Lecture 1: Introduction to Computability

Models of Computation

https://clegra.github.io/moc/moc.html

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July 7, 2025

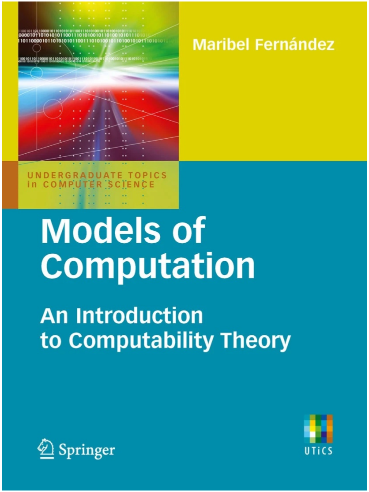
Course overview

Monday, July 7 10.30 – 12.30	Tuesday, July 8 10.30 – 12.30	Wednesday, July 9 10.30 – 12.30	Thursday, July 10 10.30 – 12.30	Friday, July 11
intro	*classic models*			*additional models*
Introduction to Computability	**Machine Models**	**Recursive Functions**	**Lambda Calculus**	
computation and decision problems, from logic to computability, overview of models of computation relevance of MoCs	Post Machines, typical features, Turing's analysis of human computers, Turing machines, basic recursion theory	primitive recursive functions, Gödel–Herbrand recursive functions, partial recursive funct's, partial recursive = = Turing-computable, Church's Thesis	λ-terms, β-reduction, λ-definable functions, partial recursive = λ-definable = Turing computable	
	imperative programming	*algebraic programming*	*functional programming*	
				14.30 – 16.30
				Three more Models of Computation
				Post's Correspondence Problem, Interaction-Nets, Fractran comparing computational power

Today

- What is computation?
  - questions where the answer may depend on computation
  - algorithm examples
  - unsolvable problems
- from logic to computability
- some models of computation
- example relevance: calculator
- fields for which models of computation are important
- recommended reading
- references

Book


Maribel Fernández
UNDERGRADUATE TOPICS
in COMPUTER SCIENCE
Models of Computation
An Introduction to Computability Theory
Springer
UTiCS

Q's where the A's depend (somehow) on computation

Q: Is $2^{20} > 1\,000\,000$?

A: Yes. (Check by computing $2^{20} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{20} = 1048576$).

Q: When was the last leap year before 1903?

A: 1896. (Use the rule: leap years are divisible by 4, but not divisible by 100 with the exception of those divisible by 400.)

Q: Given a day d in the second week of July 2025, will the sunshine percentage in L'Aquila on day d exceed $70\,\%$?

A: ??

Q: Will the rise in sea level until 2100 (worldwide yearly average) be more than 0.5 m?

A: ? A 2010 Dutch study (KNMI) projected 0.47 m.
In the meantime already ~ 1 m is being projected.

Q's where the A's depend (somehow) on computation

Q: What number comes next in the series 10, 9, 60, 90, 70, 66?

A: ?

Q: Is the diophantine equation $15x + 9y + 12 = 0$ solvable?

A: ? (. . . to be given in a moment. . .)

Q: Is $((p \to q) \to p) \to p$ a tautology of propositional calculus?

A: Yes (Peirce's law).

Q: Given a formula ϕ of propositional logic, is ϕ a tautology?

A: Yes, if the truth table for ϕ contains (in the row for ϕ) only "T"; no otherwise.

(Comput.) Yes-or-no-questions / Decision problems

Example

Tautology Problem for the propositional calculus
Instance: A formula ϕ of propositional logic.
Question: Is ϕ a tautology?

Suppose $A \subseteq E$, where E a set of finitely describable objects.

A decision method for A in E is a method by which, given an element $a \in E$, we can decide in a finite number of steps whether or not $a \in A$.

Decision problem for A in E: Find a decision method for A in E, or show that no such method can exist.

The decision problem for A in E is solvable (the set A in E is (effectively) calculable) if there exists a decision method for A in E.

