From Partial Recursive to λ-Definable Functions

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Abstract. Adapting the presentation by Sørensen en Urzyczyn in [1] to the definitions used in the lecture, we show that partial recursive functions are λ-definable.

1 Primitive recursive and partial recursive functions

We start with the definition of primitive recursive functions on the natural numbers $\mathbb{N} := \{0, 1, 2, \ldots\}$ including 0.

Definition 1. The class $\mathcal{PR}$ of *primitive recursive functions* with values in $\mathbb{N}$ is the smallest class $\mathcal{C}$ of functions contained in $\{h \mid h : \mathbb{N}^n \to \mathbb{N},\, n \in \mathbb{N}\}$ that contains the *base functions*:

- $\mathcal{O} : \mathbb{N}^0 = \{\emptyset\} \to \mathbb{N}\,,\ \emptyset \mapsto 0$ (0-ary constant-0 function);
- $\mathsf{Succ} : \mathbb{N} \to \mathbb{N}\,,\ x \mapsto x + 1$ (successor function);
- $\pi^n_i : \mathbb{N}^n \to \mathbb{N}\,,\ \vec{x} = \langle x_1, \ldots, x_n\rangle \mapsto x_i$ (projection function).

and is closed under the operations composition and primitive recursion:

- *Composition*: if $f : \mathbb{N}^k \to \mathbb{N}$, and $g_i : \mathbb{N}^n \to \mathbb{N}$ are in $\mathcal{C}$, then so is $h = f \circ (g_1 \times \ldots \times g_k) : \mathbb{N}^n \to \mathbb{N}$ defined by
$$h(\vec{x}) = f(g_1(\vec{x}), \ldots, g_k(\vec{x}))\,.$$
- *Primitive recursion*: if $f : \mathbb{N}^n \to \mathbb{N}$, $g : \mathbb{N}^{n+2} \to \mathbb{N}$ are in $\mathcal{C}$ then so is $h = \mathsf{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by:
$$\begin{aligned} h(\vec{x}, \mathit{0}) &= f(\vec{x}) \\ h(\vec{x}, y{+}\mathit{1}) &= g(\vec{x}, h(\vec{x}, y), y)\,. \end{aligned}$$

A function belonging to $\mathcal{PR}$ is called *primitive recursive*.

Next, we give the definition of the classes of partial recursive, and of total recursive, functions. For a partial function[1] $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$, and for $\vec{x} = \langle x_1, \ldots, n_n\rangle \in \mathbb{N}^n$ we write $f(\vec{x})\!\downarrow$ if $f(\vec{x})$ is defined, and $f(\vec{x})\!\uparrow$ if $f(\vec{x})$ is undefined.

[1] Note that possible partiality of f is indicated by using the harpoon symbol "$\rightharpoonup$" instead of the symbol "$\to$" in the expression $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$.

Definition 2. The class $\mathcal{P}$ of *partial recursive functions*[2] with values in $\mathbb{N}$ is the smallest class $\mathcal{C}$ of partial functions contained in $\{h \mid h : \mathbb{N}^n \rightharpoonup \mathbb{N},\, n \in \mathbb{N}\}$ that contains the base functions (see Definition 1), and is closed under the operations of composition and primitive recursion (see Definition 1) as well as of unbounded minimisation (μ-recursion):

– *Unbounded minimisation*: if $g : \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$ is in $\mathcal{C}$, then so is $\mu(g)$ defined by:

$$
\begin{aligned}
\mu(g) : \mathbb{N}^n &\rightharpoonup \mathbb{N} \\
\vec{x} &\mapsto \mu z.[g(\vec{x}, z) = 0] := \\
&\qquad \begin{cases} z & \ldots\ g(\vec{x}, z) = 0 \wedge \forall y \left(0 \le y < g(z) \to (g(\vec{x}, y){\downarrow} \neq 0) \right) \\ \uparrow & \ldots\ \neg \exists y \left(g(\vec{x}, y) = 0 \wedge \forall z \left(0 \le z < y \to (g(\vec{x}, z){\downarrow}) \right) \right) \end{cases}
\end{aligned}
$$

We denote by $\mathcal{R}$ the class of functions that consists of all partial functions in $\mathcal{P}$ that are total, that is, of all functions in $\mathcal{P}$ that are defined for all $n \in \mathbb{N}$.

