Lecture 3 — 29-01-2026

$TS = \langle S, Act, \rightarrow, I, AP, L \rangle$ transition system

$G(TS) = \langle S, E \rangle$ state graph of TS

$E ::= \{ \langle s, s' \rangle \in S \times S \mid \underbrace{s \xrightarrow{a} s' \text{ for some } a \in Act}_{s' \in Post(s)} \}$

actions needed
- for communication (handshaking)
- for defining some notions of fairness

path-fragments
paths
traces

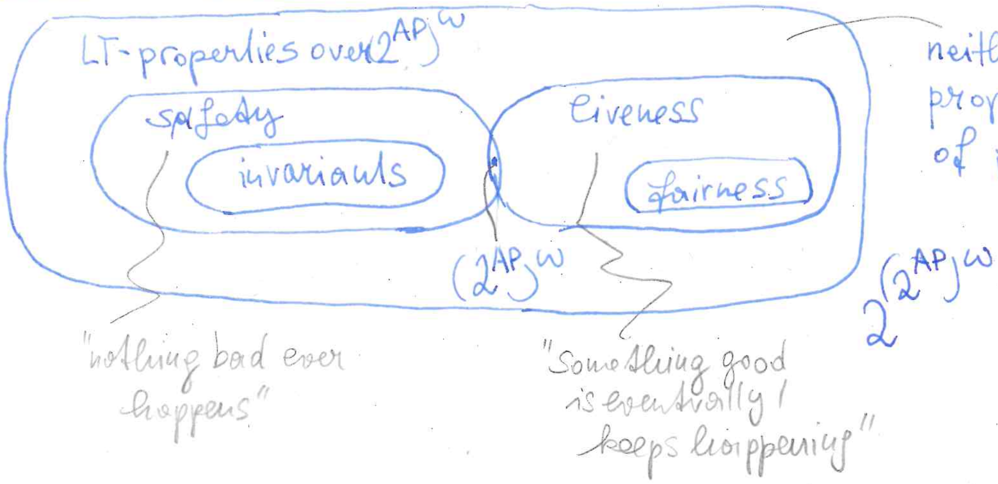
Linear-time properties (LT-property): $P \subseteq (2^{AP})^{\omega}$ i.e. $P \in 2^{(2^{AP})^{\omega}}$

transition system satisfies P

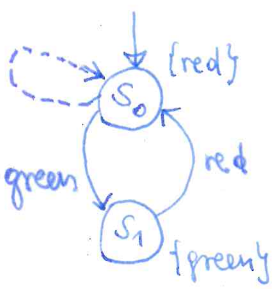
$TS \models P \iff Traces(TS) \subseteq P.$

state s satisfies P

$s \models P \iff Traces(s) \models P$



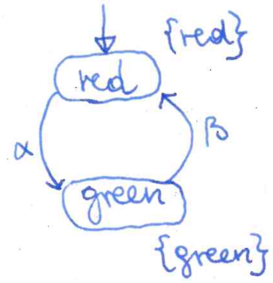
neither fairness nor liveness properties, but intersections of fairness and liveness properties.



$\boxed{TS}$

$AP = \{red, green\}$

execution fragment	: s_0 green s_1 red s_0
execution	: s_0 green s_1 red s_0 green ...
path fragment	: $s_0 s_1 s_0$, $s_1 s_0$
path	: $s_0 s_1 s_0 s_1 \ldots$
trace	: $\{red\}\{green\}\{red\}\{green\}\ldots$



$$TrLight = \langle \overbrace{\{red, green\}}^{S}, \overbrace{\{\alpha, \beta\}}^{Act}, \overbrace{\longrightarrow}^{Trans.rel.}, \overbrace{\{red\}}^{I}, \overbrace{\{green, red\}}^{AP}, L \rangle$$

labeling function $L(red) = \{red\}$, $L(green) = \{green\}$

execution fragment: green β red (not initial)
red α green (initial)

execution: red α green β red α green ...

path fragm: green red
red green

path: red green red green ...
trace: $\{red\}\{green\}\{red\}\{green\}\ldots =: t_0$

