon Post:

- ▶ "A Variant of a Recursively Unsolvable Problem"
Bulletin of the American Mathematical Society, 1946.

Instance of PCP:

$I = \{\langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle\}$, where $k \geq 1$, $g_i, g'_i \in \Sigma^+$ for $i \in \{1, \dots, k\}$.

Question: Is I solvable?

Do there exist $n \geq 1$, and $i_1, \dots, i_n \in \{1, \dots, k\}$ such that:

$$g_{i_1} g_{i_2} \dots g_{i_n} = g'_{i_1} g'_{i_2} \dots g'_{i_n} \quad ?$$

Theorem

Codings of solvable instances of PCP:

$$\left\{ \left\{ \overbrace{\langle g_1, g'_1 \rangle, \dots, \langle g_k, g'_k \rangle}^{\text{PCP instance } I} \mid k \geq 1, g_i, g'_i \in \Sigma^+ \right\} \mid I \text{ is solvable} \right\}$$

form a set that is recursively enumerable, but not recursive.

Interaction Nets

Yves Lafont (1990) [4] ([link](#) [pdf](#)) proposed:

- ▶ a programming language with a simple graph rewriting semantics

An interaction net is specified by:

- ▶ a set of agents
- ▶ a set of interaction rules

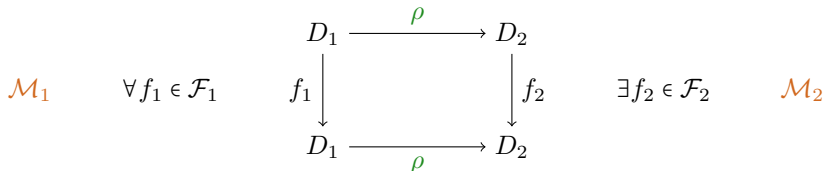
Analogy with:

- ▶ electric circuits:
 - ▶ agents $\hat{=}$ gates,
 - ▶ edges $\hat{=}$ wires
- ▶ agents as computation entities:
 - ▶ interaction rules specify behavior

Comparing computational power via encodings

► Simulation of functions:

function f_2 *simulates* function f_1 via *encoding* ρ if:



► Simulation of models of computation $\mathcal{M}_1 = \langle D_1, \mathcal{F}_1 \rangle$, $\mathcal{M}_2 = \langle D_2, \mathcal{F}_2 \rangle$:
 \mathcal{M}_2 *can simulate* \mathcal{M}_1 via ρ ($\mathcal{M}_1 \precsim_\rho \mathcal{M}_2$), if:

$$\forall f_1 \in \mathcal{F}_1 \exists f_2 \in \mathcal{F}_2 (f_2 \text{ simulates } f_1 \text{ via } \rho)$$

Weak requirements on encodings (Boker/Dershowitz)

Traditional requirements on encodings are:

- ▶ *informally computable/effective/mechanizable in principle*
- ▶ *computable* with respect to a specific model (Turing machine, ...)

Boker & Dershowitz [1]: want a ‘**robust definition that does not itself depend on the notion of computability**’, and therefore suggest as encodings:

- injective* functions
- bijective* functions

Definition (**power subsumption** pre-order [Boker/Dershowitz 2006 [1]])

- $\mathcal{M}_1 \lesssim \mathcal{M}_2$ if: there is an **injective** ρ such that $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$
- $\mathcal{M}_1 \lesssim_{\text{bijective}} \mathcal{M}_2$ if: there is a **bijective** ρ such that $\mathcal{M}_1 \lesssim_{\rho} \mathcal{M}_2$

Anomalies for decision models

However, we found anomalies of these definitions.

$\mathcal{M} = \langle D, \mathcal{F} \rangle$ is a *decision model* if $\{0, 1\} \subseteq D$, $\forall f \in \mathcal{F} (f[D] \subseteq \{0, 1\})$.

Theorem (Endrullis/G/Hendriks, [3])

Let Σ and Γ with $\{0, 1\} \subseteq \Sigma, \Gamma$ be alphabets.