Functions in $\mathcal{P}$ are called *partial recursive*, and functions in $\mathcal{R}$ are called *(total) recursive*.

The Kleene Normal Form Theorem below (due to Stephen Cole Kleene) states that every partial recursive function can be factorised into the composition of a primitive recursive function with the unbounded minimisation of a (second) primitive recursive function.

Theorem 3 (Kleene's Normal Form Theorem). *For every partial recursive function $h : \mathbb{N}^n \to \mathbb{N}$ there exist primitive recursive functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N}^{n+1} \to \mathbb{N}$ such that:*

$$
\begin{aligned}
h(x_1, \ldots, x_n) &= (f \circ \mu(g))(x_1, \ldots, x_n)\ . \\
&= f(\mu(g)(x_1, \ldots, x_n))
\end{aligned}
$$

2 λ-definable functions

In order to 'encode' natural numbers in λ-calculus as pure λ-terms, on which λ-terms that mimic functions on natural numbers are then able to operate (by application of λ-terms), we define the 'Church numerals' (due to Alonzo Church).

Definition 4. For every $n \in \mathbb{N}$, the *Church numeral* $\ulcorner n \urcorner$ for n is defined by:

$$
\begin{aligned}
\ulcorner n \urcorner &:= \lambda f x. f^n x \\
&= \lambda f x. \underbrace{f(f(\ldots(f}_{n} x) \ldots))\ .
\end{aligned}
$$

Example 5. We find: $\ulcorner 0 \urcorner = \lambda f x.x$, $\ulcorner 1 \urcorner = \lambda f x.f x$, $\ulcorner 2 \urcorner = \lambda f x.f(f x)$.

[2] Note that "recursive, partial functions" would be a more adequate name.

Based on Church numerals we now give the definition of definability in λ-calculus of total, and of partial, functions on natural numbers.

Definition 6. (i) Let $f : \mathbb{N}^n \to \mathbb{N}$ be total. A λ-term M_f *represents* f if for all $m_1, \ldots, m_k \in \mathbb{N}$:

$$M_f \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \twoheadrightarrow_\beta \ulcorner f(m_1, \ldots, m_n) \urcorner \,.$$

f is called λ-*definable* if there exists a λ-term that represents f.

(ii) Let $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$ be a partial function. A λ-term M_f *represents* f if for all $m_1, \ldots, m_n \in \mathbb{N}$:

$$f(m_1, \ldots, m_n)\downarrow \implies M_f \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \twoheadrightarrow_\beta \ulcorner f(m_1, \ldots, m_n) \urcorner \,,$$
$$f(m_1, \ldots, m_n)\uparrow \implies M_f \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \text{ has no normal form} \,.$$

f is called λ-*definable* if there exists a λ-term that represents f.

Example 7. We give a few examples of λ-terms representing operations on natural numbers:

- *successor*: $M_{\mathsf{Succ}} := \lambda nfx.f(nfx)$
- *addition*: $M_+ := \lambda mnfx.mf(nfx)$
- *multiplication*: $M_\times := \lambda mnfx.m(nf)x$
- *exponentiation*: $M_{\mathsf{E}} := \lambda mnfx.mnfx$
- *unary constant zero function*: $M_{\mathsf{C}_0^1} = \lambda m.\ulcorner 0 \urcorner$
- *projection function*: $M_{\pi_i^n} = \lambda m_1 \ldots m_n.m_i$