LT-Properties:

e.g. P_1: infinitely often green — liveness property

$$P_1 = \{A_0A_1A_2\ldots \in (2^{AP})^\omega \mid \overset{\infty}{\exists} i \geq 0 \ green \in A_i\}$$

$\ni t_0$

$\ni \{green\}\{green\}\ldots$

$\ni \{red\}\{green, red\}\{green, red\}\ldots$

P_2: starts with red

$$P_2 = \{\{red\}A_1A_2A_3\ldots \in (2^{AP})^\omega \mid A_i \in 2^{AP}\}$$ safety property

$\ni t_0$

$\ni \{red\}\emptyset\emptyset\emptyset\ldots$

LINEAR-TIME BEHAVIOUR & PROPERTIES

$TS = \langle S, Act, \rightarrow, I, AP, L\rangle$ transition system — W.l.o.g. no terminal states!

$G(TS) = \langle V, E\rangle$ with $V := S$ and $E := \{\langle s,s'\rangle \in S\times S \mid \underbrace{s \xrightarrow{a} s' \text{ for some } a\in Act}_{s'\in Post(s)}\}$

STATE GRAPH of TS

A PATH FRAGMENT is a (finite or infinite) state sequence ($\pi = s_0 s_1 s_2 \ldots$) on its state graph:

$\pi \in S^* \cup S^\omega$ such that $\forall 0 \le i \le |\pi| : \pi[i+1] \in Post(\pi[i])$.

NOTATION: let $\pi = s_0 s_1 s_2 \ldots$. $\pi[j] := s_j$ $\quad \pi[..j] := s_0 s_1 \ldots s_j$

$first(\pi) := s_0$, $\quad \pi[j..] := s_j s_{j+1} \ldots$

$\boxed{Paths_{fin}(TS)}$ if π is finite, $\pi = s_0 s_1 \ldots s_n$, then: $Last(\pi) := s_n$, $len(\pi) := n$

$\boxed{Paths(TS)}$ if π is infinite, $\pi = s_0 s_1 \ldots$, then: $Last(\pi)\uparrow$, $len(\pi) := \omega$.

π is maximal if $\underbrace{\pi \in S^* \,\&\, Post(Last(\pi)) = \emptyset}_{\pi \text{ is finite and ends in a terminal state}}$ or $\underbrace{\pi \in S^\omega}_{\pi \text{ is infinite}}$.

π is initial if $\pi[0] \in I$

π is a path if π is initial & maximal.

TRACE of π (path fragment) is $\{L(\pi[i])\}_{0\le i<|\pi|} =: trace(\pi) \subseteq (2^{AP})^{|\pi|}$

$traces(\Pi) := \{trace(\pi) \mid \pi \in \Pi\}$ for every set Π of path fragments

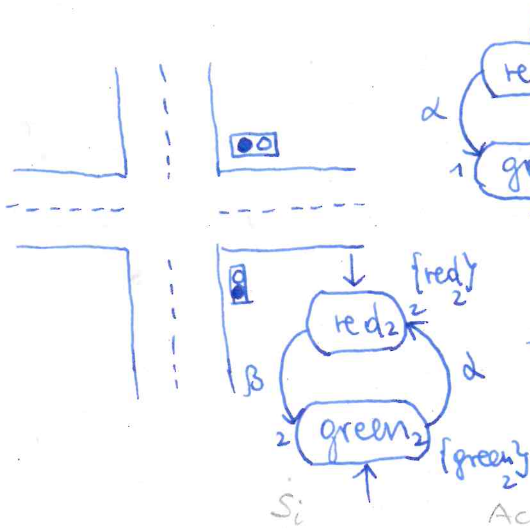
$Traces(s) := traces(Paths(s))$ for every $s \in S$
where $Paths(s)$ are all maximal path fragments from s in TS

$Traces(TS) := \bigcup_{s\in I} Traces(s)$

A LINEAR-TIME PROPERTY P over set AP of atomic propositions is a subset of $(2^{AP})^\omega$ i.e. $P \subseteq (2^{AP})^\omega$ [i.e. an infinite sequence of subsets of prop's]

Transition system TS satisfies $P \subseteq (2^{AP})^\omega$ if $Traces(TS) \subseteq P$.
NOTATION $TS \models P$