Then for every countable decision model $\mathcal{M} = \langle \Sigma^*, \mathcal{F} \rangle$, it holds:

$$\mathcal{M} \preceq \text{DFA}(\Gamma) \quad \mathcal{M} \preceq_{\text{bijective}} \text{DFA}(\Gamma)$$

$\text{TMD}(\Sigma)$: class of Turing machine deciders with input alphabet Σ

Anomaly (example)

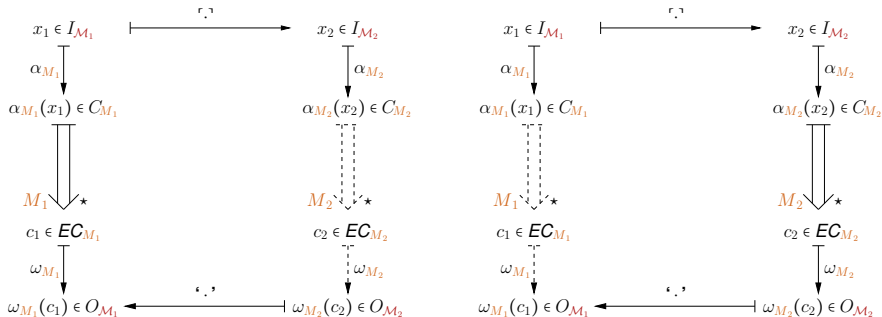
$$\text{TMD}(\Sigma) \preceq_{\text{bijective}} \text{DFA}(\Gamma)$$

These anomalies for decision models and bijective encodings:

- ▶ depend on uncomputable encodings
- ▶ can be extended to some moc's with unbounded output domain
- ▶ but do not extend to all moc's

Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ **simulate each other** with respect to **computable** coding $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and decoding $\lceil \cdot \rceil : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ if:



(defines a **Galois connection**)

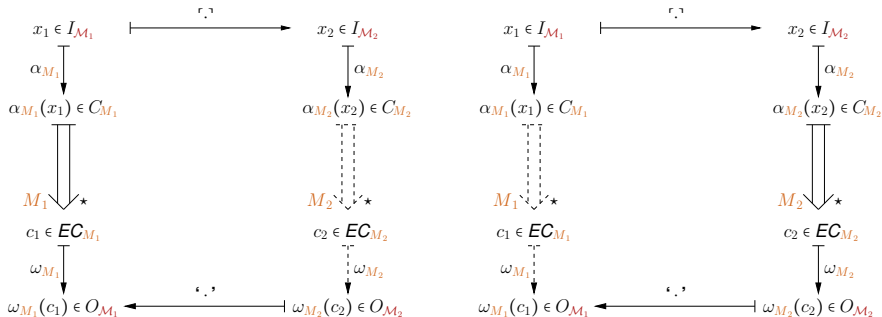
Models of computation, viewed abstractly

A(n abstractly viewed) model of computation (MoC) is a class \mathcal{M} of machines/systems/. . . such that every $M \in \mathcal{M}$ it holds:

- ▷ M has a countable set $I_{\mathcal{M}}$ of input objects, and a countable set $O_{\mathcal{M}}$ of output objects that are specific to the MoC \mathcal{M} ;
- ▷ M has a set C_M of configurations of M , which contains the subset $EC_M \subseteq C_M$ of end-configurations of M ;
- ▷ M has an injective input function $\alpha_M : I_{\mathcal{M}} \rightarrow C_M$, which maps input objects of M to configurations of M ; α_M is computable;
- ▷ M defines a one-step computation relation \Rightarrow_M on the set C_M ; the transitive closure of \Rightarrow_M is designated by \Rightarrow_M^* ;
- ▷ M has a partial output function $\omega_M : EC_M \rightarrow O_{\mathcal{M}}$, which maps some end-configurations of M to output objects of M ; ω_M is computable, and membership of end-configurations in $\text{dom}(\omega_M)$ is decidable.

Simulations between models of computation

models $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ **simulate each other** with respect to **computable** coding $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and decoding $\lceil \cdot \rceil : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ if:



(defines a **Galois connection**)

Comparing Computational Power of MoC's

Definition

Let \mathcal{M}_1 and \mathcal{M}_2 be MoC's.

- 1 The computational power of \mathcal{M}_1 is subsumed by that of \mathcal{M}_2 , denoted symbolically by $\mathcal{M}_1 \leq \mathcal{M}_2$, if:

(\exists a pair $\langle \ulcorner \cdot \urcorner, \lceil \cdot \rceil \rangle$ of **computable** encoding and decoding functions $\ulcorner \cdot \urcorner : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and $\lceil \cdot \rceil : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$

$(\forall M_1 \in \mathcal{M}_1) (\exists M_2 \in \mathcal{M}_2)$

$[M_1 \text{ and } M_2 \text{ simulate each other w.r.t. } \langle \ulcorner \cdot \urcorner, \lceil \cdot \rceil \rangle]$).