For recognising that M_{Succ} indeed represents the successor function, we find that for all $n \in \mathbb{N}$ the following $\to_\beta$-rewrite sequence:

$$\begin{aligned} M_{\mathsf{Succ}} \ulcorner n \urcorner &= (\lambda nfx.f(nfx)) \ulcorner n \urcorner \\ &\to_\beta \lambda fx.f(\ulcorner n \urcorner fx) \\ &= \lambda fx.f((\lambda fx.f^n x)fx) \\ &\to_\beta \lambda fx.f((\lambda x.f^n x)x) \\ &\to_\beta \lambda fx.f(f^n x) \\ &= \lambda fx.f^{n+1} x \\ &= \ulcorner n+1 \urcorner \,. \end{aligned} \tag{1}$$

3 Primitive recursive functions are λ-definable

In this section we verify that all primitive recursive functions are λ-definable.

For use in the proofs below, we start by defining how pairs of λ-terms can be coded as λ-terms.

Definition 8. For all λ-terms M, N we define the λ-term *pair* $\langle M, N\rangle$ representing M and N by:

$$\langle M, N\rangle := \lambda x.xMN$$

and the *unpairing projections* ρ_1 and ρ_2 by:

$$\rho_1 := \lambda p.p(\lambda xy.x)$$
$$\rho_2 := \lambda p.p(\lambda xy.y)$$

Based on this definition, the following proposition is easy to check.

Proposition 9. *For all λ-terms M_1, M_2 and $i = 1, 2$ it holds:*

$$\rho_i(\langle M_1, M_2\rangle) \twoheadrightarrow_\beta M_i \,.$$

Having assembled some essential tools, we can now formulate, and then prove, the statement on λ-definability of the primitive recursive functions.

Theorem 10. *Every primitive recursive function is λ-definable.*

Proof. We show the theorem by proving that the class of primitive recursive functions is a subset of the class of λ-definable total functions.

First we have to show that the class of λ-definable functions contains the base functions of Definition 1:

- ▷ The 0-ary function $\mathcal{O}$ can be represented by $\ulcorner 0 \urcorner$, the Church numeral for 0.
- ▷ The successor function Succ can be represented by the λ-term $M_{\mathsf{Succ}} := \lambda nfx.f(nfx)$, as we saw above in (1).
- ▷ Every projection function $\pi^n_i : \mathbb{N}^n \to \mathbb{N}$, can be represented by the λ-term $M_{\pi^n_i} = \lambda m_1 \dots m_n.m_i$, as is straightforward to check.

Second, we have to show that the class of λ-definable total functions is closed under composition. For this we let $f : \mathbb{N}^k \to \mathbb{N}$, and $g_i : \mathbb{N}^n \to \mathbb{N}$, for all $i \in \{1, \dots, k\}$, be arbitrary λ-definable functions. We have to show that $h = f \circ (g_1 \times \dots \times g_k) : \mathbb{N}^n \to \mathbb{N}$ is λ-definable as well. Suppose that f and g_1, …, g_k are represented by the λ-terms M_f, M_{g_1}, …, M_{g_k}, respectively. Then it is easy to check that the λ-term:

$$M_h := \lambda x_1 \dots x_n.M_f(M_{g_1}x_1 \dots x_n) \dots (M_{g_k}x_1 \dots x_n)$$

represents h.

Finally, we have to establish that the class of λ-definable total functions is closed under primitive recursion. For this, let $f : \mathbb{N}^n \to \mathbb{N}$ and $g : \mathbb{N}^{n+2} \to \mathbb{N}$ be arbitrary λ-definable (total) functions. Suppose that f and g are represented by λ-terms M_f and M_g, respectively. We have to show that the function $h := \mathsf{pr}(f; g) : \mathbb{N}^{n+1} \to \mathbb{N}$ defined from f and g by primitive recursion via:

$$\begin{aligned} h(\vec{x}, \mathit{0}) &= f(\vec{x}) \\ h(\vec{x}, y{+}\mathit{1}) &= g(\vec{x}, h(\vec{x}, y), y) \end{aligned} \tag{2}$$

is λ-definable as well.