State $s \in S$ satisfies $P \subseteq (2^{AP})^\omega$ if $Traces(s) \subseteq P$.
NOTATION $s \models P$



$TrLight_1$

$TrLight_2$

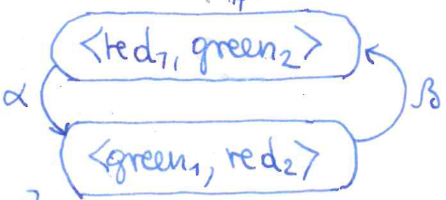
$$\frac{x \xrightarrow{a}_1 x' \quad y \xrightarrow{a}_2 y'}{x \parallel_H y \xrightarrow{a} x' \parallel_H y'} \text{ if } a \in H$$

$\boxed{\text{handshaking communication}}$

We consider

$TrLight_1 \parallel_H TrLight_2$

$$red_1 \parallel_H green_2 \xrightarrow{\alpha} green_1 \parallel_H red_2 \xrightarrow{\beta} red_1 \parallel_H green_2$$



$$TrLight_i = \langle \overbrace{\{red_i, green_i\}}^{S_i}, \overbrace{\{\alpha, \beta\}}^{Act}, \rightarrow_i, \underbrace{\{red_i, green_i\}}_{I_i}, \underbrace{\{red_i, green_i\}}_{AP_i}, L_i \rangle$$

$L_i(red_i) := \{red_i\}$
$L_i(green_i) := \{green_i\}$

$$TrLights = TrLight_1 \parallel_H TrLight_2 = \langle S_1 \times S_2, Act, \rightarrow, I_1 \cup I_2, AP_1 \cup AP_2, L \rangle$$

$H := \{\alpha, \beta\}$

$L(\langle s_1, s_2 \rangle) = L(s_1) \cup L(s_2)$

Path fragments:

in $TrLight_1$: $red_1\, green_1\, red_1$ — initial, not maximal / path; $green_1\, red_1\, green_1\, red_1 \dots$ — maximal / initial / path

in $TrLight_1 \parallel_H TrLight_2$:

$\langle red_1, red_2 \rangle$ initial / maximal / path

$\langle green_1, red_2 \rangle \langle red_1, green_2 \rangle$ initial / not maximal / path

$\langle red_1, green_2 \rangle \langle green_1, red_2 \rangle \langle red_1, green_2 \rangle \dots$ initial / maximal / path.

Traces:

in $TrLight_1$: $\{red_1\}\{green_1\}, \{red_1\}$ trace of a not maximal path fragment

LT-properties: P: the first traffic light is green infinitely often

$\{red_1, green_2\}\{green_1, red_2\}\{red_1, green_2\} \dots \in P$

$\emptyset\ \{green_1\}\ \emptyset\ \{green_1\}\ \emptyset\ \{green_1\} \dots \in P$

$\{green_1, green_2\}\{green_1, green_2\} \dots \in P$

The importance of traces

$\langle S, Act, \rightarrow, I, AP, L \rangle$ $(\rightarrow \subseteq S \times Act \times S,\ I \subseteq S,\ L: S \rightarrow 2^{AP})$

WLOG: no terminal states in TS (hence all maximal path fragments are infinite)

The trace of a maximal path fragment π of TS is $trace(\pi) := \{L(\pi[i])\}_{i \geq 0} \in 2^{AP}$.

$$TS \models P \quad :\Leftrightarrow \quad Traces(TS) \subseteq P, \quad \text{for all } P \subseteq (2^{AP})^{\omega}$$

$$s \models P \quad :\Leftrightarrow \quad Traces(s) \subseteq P, \quad \text{" ———— "}$$

TS_1, TS_2 two transition systems over the same set AP of atomic propositions

Then: $\underset{\text{refinement}}{Traces(TS_1)} \subseteq \underset{\text{abstract model}}{Traces(TS_2)}$ could mean: "TS_1 implements TS_2 (correctly)".

