Lecture 2: Machine Models, Basic Computability Theory

Models of Computation

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Ph.D. Program, Advanced Courses Period
Gran Sasso Science Institute
L'Aquila, Italy

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Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Overview

Reading recommended (for today)

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [2].

2. Turing machine motivation:
   Turing's analysis of a human computer:
   Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [3].

Emil Post

Emil Leon Post (1897–1954)

Post about . . .

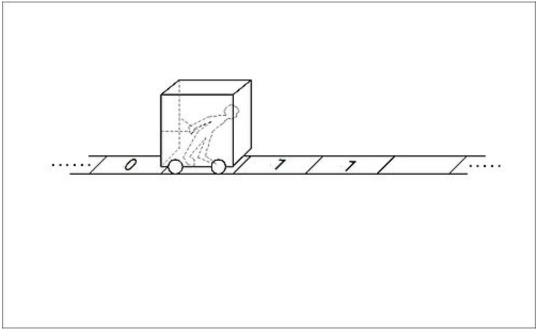
. . . a result of his from 1921 similar to the Incompleteness Theorem:

Theorem (Gödel, 1931 (paraphrased here))

Every axiomatisable, consistent first-order-logic system of number theory is incomplete: it contains true, but unprovable formulas.

> *"For full generality a complete analysis would have to be given of all possible ways in which the human mind could set up finite processes for generating sequences."*

Post machine (1936)



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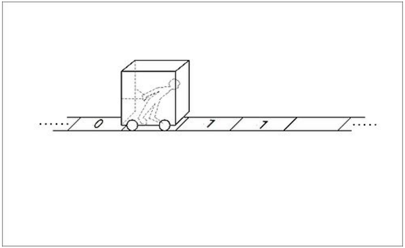

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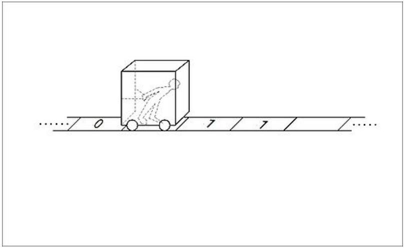

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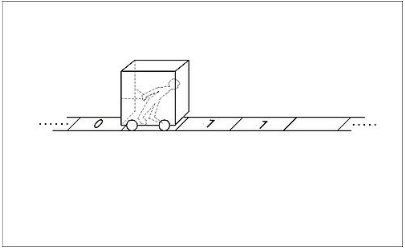

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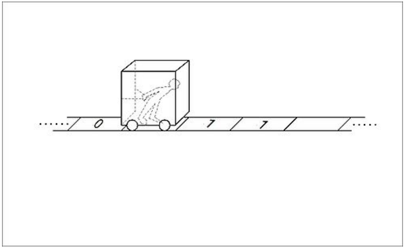

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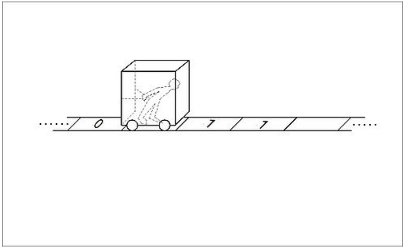

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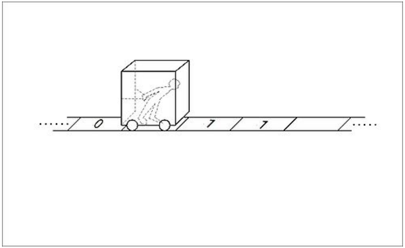

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(e) Determining whether the box he is in, is or is not marked.”

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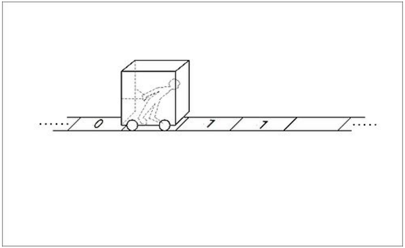

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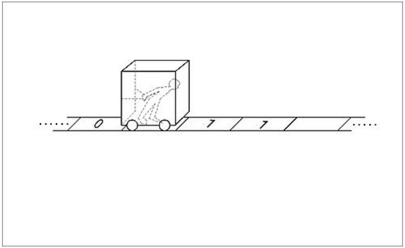

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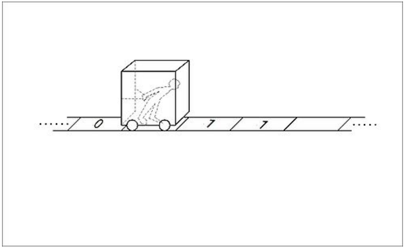

