Halting Problem

$$H := \{ w \in \{0,1\}^* \mid w = \langle w_n, w_m \rangle,\ M_n \text{ halts on } w_m \}$$

$$= \{ \langle \underbrace{\langle M_n \rangle}_{\in \{0,1\}^*}, w_m \rangle \mid M_n \text{ halts on } w_m \}$$

$$= \{ \langle \langle M \rangle, w \rangle \mid \underbrace{M \in TM_{\{0,1\}},\ w \in \{0,1\}^*,\ M \text{ halts on } w}\}$$

$\boxed{\text{Codings necessary:}}$ (easy computable)

- $\langle \cdot \rangle : TM_{\{0,1\}} \longrightarrow \{0,1\}^*$ bijective of Turing machines
- $\langle \cdot, \cdot \rangle : \{0,1\}^* \times \{0,1\}^* \longrightarrow \{0,1\}^*$ bijective of pairs of binary words

$M_n := \langle \cdot \rangle^{-1}(w_n)$ n-th machine hence $\langle M_n \rangle = w_n$

for an enumeration $\{0,1\}^* = \{\underset{\varepsilon}{w_0}, \underset{0}{w_1}, \underset{1}{w_2}, \dots\}$ of binary words

Theorem. H is r.e., but not recursive.

(i) H is recursively enumerable:
We can build a machine $M^{(h)}$ that accepts H as follows:
Given $w \in \{0,1\}^*$, we obtain $w_n = \langle M_n \rangle$ and w_m such that $\langle w_n, w_m \rangle = w$ (by inversing the codings). Then $M^{(a)}$ runs M_n on w_m. If M_n halts on w_m, the $M^{(h)}$ accepts $w = \langle\langle M_n\rangle, w_m\rangle$.
Otherwise $M^{(a)}$ runs forever (and hence does not accept).
Then $L(M^a) = H$.

(ii) H is not recursive:
$\boxed{\text{Suppose that } H \text{ is recursive.}}$ $\boxed{\text{Assm}}$ Then there exists $M^{(H)}$ such that $L(M^{(H)}) = H$ and $M^{(H)}$ halts on all of its inputs.
Then $M^{(H)}$ permits us to fill in the following table and diagonalize out of it:

	w_0	w_1	w_2	....
$\langle M_0\rangle = w_0$, M_0	$h(M_0, w_0)$	$h(M_0, w_1)$	$h(M_0, w_2)$	....
$\langle M_1\rangle = w_1$, M_1	$h(M_1, w_0)$	$h(M_1, w_1)$	$h(M_1, w_2)$	....
$\langle M_2\rangle = w_2$, M_2	$h(M_2, w_0)$	$h(M_2, w_1)$	$h(M_2, w_2)$	...
$\langle M_3\rangle = w_3$, M_3	$\vdots$	$\vdots$	$\vdots$	$h(M_3, w_3)$
$\vdots$				

$$h(M_i, w_j) = \begin{cases} 1 \dots M_i \text{ halts on } w_j \\ 0 \dots M_i \text{ does not halt on } w_j \end{cases}$$

We now build a machine $\tilde{M}^{(H)}$ that halts / does not halt opposite to all machines in the list:

$$\tilde{M}^{(H)} : \quad w_i \longmapsto \langle w_i, w_i \rangle \xmapsto{M^{(H)}} \begin{cases} \text{cycles forever} \dots M^{(H)} \text{ accepts } \langle w_i, w_i \rangle \\ \text{accepts} \dots M^{(H)} \text{ rejects} \end{cases}$$

The machine $\tilde{M}_H$ has a halting behaviour that is different from any Turing machine. This is a contradiction. ↯ Hence (Assm) cannot be sustained. $\Rightarrow H$ not recursive

Further explanation:

$\widetilde{M}^{(H)} = M_{\tilde{h}}$ for some $\tilde{h} \in \mathbb{N}$.

$\langle\langle M_i\rangle, \langle M_i\rangle\rangle$
$\overset{\shortparallel}{\langle\langle M_i\rangle, w_i\rangle}$

$\widetilde{M}^{(H)}$ has the property by its construction:

(1) $\left\{\begin{array}{l} \widetilde{M}^{(H)} \text{ halts on } w_i \iff M^{(H)} \text{ rejects } \langle w_i, w_i\rangle \\ \qquad \iff \langle\langle M_i\rangle, \langle M_i\rangle\rangle \notin L(M^{(H)}) = H \\ \qquad \text{Assm } L(M^{(H)}) = H \\ \qquad \iff M_i \text{ does not hold on } \langle M_i\rangle \end{array}\right.$

Now: What about $\widetilde{M}^{(H)} = M_{\tilde{h}}$ halting on $\langle \widetilde{M}^{(H)}\rangle = \langle M_{\tilde{h}}\rangle$?

$\widetilde{M}^{(H)} = M_{\tilde{h}}$ <u>halts on</u> $\overset{\text{input}}{\langle M_{\tilde{h}}\rangle} \overset{\text{due to (1)}}{\iff} M_{\tilde{h}}$ does not hold on input $\langle M_{\tilde{h}}\rangle$

$\lightning$

This is a contradiction.

Therefore our assm. Assm that H is recursive.

$\Rightarrow$ H is not recursive!