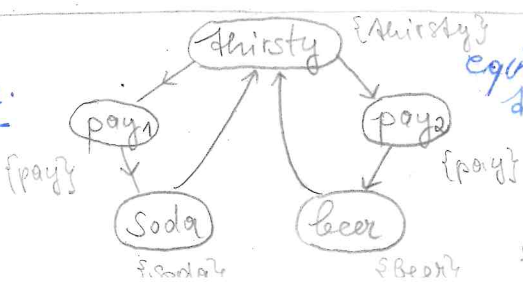
Theorem. $Traces(TS_1) \subseteq Traces(TS_2) \Leftrightarrow \forall$ LT-properties P over AP $[TS_2 \models P \Rightarrow TS_1 \models P]$

Proof. $(\Rightarrow)$ Suppose $Traces(TS_1) \subseteq Traces(TS_2)$ (assm).
Let $P \subseteq (2^{AP})^{\omega}$ be an LT-property over AP.
Then: $TS_2 \models P \underset{\text{by def.}}{\Longrightarrow} Traces(TS_2) \subseteq P$
$\underset{\text{by assm}}{\Longrightarrow} Traces(TS_1) \subseteq Traces(TS_2) \subseteq P$
$\underset{\text{by def.}}{\Longrightarrow} TS_1 \models P$

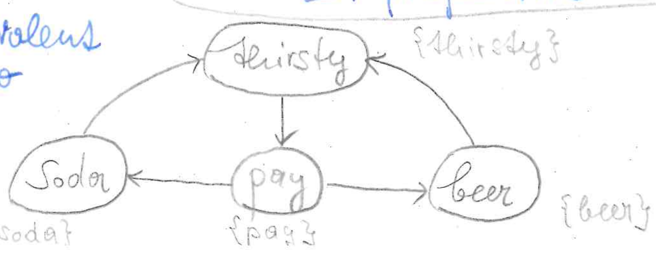
$(\Leftarrow)$ Suppose $TS_2 \models P \Rightarrow TS_1 \models P$ holds for all LT-properties over AP. Then it also holds for $P := Traces(TS_2)$. As $TS_2 \models Traces(TS_2)$ holds obviously (as it means $Traces(TS_2) \subseteq Traces(TS_2)$), we conclude $TS_1 \models Traces(TS_2)$, which means: $Traces(TS_1) \subseteq Traces(TS_2)$.

Corollary. $\underbrace{Traces(TS_1) = Traces(TS_2)}_{\text{trace-equivalence of } TS_1, TS_2} \Leftrightarrow \underbrace{\forall \text{ LT-properties over AP } [TS_1 \models P \Leftrightarrow TS_2 \models P]}_{TS_1, TS_2 \text{ fulfill the same LT properties}}$.

Example.



equivalent to



Taxonomy of LT-properties

Invariant $\subsetneq$ Safety "nothing bad ever happens" Liveness "something good is eventually / keeps happening"

An LT-property P_{inv} over AP (i.e. $P_{inv} \subseteq 2^{AP}$) is an invariant:

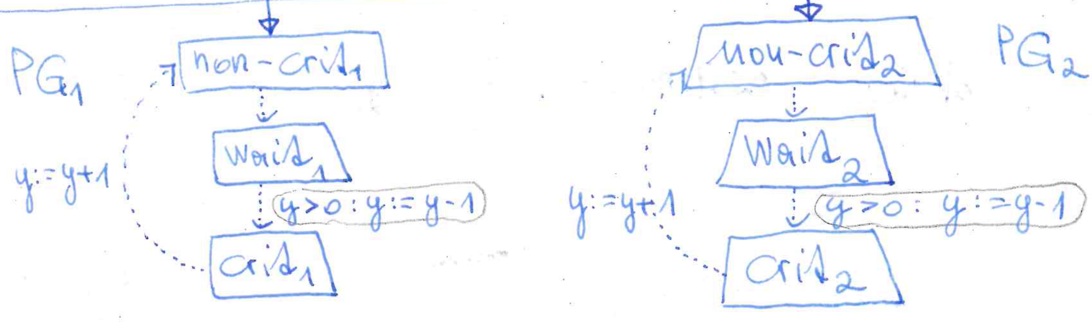
$:\Leftrightarrow$ $P_{inv} = \{A_0 A_1 A_2 \dots \in 2^{AP} \mid A_i \models \Phi\}$ for some formula Φ of propositional calculus over AP.

