Linear Temporal Logic (propositional)

redundant if $AP \neq \emptyset$

Syntax $\varphi ::= \text{true} \mid a \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi$ — logical connectives ($a \in AP$)

$\mid \circ\varphi \mid \varphi_1 \,\mathsf{U}\, \varphi_2$ — temporal modalities

right associative

Obs false, $\vee$, $\rightarrow$, $\leftrightarrow$, $\oplus$ obtained as usual eg

$$\varphi_1 \oplus \varphi_2 \overset{def}{=} (\varphi_1 \wedge \neg\varphi_2) \vee (\neg\varphi_1 \wedge \varphi_2)$$

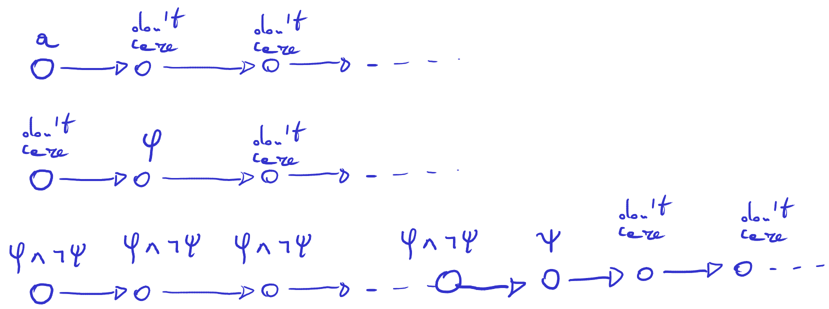
Intuitive semantics

An LTL f.la expresses a property of an infinite "path"
(i.e. the models of an LTL f.la are infinite sequences of 2^{AP} (= states))

$a \rightsquigarrow$

$\circ\varphi \rightsquigarrow$

$\varphi \,\mathsf{U}\, \psi \rightsquigarrow$



Formal Semantics

Let $\sigma \in (2^{AP})^\omega$ and $\sigma = A_0 \ldots A_i A_{i+1} \ldots$ then $\begin{cases} \sigma_{\geq i} = A_i A_{i+1} \ldots \\ \sigma[i] = A_i \end{cases}$

$\sigma \in (2^{AP})^\omega$ models $\varphi \in LTL$ if $\sigma \models \varphi$ can be derived from the following statements

$\sigma \models \text{true}$

$\sigma \models a$ iff $a \in \sigma[0]$ $\quad (\equiv \sigma[0] \models a)$

$\sigma \models \varphi \wedge \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$

$\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$

$\sigma \models \circ\varphi$ iff $\sigma_{\geq 1} \models \varphi$

$\sigma \models \varphi \,\mathsf{U}\, \psi$ iff $\exists j \geq 0 : \sigma_{\geq j} \models \psi$ and $\forall\, 0 \leq i < j : \sigma[i] \models \varphi$

$$\text{Words}(\varphi) = \{\sigma \in (2^{AP})^\omega \mid \sigma \models \varphi\}$$

Some important derived modalities

"eventually" $\Diamond$

$\Diamond \varphi \equiv \text{true}\ U\ \varphi$

$\equiv \neg \Box \neg \varphi$

"always" $\Box$

$\Box \varphi \equiv \varphi\ U\ \text{false}$

$\equiv \neg \Diamond \neg \varphi$

Exercise Define "infinitely often". A: $\Box \Diamond \varphi$

"eventually forever" $\Diamond \Box \varphi$

Exercise Which of the following equivalences are correct:

a) $\Box(\varphi \to \Diamond \psi) \equiv \varphi\ U\ (\varphi \wedge \neg \varphi)$

b) $\circ \Diamond \varphi \equiv \Diamond \circ \varphi$

c) $\Box(\varphi \wedge \circ \Diamond \varphi) \equiv \Box \varphi$

d) $\Diamond(\varphi \wedge \psi) \equiv \Diamond \varphi \wedge \Diamond \psi$

e) $\Box(\varphi \wedge \psi) \equiv \Box \varphi \wedge \Box \psi$

f) $\Box \Box (\psi \to \varphi) \equiv \neg \Diamond (\neg \varphi \wedge \psi)$

Exercise Give an LTL f.la expressing safety & liveness of the mutual exclusion problem

Formal semantics: Interpretation over transition systems

Recall:

$$\text{Words}(\varphi) = \{ \sigma \in (2^{AP})^\omega \mid \sigma \models \varphi \}$$

We now want to interpret LTL f.lae over transition systems. An obvious way is to first interpret LTL over paths and states.

π infinite path fragment of TS

$$\pi \models \varphi \iff \text{trace}(\pi) \models \varphi$$
$$\iff \text{trace}(\pi) \in \text{Words}(\varphi)$$

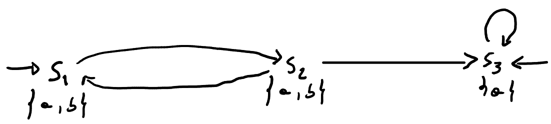
Hence we can define

$$s \models \varphi \iff \forall \pi \in \text{Path}(s) : \pi \models \varphi \qquad \text{where } s \in S$$

And finally

$$TS \models \varphi \iff \forall s \in I : s \models \varphi$$
$$\iff TS \models \text{Words}(\varphi)$$

Exercise Let $AP = \{a, b\}$ and TS =



a) Is TS deterministic?
b) $s_1 \models \circ (a \wedge b)$?
c) $s_2 \models \circ (a \wedge b)$?
d) $TS \models \Box a$?
e) $TS \models \Box(\neg b \rightarrow a)$?

A note on negation

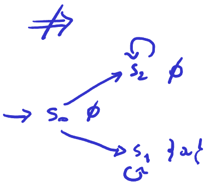
$$\text{Words}(\neg\varphi) = (2^{AP})^\omega \setminus \text{Words}(\varphi) \qquad \text{hence} \qquad \pi \models \varphi \iff \pi \not\models \neg\varphi$$

However negation is weird

Exercise Show that $TS \not\models \varphi \;\not\Leftrightarrow\; TS \models \neg\varphi$

$\Leftarrow$ holds, but $\not\Rightarrow$

Take TS =



then $TS \not\models \Diamond a$ because of $s_0 s_2^\omega$
&
$TS \not\models \neg\Diamond a$ because of $s_0 s_1^\omega$