(Comput.) What-questions / Computation Problems

Example

Computing the greatest common divisor
Instance: a pair $\langle a, b\rangle$ of numbers $a, b \in \mathbb{N}$ with $a, b > 0$.
Question: What is $\gcd(a, b)$, the greatest common divisor of a and b?

Suppose $F : A \to B$ is a mapping, where the elements of A, B are finitely describable objects.

A computation method for F is a method by which, given an element $a \in A$, we can obtain solution $F(a)$ in a finite number of steps.

Computation problem for F: Find a computation method for F, or show that no such method can exist.

A mapping F is calculable if there exists a computation method for F.

Representing function

Let $P(a_1, \ldots, a_n)$ be an n-ary number-theoretic predicate.

The representing function f of P:

$$f(a_1, \ldots, a_n) := \begin{cases} 1 & \ldots P(a_1, \ldots, a_n) \text{ is true} \\ 0 & \ldots P(a_1, \ldots, a_n) \text{ is false} \end{cases}$$

Hence:
A decision procedure can be handled as a computation procedure f by taking ‘0’ for ‘yes’, and ‘1’ for ‘no’.

Decision / Computation procedures (steps)

What is a computation method (procedure) more precisely, with respect to its steps?

– A mechanical, algorithmic computation procedure that:

- can be carried out by a machine M (ideal, not limited by resource problems, mechanical breakdown, etc.).
- for computing a function F on an argument a,
  - a is placed on the input device of the M,
  - which then produces $F(a)$ after finitely many steps.
- for computing a function F,
  - the machine M that is chosen for obtaining $F(a)$ may not be different for different arguments a

– Similar for a decision methods.

Solvability by an effective procedure

Q: Is the diophantine equation $15x + 9y + 12 = 0$ solvable?
(I.e. solvable for $x, y \in \mathbb{Z}$?)

From elementary number theory we know:

$$ax + by + c = 0 \text{ solvable in } \mathbb{Z} \iff \gcd(a, b) \mid c \qquad (\star)$$

Using Euclid's algorithm we calculate $\gcd(15, 9)$:

15	:	9	=	1	rem	6
9	:	6	=	1	rem	3
6	:	3	=	2	rem	0

We find: $\gcd(15, 9) = 3$.
Due to $3 \mid 12$ and ($\star$) we conclude:

A: Yes. (Infinitely many solutions, e.g. $x = 4$ and $y = -8$.)

Not effectively calculable

Examples (Shoenfield)

- methods that involve chance procedures: tossing a coin
- methods involving magic: asking a fortune teller
- methods that require (unformalised, unmechanised) insight

Effectively calculable? – No!

Example

Hilbert's 10th Problem
Instance: An equation $p(x_1, \ldots, x_n) = 0$, where
p a polynomial with integer coefficients.
Question: Is the equation solvable for $x_1, \ldots, x_n \in \mathbb{Z}$?

Instances based on quadratic polynomials are of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in \mathbb{Z}$.

Theorem (Matijasevic, 1970)

Hilbert's 10th Problem is unsolvable.

David Hilbert (1862–1943)

Problem (Entscheidungsproblem, 1928)

Is there a method for deciding, given a formula ϕ of the predicate calculus, whether or not ϕ is a tautology?

Timeline: From logic to computability

1900	Hilbert's 23 Problems in mathematics
1921	Schönfinkel: Combinatory logic
1928	Hilbert/Ackermann: formulate completeness/decision problems for the predicate calculus (the latter called 'Entscheidungsproblem')
1929	Presburger: completeness/decidability of theory of addition on $\mathbb{Z}$
1930	Gödel: completeness theorem of predicate calculus
1931	Gödel: incompleteness theorems for first-order arithmetic
1932	Church: λ-calculus
1933/34	Herbrand/Gödel: general recursive functions
1936	Church/Kleene: λ-definable $\sim$ general recursive Church Thesis: 'effectively calculable' be defined as either Church shows: the 'Entscheidungsproblem' is unsolvable
1937	Post: machine model; Church's thesis as 'working hypothesis' Turing: convincing analysis of a 'human computer' leading to the 'Turing machine'

Calculable functions?