- 2 The computational power of \mathcal{M}_1 is equivalent to that of \mathcal{M}_2 , denoted by $\mathcal{M}_1 \sim \mathcal{M}_2$, if both $\mathcal{M}_1 \leq \mathcal{M}_2$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ hold.

Comparing Computational Power of MoC's

Theorem

For all models \mathcal{M}_1 and \mathcal{M}_2 , and encoding and decoding functions $\lceil \cdot \rceil : I_{\mathcal{M}_1} \rightarrow I_{\mathcal{M}_2}$ and $\lceil \cdot \rceil : O_{\mathcal{M}_2} \rightarrow O_{\mathcal{M}_1}$ it holds:

$$\mathcal{M}_1 \leq_{(\lceil \cdot \rceil, \lceil \cdot \rceil)} \mathcal{M}_2 \implies \mathcal{F}(\mathcal{M}_1) \subseteq \{ \lceil \cdot \rceil \circ f \circ \lceil \cdot \rceil \mid f \in \mathcal{F}(\mathcal{M}_2) \}.$$

Turing completeness and equivalence

By $\mathcal{TM}(\Sigma)$ we mean the model of Turing machines over input alphabet Σ .

Definition

Let \mathcal{M} a model of computation.

\mathcal{M} is **Turing-complete** if $\mathcal{TM}(\Sigma) \leq \mathcal{M}$ for some alphabet Σ with $\Sigma \neq \emptyset$.

\mathcal{M} is **Turing-equivalent** if $\mathcal{M} \sim \mathcal{TM}(\Sigma)$ for some alphabet $\Sigma \neq \emptyset$.

Fractran

John Horton Conway:

- ▶ article:
 - ▶ **FRACTRAN:**
A Simple Universal Programming Language for Arithmetic
- ▶ talk video:
 - ▶ "Fractran: A Ridiculous Logical Language"

Summary

- ▶ [Post's Correspondence Problem](#) (by [Emil Post](#), 1946, [[6](#)])
- ▶ [Interaction Nets](#) (by [Yves Lafont](#), 1990, [[4](#)])
- ▶ Compare computational power of models of computation
- ▶ [Fractran](#) (by [John Horton Conway](#), 1987, [[2](#)])

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ -calculus Herbrand–Gödel recursive functions partial-recursive/ μ -recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	<i>classical</i>
	Fractran	<i>less well known</i>
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ζ -calculus evolutionary programming/genetic algorithms	<i>modern</i>
	abstract state machines	
	hypercomputation	<i>speculative</i>
	quantum computing bio-computing reversible computing	<i>physics-/biology- inspired</i>

Course overview

<i>intro</i>	<i>classic models</i>			<i>additional models</i>
Introduction to Computability	Machine Models	Recursive Functions	Lambda Calculus	Three more Models of Computation
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ -terms, β -reduction, λ -definable functions, partial recursive = λ -definable = Turing computable	Post's Correspondence Problem, Interaction-Nets, Fractran
	<i>imperative programming</i>	<i>algebraic programming</i>	<i>functional programming</i>	

References I



Udi Boker and Nachum Dershowitz.

Comparing computational power.

Logic Journal of the IGPL, 14(5):633–647, 10 2006.



John Horton Conway.

FRACTRAN: A Simple Universal Programming Language for Arithmetic.

58(2):345–363, April 1936.



Jörg Endrullis, Clemens Grabmayer, and Dimitri Hendriks.

Regularity-Preserving but not Reflecting Encodings.

In *Proceedings of the 30th Annual ACM/IEEE Symposium on Logic in Computer Science 2015 (Kyoto, Japan, July 6–10, 2015)*, pages 535–546, July 2015.



Yves Lafont.

Interaction Nets.

Proceedings of POPL'90, pages 95–108, 1990.

References II



Vincent van Oostrom, Kees-Jan van de Looij, and Marijn Zwitterlood.

Lambdascope.

Extended Abstract, Workshop ALPS, Kyoto, April 10th 2004, 2004.

<http://www.phil.uu.nl/~oostrom/publication/pdf/lambdascope.pdf>.



Emil Leon Post.

A Variant of a Recursively Unsolvable Problem.

Bulletin of the American Mathematical Society, 52:264–268, 1946.

References III



Jan Rochel.

graph-rewriting-lambdascope: Lambdascope, an optimal evaluator of the lambda calculus.

Haskell package on Hackage, <https://hackage.haskell.org/package/graph-rewriting-lambdascope>, 2010.

Lambdascope interaction-net animation tool.