In order to establish this, we let:

$$\begin{aligned}\mathsf{Init} &:= \langle \ulcorner 0 \urcorner, M_f\, x_1 \ldots x_n \rangle \\ \mathsf{Step} &:= \lambda p. \langle M_{\mathsf{Succ}}(\rho_1(p)), M_g x_1 \ldots x_n(\rho_2(p))(\rho_1(p)) \rangle\end{aligned}$$

and will show that the λ-term M_h defined by:

$$M_h := \lambda x_1 \ldots x_n x. \rho_2((x\, \mathsf{Step}\ \mathsf{Init})) \tag{3}$$

represents h.

Let $m_1, \ldots, m_n \in \mathbb{N}$ be arbitrary. We have to show that for all $k \in \mathbb{N}$ there are β-reduction sequences of the form:

$$M_h \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner k \urcorner \twoheadrightarrow_\beta \ulcorner h(m_1, \ldots, m_n, k) \urcorner \tag{4}$$

As a (crucial) step towards showing (4), we first define the following two abbreviations for λ-terms:

$$\begin{aligned}\mathsf{Init}' &:= \mathsf{Init}[x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \langle \ulcorner 0 \urcorner, M_f\, x_1 \ldots x_n \rangle [x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \langle \ulcorner 0 \urcorner, M_f \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \rangle \end{aligned} \tag{5}$$

$$\begin{aligned}\mathsf{Step}' &:= \mathsf{Step}[x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \big(\lambda p. \big\langle M_{\mathsf{Succ}}(\rho_1(p)), M_g x_1 \ldots x_n(\rho_2(p))(\rho_1(p)) \big\rangle\big) [x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \lambda p. \big\langle M_{\mathsf{Succ}}(\rho_1(p)), M_g \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner (\rho_2(p))(\rho_1(p)) \big\rangle \end{aligned} \tag{6}$$

and prove, for all $k \in \mathbb{N}$, the existence of β-reduction sequences of the form:

$$(\mathsf{Step}')^k\, \mathsf{Init}' \twoheadrightarrow_\beta \big\langle \ulcorner k \urcorner, \ulcorner h(m_1, \ldots, m_n, k) \urcorner \big\rangle \tag{7}$$

by induction on k.

For showing the base case $k = 0$ for the proof by induction of (7), we construct the following β-reduction sequence which uses that the λ-term M_f represents f:

$$\begin{aligned} &(\mathsf{Step}')^0\, \mathsf{Init}' \\ &= \mathsf{Init}' \\ &= \mathsf{Init}[x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \langle \ulcorner 0 \urcorner, M_f\, x_1 \ldots x_n \rangle [x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \\ &= \langle \ulcorner 0 \urcorner, M_f \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \rangle \\ &\twoheadrightarrow_\beta \langle \ulcorner 0 \urcorner, \ulcorner f(m_1, \ldots, m_n) \urcorner \rangle \quad \text{(since } M_f \text{ represents } f\text{)} \\ &= \big\langle \ulcorner 0 \urcorner, \ulcorner h(m_1, \ldots, m_n, 0) \urcorner \big\rangle \end{aligned}$$

For showing the induction step from k to $k+1$ in our proof, we assume that (7) holds for k, and in order to show that (7) holds also for $k+1$ substituted for k, we construct the following β-reduction sequence:

$$\begin{aligned} &(\mathsf{Step}')^{k+1}\, \mathsf{Init}' \\ &= \mathsf{Step}'((\mathsf{Step}')^{k+1}\, \mathsf{Init}) \end{aligned}$$