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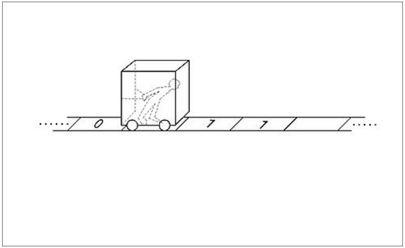

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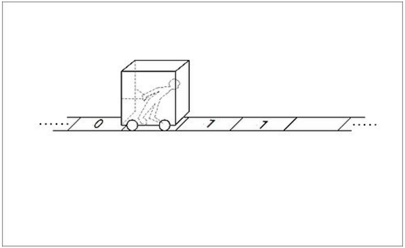

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  - (C) Stop.

Exercise

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Construct a Post machine that adds one to a natural number in unary representation.

Typical features of ‘computationally complete’ MoC’s

- storage (unbounded)

(Credits due to: Vincent van Oostrom)

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- stopping condition

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Turing computability

Alan Turing (1912 –1954)

Turing's analysis of a human 'computer'

Section 9 in Turing's 1937 paper 'On computable numbers, with an application to the Entscheidungsproblem' [3].

A direct appeal to intuition in analysing human computation:

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- behaviour of computer at any time is determined by:
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- bound B on the number of symbols/squares the computer can observe at any moment
- number of 'states of mind' of the computer is finite

Turing's analysis of a human 'computer'

- modification of tape symbols
  - in a simple operation only one symbol is altered
  - only 'observed' symbols can be altered
- modification of observed squares
  - new observed squares are within L squares of a previously observed square
  - other directly observable squares? – T. argues: not necessary
- modification of 'state of mind'

Turing's analysis of a human 'computer'

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  (A) A change (a) of symbol with a possible change of state of mind
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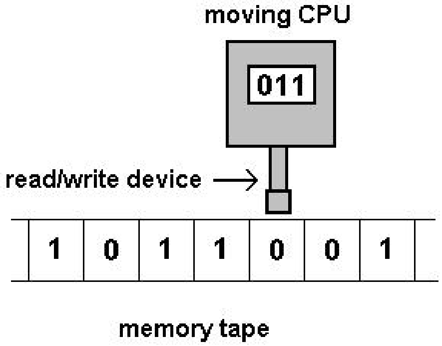
"It is my contention that these operations include all those which are used in the computation of a number."

Turing machine



Church–Turing Thesis

Thesis (Church–Turing, 1937)

Every effectively calculable function is computable by a Turing-machine.

Turing machine: formal definition

Definition

A Turing machine is a tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not{b}, F \rangle$ where:

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- $F \subseteq Q$ is the set of final or accepting states.

Turing machine: definition notions

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Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \not b, F \rangle$ be a Turing machine.

A configuration of M is elements $w_1 q w_2 \in \Gamma^* \times Q \times \Gamma^*$ such that the first letter in w_1 and the last letter in w_2 are different from $\not b$.

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- M halts on (input) w if $q_0 w \vdash_M^* uqv$ for some end-config. uqv.

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$L(M) := \{w \in \Sigma^* \mid M \text{ accepts } w\}$ is the language accepted by M.

Recursively enumerable/recursive languages

Definition

Let $L \subseteq \Sigma^*$ a language.

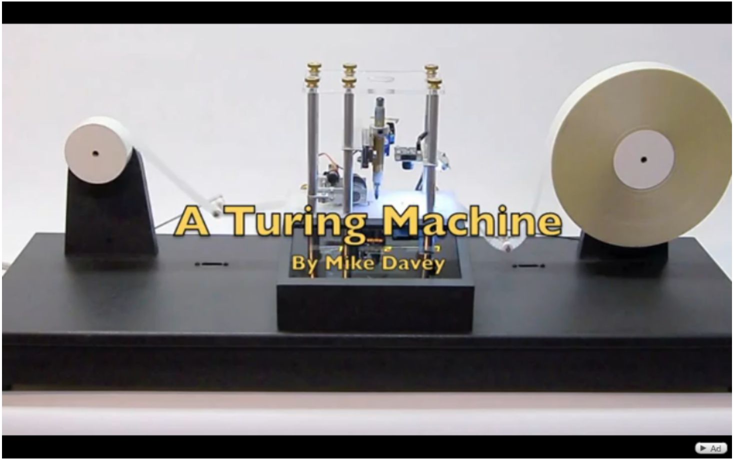
L is called recursively enumerable if

- $L = L(M)$ for some Turing machine M with input symbols Σ.

L is called recursive if

- there is a Turing machine M with input symbols Σ such that
  1. $L = L(M)$
  2. M halts on all of its inputs.