Example: $AP = \{a,b,c\}$, $\Phi = a \vee b$
$P_1 := \{A_0 A_1 A_2 \dots \in (2^{AP})^\omega \mid A_i \models \Phi \text{ for all } i \geq 0\}$
$= \{A_0 A_1 A_2 \dots \in (2^{AP})^\omega \mid (a \in A_i \text{ or } b \in A_i) \text{ for all } i \geq 0\}$

For such a property P_{inv} given by a prop. formula Φ:

$$
\begin{aligned}
TS \models P_{inv} &\Leftrightarrow \text{Traces}(TS) \subseteq P_{inv} \\
&\Leftrightarrow \forall \pi \text{ path of } G(TS): \text{trace}(\pi) \in P_{inv} \\
&\Leftrightarrow \forall s \text{ state of } TS \text{ on a path } \pi \text{ of } TS: \\
&\qquad s \models \Phi \\
&\Leftrightarrow \forall s \text{ reachable state of } TS: s \models \Phi.
\end{aligned}
$$

Thus invariants are state-properties that can be decided for a transition system by checking them in every reachable state.



Another invariant example: Deadlock-freedom in the Dining Philosophers' Problem: The state in which every philosopher waits for the second stick should not occur.

Over $AP = \{crit_i, wait_i, non\text{-}crit_i \mid i \in \{1,2\}\}$ the desired property Φ of a mutual exclusion algorithm can be expressed by the invariant:

$$\neg crit_1 \vee \neg crit_2$$

$$\equiv \neg(crit_1 \wedge crit_2)$$

Safety

Safety properties impose conditions on <u>finite path fragments</u> of executions

e.g. "before withdrawing money, a correct PIN is entered" (*)
at an ATM $BP = (\neg p^+ w)^* w(true)^*$

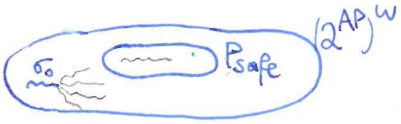
<u>Intuition:</u> an infinite execution violating (*) has a finite prefix that already violates (*)

P_{safe} is a <u>safety property</u> $:\Leftrightarrow$ $\exists \underbrace{BP}_{\text{bad prefixes}} \subseteq (2^{AP})^*.\ P_{safe} = (2^{AP})^\omega \setminus \underbrace{(BP.(2^{AP})^\omega)}_{\text{bad traces have bad prefixes}}$

$\Leftrightarrow$ (book) $\forall \sigma \in (2^{AP})^\omega \setminus P_{safe}.\ \exists n \geq 0.\ \left[(\sigma_{<n}.(2^{AP})^\omega) \cap P_{safe} = \phi \right]$.

(every "unsafe" trace has a bad prefix)

$BP(P_{safe}) = \{ \sigma_0 \in (2^{AP})^* / \ \sigma_0.(2^{AP})^\omega \cap P_{safe} = \phi \}$



<u>Lemma.</u> $TS \models P_{safe} \Leftrightarrow Traces_{fin}(TS) \cap BP(P_{safe}) = \phi$

Proof. $(\Rightarrow)$ If $\sigma_0 \in Traces_{fin}(TS) \cap BP(P_{safe})$
we proceed indirectly
$\Rightarrow \exists \sigma \in Traces(TS)\ \exists n\ (\sigma_{<n} = \sigma_0 \wedge \underbrace{\sigma_0.(2^{AP})^\omega}_{\sigma\in} \cap P_{safe} = \phi)$

$\Rightarrow \exists \sigma \in Traces(TS).\ \sigma \notin P_{safe}$

$\Rightarrow Traces(TS) \not\subseteq P_{safe}$

$\Rightarrow TS \not\models P_{safe}$

$(\Leftarrow)$ If $TS \not\models P_{safe}$ $\Rightarrow Traces(TS) \not\subseteq P_{safe}$
we proceed indirectly
$\Rightarrow \exists \sigma \in Traces(TS)\ \sigma \notin P_{safe}$

$\Rightarrow \exists \sigma \in Traces(TS)\ \exists n \geq 0.\ \sigma_{<n} \in BP(P_{safe})$

$\Rightarrow \exists \sigma \in Traces(TS)\ \exists n\ \sigma_{<n} \in Traces_{fin}(TS) \cap BP(P_{safe})$

$\Rightarrow Traces_{fin}(TS) \cap BP(P_{safe}) \neq \phi$.