$$\begin{aligned}
&\twoheadrightarrow_\beta \mathsf{Step}' \left\langle \ulcorner k \urcorner, \ulcorner h(m_1,\ldots,m_n,k) \urcorner \right\rangle \qquad \text{(by induction hypothesis)} \\
&= \left(\lambda p. \left\langle M_{\mathsf{Succ}}\,(\rho_1 p), M_g \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner (\rho_2 p)(\rho_1 p) \right\rangle\right) \left\langle \ulcorner k \urcorner, \ulcorner h(m_1,\ldots,m_n,k) \urcorner \right\rangle \\
&\twoheadrightarrow_\beta \left\langle M_{\mathsf{Succ}} \ulcorner k \urcorner, M_g \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner h(m_1,\ldots,m_n,k) \urcorner \ulcorner k \urcorner \right\rangle \\
&\twoheadrightarrow_\beta \left\langle M_{\mathsf{Succ}} \ulcorner k \urcorner, g(m_1,\ldots,m_n,h(m_1,\ldots,m_n,k),k) \right\rangle \quad \text{(since } M_g \text{ represents } g\text{)} \\
&= \langle \ulcorner k+1 \urcorner, \ulcorner h(m_1,\ldots,m_n,k+1) \urcorner \rangle
\end{aligned}$$

This β-reduction sequence guarantees that (7) holds also for $k+1$ substituted for k, By this argument we have shown the induction step.

Having, in this way, concluded the proof by induction of (7), we have established that (7) holds for all $k \in \mathbb{N}$.

With this preparation, we finally are in the position to prove (4), for all $k \in \mathbb{N}$, that is, that M_h as defined in (3) represents h. Let $k \in \mathbb{N}$ be arbitrary. We construct the following β-reduction sequence:

$$\begin{aligned}
&M_h \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner k \urcorner \\
&= \left(\lambda x_1 \ldots x_n x. \rho_2 (x\, \mathsf{Step}\, \mathsf{Init})\right) \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner k \urcorner \\
&\twoheadrightarrow_\beta \rho_2 \left(\ulcorner k \urcorner\, \mathsf{Step}[x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner]\, \mathsf{Init}[x_1 := \ulcorner m_1 \urcorner, \ldots, x_n := \ulcorner m_n \urcorner] \right) \\
&= \rho_2 \left(\ulcorner k \urcorner\, \mathsf{Step}'\, \mathsf{Init}' \right) \qquad \text{(by definition of } \mathsf{Init}' \text{ and } \mathsf{Step}' \text{ in (5) and (6))} \\
&= \rho_2 \left((\lambda f x. f^k x)\, \mathsf{Step}'\, \mathsf{Init}' \right) \\
&\twoheadrightarrow_\beta \rho_2 \left((\mathsf{Step}')^k\, \mathsf{Init}' \right) \\
&\twoheadrightarrow_\beta \rho_2 \left(\langle \ulcorner k \urcorner, \ulcorner h(m_1,\ldots,m_n,k) \urcorner \rangle \right) \qquad \text{(due to (7))} \\
&\twoheadrightarrow_\beta \ulcorner h(m_1,\ldots,m_n,k) \urcorner
\end{aligned}$$

Since $k \in \mathbb{N}$ was arbitrary in this argument, we now have established (4) for all $k \in \mathbb{N}$. Since also $m_1,\ldots,m_n \in \mathbb{N}$ were arbitrary, we have shown that the λ-term M_h indeed represents h as defined according to (2) from f and h, using the assumption that M_f represents f, and M_g represents f. ☐

4 Partial recursive functions are λ-definable

In this section we prove that all partial recursive functions are λ-definable.

For use in the proof below, we define codings of the Boolean truth values, a test function for equality with zero, and the if-then-else construct in λ-calculus.

Definition 11. For representing the Boolean truth values "true" and "false" we define λ-terms **true** and **false**, and for representing a predicate that tests on λ-terms for being equal to the Church numeral $\ulcorner 0 \urcorner$ we define the λ-term **zero?** as follows:

$$\mathbf{true} := \lambda xy.x \qquad \mathbf{false} := \lambda xy.y \qquad \mathbf{zero?} := \lambda x.x(\lambda y.\mathbf{false})\mathbf{true}$$