Mike Davey's Turing machine (link)



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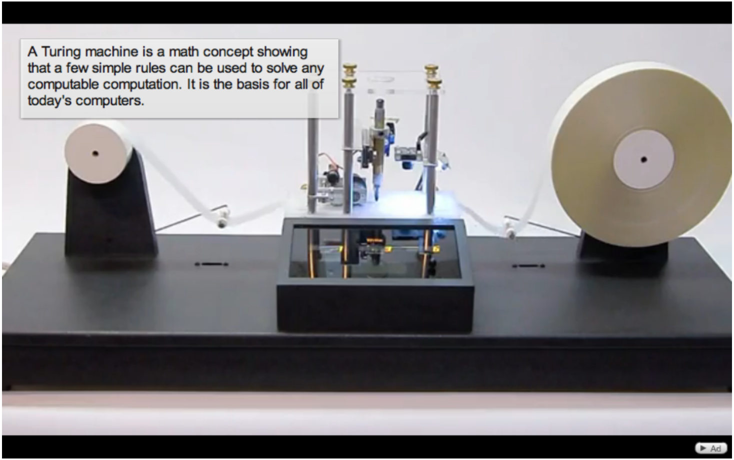

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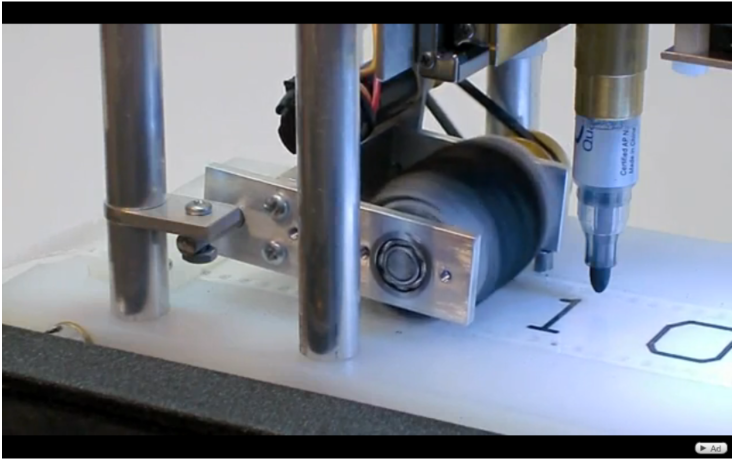

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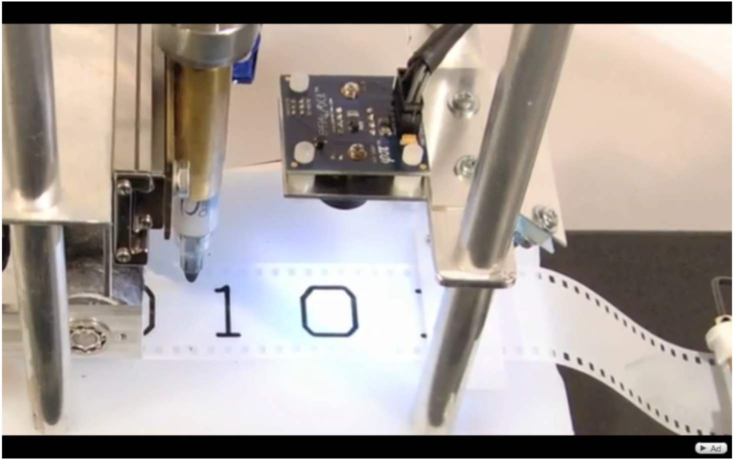

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Exercises

Exercise

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(In the film this Turing machine is executed five times consecutively.)

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Exercise

Construct a Turing machine that, if started on the empty tape, writes the sequence

$$010110111011110111110\ldots$$

on the tape, but does not halt.

(Compare your machine with Turing's machine for this purpose.)

Variants of Turing machines

- TM's with semi-infinite tapes (infinite in only one direction)
- TM's with multiple tapes
  - Input/Output Turing machines (with input- and output tapes)
- non-deterministic TM's: $\delta \subseteq ((Q \times \Gamma) \times (Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}))$
- tape-bounded TM's (by $f(n)$ for inputs of length n)
- oracle Turing machines
- Turing machines with advice
- alternating Turing machines
- ...
- interactive/reactive TM's

An unsolvable problem

The diagonalisation language:

$$L_d := \{w \mid w = \langle M \rangle, w \notin L(M)\}$$

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Proposition

L_d *is not recursively enumerable.*

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Proof.