Questions/Exercises

1. Suppose $P(a, b)$ is a calculable predicate.
   Why does $(\exists x)P(a, x)$ not have to be calculable?
2. Let $f : \mathbb{N} \to \mathbb{N}$ defined by

   $$n \longmapsto \begin{cases} 0 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is false} \\ 1 & \ldots n = 0 \;\&\; \text{Goldbach's conjecture is true} \\ n+1 & \ldots n > 0 \end{cases}$$

   Is f calculable?
3. Can computation problems for mappings $F : \mathbb{N}^n \to \mathbb{N}^m$ always be represented by decision problems?

Some Models of Computation

machine model	mathematical model	sort
Turing machine Post machine register machine	Combinatory Logic λ-calculus Herbrand–Gödel recursive functions partial-recursive/μ-recursive functions Post canonical system (tag system) Post's Correspondence Problem Markov algorithms Lindenmayer systems	*classical*
	Fractran	*less well known*
cellular automata neural networks	term rewrite systems interaction nets logic-based models of computation concurrency and process algebra ς-calculus evolutionary programming/genetic algorithms	*modern*
abstract state machines		*modern*
hypercomputation		*speculative*
quantum computing bio-computing reversible computing		*physics-/biology-inspired*

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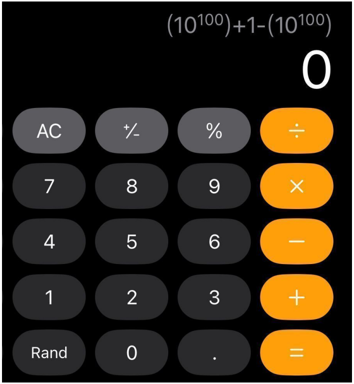
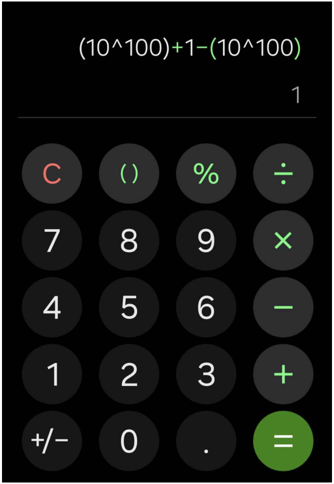
Example MoC relevance: Calculator (1/5)

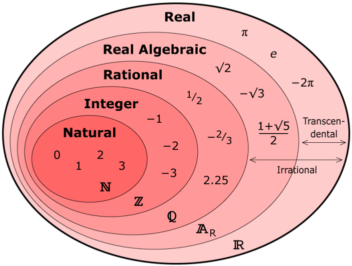


iOS

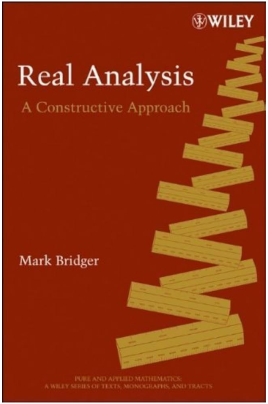


Android

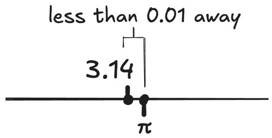
Calculator (2/5): constructive real numbers



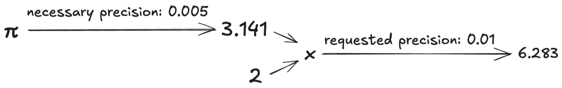
subclasses of real numbers $\mathbb{R}$



Calculator (3/5): constructive real numbers



approximating π within 0.01



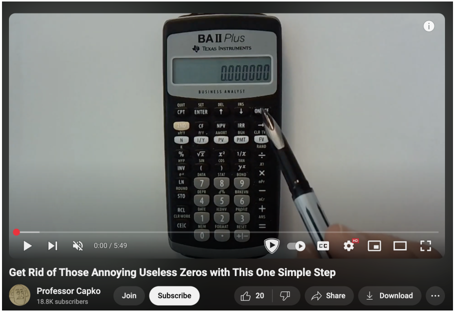
approximating 2π within 0.01

Definition

A real number $x \in \mathbb{R}$ is constructive if:

- there exists a program P_x that for every bound $0 < \delta \in \mathbb{Q}$ returns a rational approximation $P_x(\delta) \in \mathbb{Q}$ of x with $|x - P_x(\delta)| < \delta$.