Furthermore we define, for all λ-terms P, Q, and R, the λ-term **if** P **then** Q **else** R as follows:

$$\mathbf{if}\ P\ \mathbf{then}\ Q\ \mathbf{else}\ R := PQR$$

Proposition 12. *For all λ-terms Q and R, and for all $n \in \mathbb{N}$ it holds:*

$$\begin{aligned} \textbf{if true then } Q \textbf{ else } R &\twoheadrightarrow_\beta Q \\ \textbf{if false then } Q \textbf{ else } R &\twoheadrightarrow_\beta R \\ \textbf{zero?}\ulcorner 0 \urcorner &\twoheadrightarrow_\beta \textbf{true} \\ \textbf{zero?}\ulcorner n+1 \urcorner &\twoheadrightarrow_\beta \textbf{false} \end{aligned}$$

Proof. These properties are easy to verify by using β-reduction.

We now set out to proving λ-definability for all partial recursive functions.

Theorem 13. *Every partial recursive function is λ-definable.*

Proof. Let $h : \mathbb{N}^{n+1} \to \mathbb{N}$ be an arbitrary partial recursive function. Then by Theorem 3, Kleene's normal form theorem, there exist $g : \mathbb{N}^{n+1} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ such that:

$$h(\vec{x}) = f \circ \mu(g)(\vec{x}) = f(\mu z.[g(\vec{x}, z) = 0]) .$$

Let M_f and M_g be λ-terms representing f and g, respectively. Let:

$$W := \lambda y.\textbf{if } (\textbf{zero?}\, M_g x_1 \ldots x_n y) \textbf{ then } (\lambda w.M_f y) \textbf{ else } (\lambda w.w(M_{\mathsf{Succ}} y)w) .$$

We will show that the following λ-term M_h represents h:

$$M_h := \lambda x_1 \ldots x_n.W \ulcorner 0 \urcorner W .$$

For this we first observe:

$$M_h \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \twoheadrightarrow_\beta W' \ulcorner 0 \urcorner W' \tag{8}$$

for $W' := W[x_1 := \ulcorner m_1 \urcorner] \ldots [x_n := \ulcorner m_n \urcorner]$.

Furthermore, for $\vec{m} = \langle m_1, \ldots, m_n \rangle \in \mathbb{N}^n$ and $l \in \mathbb{N}$ such that $g(\vec{m}, l) = 0$ we use that the λ-term M_g represents g to build the following β-reduction rewrite sequence:

$$\begin{aligned} W' \ulcorner l \urcorner W' &\to_\beta \underbrace{(\textbf{zero?}\ \underbrace{M_g \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner l \urcorner}_{\twoheadrightarrow_\beta \ulcorner g(m_1,\ldots,m_n,l) \urcorner = \ulcorner 0 \urcorner})}_{\twoheadrightarrow_\beta \textbf{true}} (\lambda w.M_f \ulcorner l \urcorner)(\lambda w.w(M_{\mathsf{Succ}} \ulcorner l \urcorner)w)W' \\ &\twoheadrightarrow_\beta \textbf{true}(\lambda w.M_f \ulcorner l \urcorner)(\lambda w.w(M_{\mathsf{Succ}} \ulcorner l \urcorner)w)W' \\ &\twoheadrightarrow_\beta (\lambda w.M_f \ulcorner l \urcorner)W' \\ &\to_\beta M_f \ulcorner l \urcorner \\ &\twoheadrightarrow_\beta \ulcorner f(l) \urcorner . \end{aligned} \tag{9}$$