<u>Thm</u> $Traces_{fin}(TS_1) \subseteq Traces_{fin}(TS_2) \Leftrightarrow \forall$ safety prop. $P\ (TS_2 \models P \Rightarrow TS_1 \models P)$

Proof. $(\Rightarrow)$ Let P be a safety property, and (hyp) $Traces_{fin}(TS_1) \subseteq Traces_{fin}(TS_2)$
Then: $TS_2 \models P \overset{(L)}{\Leftrightarrow} Traces_{fin}(TS_2) \cap BP(P_{safe}) = \phi$
$\Rightarrow Traces_{fin}(TS_1) \cap BP(P_{safe}) = \phi$
$\overset{(L)}{\Leftrightarrow} TS_1 \models P$.

Thm (%) $Traces_{fin}(TS_1) \subseteq Traces_{fin}(TS_2) \iff$
$\iff \forall P$ safety property: $TS_2 \models P \Rightarrow TS_1 \models P$

($\Leftarrow$) Lemma. (i) P is safety property $\iff P = \underbrace{closure(P)}$ $=: \{\sigma \in (2^{AP})^\omega \mid pref(\sigma) \subseteq pref(P)\}$

(ii) $closure(closure(P)) = closure(P)$ for all $P \subseteq (2^{AP})^\omega$

For showing "$\Leftarrow$", we assume that

(hyp) $(TS_2 \models P \Rightarrow TS_1 \models P)$ holds for all safety properties P.

We let $P := closure(Traces_{fin}(TS_2))$. Then P is a safety property by the Lemma. Also $TS_2 \models P$ because:
$Traces(TS_2) \subseteq closure(Traces_{fin}(TS_2)) = P$.

Then $TS_1 \models P$ by (hyp), and therefore $Traces(TS_1) \subseteq P$.

Now we conclude:

$$\begin{aligned} Traces_{fin}(TS_1) &= pref(Traces(TS_1)) \\ &\subseteq pref(P) \\ &= pref(closure(Traces_{fin}(TS_2))) \\ &= Traces_{fin}(TS_2). \end{aligned}$$

Corollary. $Traces_{fin}(TS_1) = Traces_{fin}(TS_2) \iff$
$\iff \forall P$ safety property: $TS_2 \models P \Rightarrow TS_1 \models P$.

Prop. Every invariant is a safety property.

$P = \{A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid A_i \models \Phi \text{ for all } i \geq 0\}$ for some propositional formula Φ over AP

$= (2^{AP})^\omega \setminus (mBP . (2^{AP})^\omega)$

where $\underbrace{mBP}_{\text{minimal bad prefixes}} = \{A_0 A_1 A_2 \ldots A_{n-1} A_n \mid A_i \models \Phi \text{ for all } i \in \{0,1,\ldots,n-1\},\ A_n \not\models \Phi, \text{ where } n \geq 0\}$

Example. vending machine gives 3 soda initially $AP = \{beer, soda\}$

$P_{3\text{-soda}} = \{\{soda\}\{soda\}\{soda\} A_3 A_4 \ldots \mid A_3 A_4 \ldots \subseteq AP\}$

is a safety property

$$BP = (2^{AP} \setminus \{soda\})(2^{AP})^* + \{soda\} . (2^{AP} \setminus \{soda\})(2^{AP})^* + \{soda\}\{soda\}(2^{AP} \setminus \{soda\})(2^{AP})^*$$

$$mBP = (2^{AP} \setminus \{soda\}) + \{soda\} . (2^{AP} \setminus \{soda\}) + \{soda\}\{soda\}(2^{AP} \setminus \{soda\})$$