Calculator (4/5): constructive real numbers



Undecidable problem

Article Talk

This article needs additional citations for verification. *(July 2019)* Learn more

In computability theory and computational complexity theory, an **undecidable problem** is a decision problem for which it is proved to be impossible to construct an algorithm that always leads to a correct yes-or-no answer. The halting problem is an example: it can be proven that there is no algorithm that correctly determines whether an arbitrary program eventually halts when run.[1]

- How to recognize that 2 constructive reals x and y are the same?
- Does there exist an program *Compare* that given P_x and P_y decides whether $x = y$?
- No! This problem is undecidable.
- Therefore $x - y = 0$ can not always be decided.

Calculator (5/5): Böhm's full precision calculator

Rational

Can only represent fractions
Exact and easy to work with

RRA

Can represent any computable real
Inexact and impossible to check equality

- Hans-Jürgen Böhm's Android full precision calculator
- uses products of:
  - full-precision rational arithmetic,
  - either of:

  (a) symbolic representations of π, e, and natural numbers, such $\sqrt{x}$, $e^x, \ln(x), \log_{10}(x), \sin(\pi x), \tan(\pi x)$ for $x \in \mathbb{Q}$.

  (b) constructive real numbers
- Equality of products with symbolic representations can be decided! (But not equality of products with at least one constructive real number)
- Credits: tech-blogger Chad Nauseam (link) for post *"A calculator app? Anyone could make that."* (link) [2].

Some fields in which MoC's are important (I)

Complexity theory

- recognize problems as being decidable
- study the computational complexity of decidable problems (classification of problems into hierarchies)

Recursion theory

- a theory of computability for sets and functions on $\mathbb{N}$ (including degrees of unsolvability of decidable problems)

Logic and Philosophy

- MoC's important for studying un-/decidability of logical theories

Rewriting

- study in a systematic way the operational and denotational aspects of MoC's like λ-calculus, CL, string rewriting, term rewriting, interaction nets

Some fields in which MoC's are important (II)

Computer Science

- e.g. functional programming: using/implementing λ-calculus

Neuro-psychology, Cognitive Modelling

- e.g. developing formal platforms for studying human cognition

Artificial Intelligence

- use knowledge of human mind to model it in an artificial system
- modeling by machines to better understand the human mind
- understand the inherent complexity of problems (un-/decidable?)

Linguistics

- e.g. formal calculi for discovering the structure of human languages related to subclasses in the Chomsky hierarchy

Recommended reading

1. Post machine: Page 1 + first paragraph on page 2 of:
   - Emil Post: *Finite Combinatory Processes – Formulation 1*, Journal of Symbolic Logic (1936), [3], https://www.wolframscience.com/prizes/tm23/images/Post.pdf.

2. Turing machine motivation: Turing's analysis of a human computer: Part I of Section 9, pp. 249–252 of:
   - Alan M. Turing's: *On computable numbers, with an application to the Entscheidungsproblem*', Proceedings of the London Mathematical Society (1936), [4], http://www.wolframscience.com/prizes/tm23/images/Turing.pdf.

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References I

- Maribel Fernández.
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- Chad Nauseam.
  A calculator app? Anyone could make that.".
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- Emil Leon Post.
  Finite Combinatory Processes – Formulation 1.
  Journal of Symbolic Logic, 1(3):103–105, 1936.
  https://www.wolframscience.com/prizes/tm23/images/Post.pdf.

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