By diagonalisation. □

An unsolvable problem

The diagonalisation language:

$$L_d := \{w \mid w = \langle M \rangle, w \notin L(M)\}$$

Proposition

L_d is not recursively enumerable.

Proof.

By diagonalisation. $\square$

Membership in the diagonalisation language
Instance: w a binary word.
Question: Does $w \in L_d$ hold? (Does Tm. M with $\langle M \rangle = w$ accept w?)

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Theorem

There exist unsolvable decision problems.

Exercise: Halting Problem

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Try to adapt the diagonalisation argument to show that for the Halting Problem

$$H = \{w \mid w = \langle w_n, w_m \rangle,\ M_n \text{ halts on input } w_m\}$$

it holds:

- H is not recursive

and show that:

- H is recursively enumerable

Properties of r.e./recursive sets (I)

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M is modified as follows to obtain $\bar{M}$:

1. the accepting states of M are made non-accepting in $\bar{M}$.
2. $\bar{M}$ has a new accepting state r.
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It follows that $\bar{L} = L(\bar{M})$, and that $\bar{M}$ halts on all inputs. $\square$

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Proof.

Let M_1 and M_2 be Tm's such that $L = L(M_1)$ and $\bar{L} = L(M_2)$.

To decide, for a given $w \in \Sigma^*$, whether $w \in L$, build a Tm M that executes M_1 and M_2 on w in parallel, and such that:

- if M_1 accepts w, then also M accepts w.
- if M_2 accepts w, then also M halts, but does not accept w.

Hence M accepts w iff $w \in L(M_1) = L$. Thus $L(M) = L$.

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Since for all w, either $w \in L$ or $w \in \bar{L}$, it follows that either M_1 or M_2 halts on w, and hence M halts on all inputs.

Hence $L = L(M)$ is recursive.

Universal language

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1. L_u is r.e.: $L_u = L(M_u)$ for an universal machine M_u.

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Theorem

L_u is r.e., but not recursive.

Proof.

1. L_u is r.e.: $L_u = L(M_u)$ for an universal machine M_u.
2. L_u is not recursive:

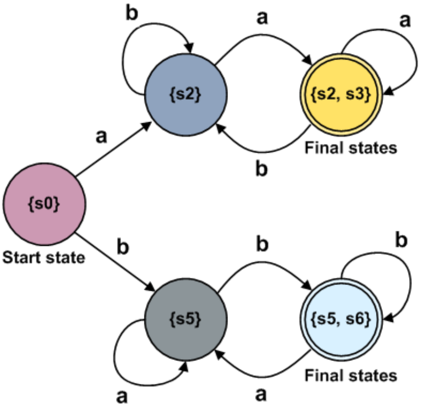
   Suppose that L_u is recursive. Then $\overline{L_u}$ is recursive, and hence there exists a Tm. M such that $\overline{L_u} = L(M)$.

   M can be used to build a Tm. M' that accepts the diagonalisation language L_d, entailing $L_u = L(M')$.

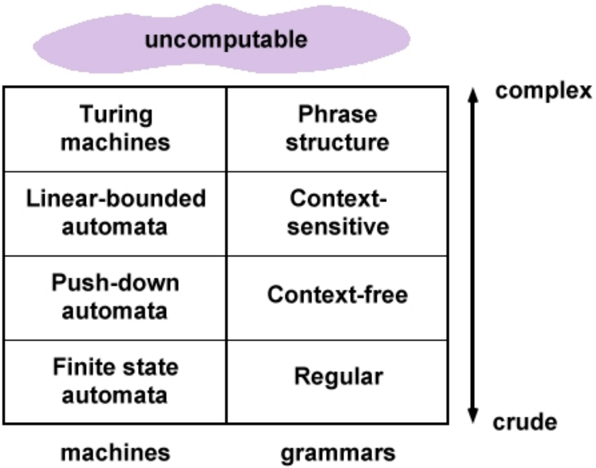
   [picture of M' to be given]

   But then L_u would actually be r.e., in contradiction with what we proved last time.

Finite-state automaton



Formal-languages Chomsky hierarchy



Summary

Example suggestions

Examples

1.
2.
3.

Recommended reading

1. Recursive and primitive-recursive functions:

   Chapter 4, Recursive Functions of the book:

   - Maribel Fernández [1]: *Models of Computation (An Introduction to Computability Theory)*, Springer-Verlag London, 2009.

Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

References

Maribel Fernández.
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Springer, Dordrecht Heidelberg London New York, 2009.

Emil L. Post.
Finite Combinatory Processes – Formulation 1.
Journal of Symbolic Logic, 1(3):103–105, 1936.
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On Computable Numbers, with an Application to the Entscheidungsproblem.
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