AP = {red, green, yellow}   traffic light

P_1: "at least one light is always on"

$$P_1 = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid \underbrace{|A_i| \geq 1}_{\Leftrightarrow A_i \neq \emptyset} \text{ for all } i \in \mathbb{N} \}$$

$$BP = \{ A_0 \ldots A_n \in (2^{AP})^* \mid n \geq 0,\ A_i = \emptyset \text{ for some } i \in \{0, \ldots, n\} \}$$

$$\text{min-}BP = \{ A_0 \ldots A_n \in (2^{AP})^* \mid n \geq 0,\ A_0 = \ldots = A_{n-1} \neq \emptyset,\ A_n = \emptyset \}$$

P_2: "it is never the case that 2 lights are switched on at the same time"

$$P_2 = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid |A_i| \leq 1 \text{ for all } i \in \mathbb{N} \}$$

$$BP = \{ A_0 \ldots A_n \in (2^{AP})^* \mid n \geq 0,\ |A_i| \geq 2 \text{ for } i \in \{0 \ldots n\} \}$$

$$\text{min-}BP = \{ A_0 \ldots A_n \in (2^{AP})^* \mid n \geq 0,\ |A_0|, \ldots, |A_{n-1}| \leq 1,\ |A_n| \geq 2 \}$$

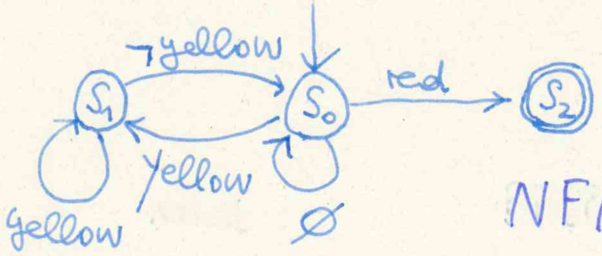
P_3: "a red phase must immediately be preceded by a yellow phase"

$$BP(P_3) = \{ A_0 A_1 \ldots A_n \in (2^{AP})^* \mid \exists\, 0 \leq i \leq n.\ A_i \ni red \wedge (i = 0 \vee i > 0 \wedge yellow \notin A_{i-1}) \}$$

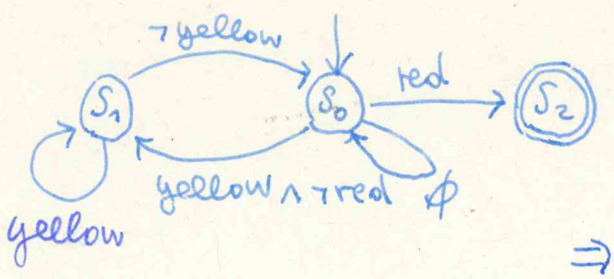
$$P_3 = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid \forall i \geq 0.\ (red \in A_i \Rightarrow i > 0 \wedge A_{i-1} \ni yellow) \}$$

{red}, $\emptyset\emptyset$ {red}, $\emptyset$ {red} $\in BP(P_3)$

{yellow} {yellow} {red} {red} $\emptyset$ {red} $\in BP(P_3)$
$\notin mBP(P_3)$
not a minimal bad prefix



NFA for $BP(P_3)$



DFA for $mBP(P_3)$

$\Rightarrow$ P_3 is a regular safety property

Beverage vending machine: "the house never Looses"

P: "The number of inserted coins is always at least the number of dispensed drinks" AP := {pay, drink}

$$P = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, \text{forall } i \geq 0: |\{0 \leq j \leq i \,/\, pay \in A_j\}| \geq |\{0 \leq j \leq i \,/\, drink \in A_j\}| \}$$

bad prefixes:
$$\begin{cases} \emptyset \{pay\} \{drink\} \{drink\} \notin P \\ \emptyset \{pay\} \{drink\} \emptyset \{pay\} \{drink\} \{drink\} \notin P \end{cases}$$

is not a regular safety property
because bad prefixes cannot be recognized
by a finite-state automaton
(it is, however, a context-free safety property,
because bad prefixes can be recognized
by a push-down automaton)

Safety constrains finite behaviour while liveness constrains infinite behaviour

$\boxed{\text{Liveness}}$ P_{live} $\forall w \in (2^{AP})^* \ \exists \sigma \in (2^{AP})^{\omega} : \ w\sigma \in P_{live}$