For $\vec{m} = \langle m_1, \ldots, m_n \rangle \in \mathbb{N}^n$ and $l \in \mathbb{N}$ such that $g(\vec{m}, l) \neq 0$, we construct the following β-reduction rewrite sequence, again by using the M_g represents g:

$$W' \ulcorner l \urcorner W' \to_\beta \underbrace{(\textbf{zero?}\ \underbrace{M_g \ulcorner m_1 \urcorner \ldots \ulcorner m_n \urcorner \ulcorner l \urcorner}_{\twoheadrightarrow_\beta \ulcorner g(m_1,\ldots,m_n,l) \urcorner \neq \ulcorner 0 \urcorner})}_{\twoheadrightarrow_\beta \textbf{false}} (\lambda w.M_f \ulcorner l \urcorner)(\lambda w.w(M_{\mathsf{Succ}} \ulcorner l \urcorner)w)W'$$

$$
\begin{aligned}
&\twoheadrightarrow_\beta \mathbf{false}(\lambda w.M_f\ulcorner l\urcorner)(\lambda w.w(M_{\mathsf{Succ}}\ulcorner l\urcorner)w)W' \\
&\twoheadrightarrow_\beta (\lambda w.w(M_{\mathsf{Succ}}\ulcorner l\urcorner)w)W' \\
&\rightarrow_\beta W'(M_{\mathsf{Succ}}\ulcorner l\urcorner)W' \\
&\twoheadrightarrow_\beta W'\ulcorner l+1\urcorner W' \quad \text{(since } M_{\mathsf{Succ}} \text{ represents } \mathsf{Succ}\text{)}
\end{aligned} \tag{10}
$$

Let now $m_1, \ldots, m_n \in \mathbb{N}$ be arbitrary.

Suppose that $h(m_1, \ldots, m_n)\downarrow$. Then it follows that $\mu(g)(m_1, \ldots, m_n)\downarrow$, and hence there exists $m \in \mathbb{N}$ such that $g(m_1, \ldots, m_n, m) = 0$ and such that $g(m_1, \ldots, m_n, l)\downarrow \neq 0$ for all $l \in \mathbb{N}$ with $l < m$. Then by (8) and by repeated application of the existence statement of a β-reduction sequences in (and above) (10) followed by a single application of the existence statement of a β-reduction sequence in (and above) (9) we obtain:

$$
\begin{aligned}
M_h\ulcorner m_1\urcorner\ldots\ulcorner m_n\urcorner \twoheadrightarrow_\beta W'\ulcorner 0\urcorner W' \twoheadrightarrow_\beta W'\ulcorner 1\urcorner W' \twoheadrightarrow_\beta \ldots \twoheadrightarrow_\beta W'\ulcorner m\urcorner W' \\
\twoheadrightarrow_\beta \ulcorner f(m)\urcorner = \ulcorner f(\mu(g)(m_1, \ldots, m_n))\urcorner \\
= \ulcorner h(m_1, \ldots, m_n)\urcorner\,.
\end{aligned}
$$

Suppose now that $h(m_1, \ldots, m_n)\uparrow$. Then it follows that $\mu(g)(m_1, \ldots, m_n)\uparrow$, and hence for all $m \in \mathbb{N}$ it holds that $g(m_1, \ldots, m_n, m) \neq 0$.. Then it follows by (8) and by repeated application of the statement connected to (10) that there is the following infinite rewrite sequence:

$$
\begin{aligned}
M_h\ulcorner m_1\urcorner\ldots\ulcorner m_n\urcorner \twoheadrightarrow_\beta W'\ulcorner 0\urcorner W' \twoheadrightarrow_\beta W'\ulcorner 1\urcorner W' \twoheadrightarrow_\beta \ldots \\
\twoheadrightarrow_\beta W'\ulcorner n\urcorner W' \twoheadrightarrow_\beta W'\ulcorner n+1\urcorner W' \twoheadrightarrow_\beta \ldots\,.
\end{aligned}
$$

Since this rewrite sequence is a maximal left-most rewrite sequence, and since maximal left-most rewrite sequences in λ-calculus are known to be normalizing (that is, they always lead to a normal form whenever there exists one), it follows that $M_h\ulcorner m_1\urcorner\ldots\ulcorner m_n\urcorner$ has no normal form.

By what we showed in particular in the last two paragraphs, we have established that M_h indeed represents h. ☐

References

1. Morten Heine Sørensen and Paweł Urzyczyn. *Lectures on the Curry–Howard Isomorphism.* Elsevier, 2006.