"something good happens (always) eventually"

$$\Updownarrow$$

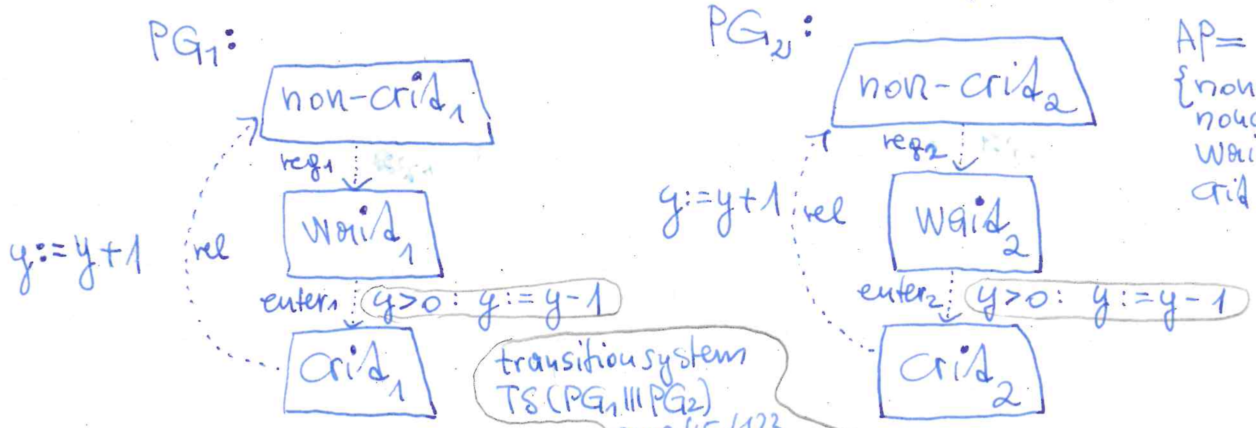
$$pref(P_{live}) = (2^{AP})^*$$

$$\Updownarrow$$

$$(2^{AP})^* . P_{live} = P_{live}$$

Exercise. Semaphore-based mutual exclusion.

$AP = \{crit_i, non\text{-}crit_i, wait_i \ / \ i \in \{1,2\}\}$



$AP = \{noncrit_1, noncrit_2, wait_1, wait_2, crit_1, crit_2\}$

transition system $TS(PG_1 ||| PG_2)$ on p. 45/133 in the book!

Typical liveness conditions

- (eventually) each process will eventually enter its critical section

  $P_1 = \{A_0A_1A_2 \ldots \in (2^{AP})^{\omega} \ / \ (\exists j \geq 0.\ crit_1 \in A_j) \wedge (\exists j \geq 0.\ crit_2 \in A_j)\}$

- (repeated eventually) each process will enter its critical section infinitely often

  $P_2 = \{A_0A_1A_2 \ldots \in (2^{AP})^{\omega} \ / \ \forall k \geq 0 \ \exists j \geq k \ (crit_1 \in A_j) \wedge \underbrace{\forall k \geq 0 \ \exists j \geq k \ (crit_2 \in A_j)}_{\overset{\infty}{\exists} j \geq 0.\ crit_2 \in A_j}\}$

- (starvation freedom) each waiting process will eventually enter its critical section

  $P_3 = \{A_0A_1A_2 \ldots \in (2^{AP})^{\omega} \ / \ \forall k \geq 0 \ (wait_1 \in A_k \Rightarrow \exists j > k.\ crit_1 \in A_j) \wedge \forall k \geq 0 \ (wait_2 \in A_k \Rightarrow \exists j > k.\ crit_2 \in A_j)\}$

if we would not have PG_1, PG_2 but processes that could stop waiting without going in their critical sections, then a stronger formulation could be:

$P_3' = \{A_0A_1A_2 \ldots \in (2^{AP})^{\omega} \ / \ \neg \exists k \geq 0 \ \forall j \geq k \ (wait_1 \in A_{j+k} \wedge crit_1 \notin A_{j+k}) \wedge \neg \exists k \geq 0 \ \forall j \geq k \ (wait_2 \in A_{j+k} \wedge crit_2 \notin A_{j+k})\}$

Safety versus Liveness Properties

- Are safety and liveness properties disjoint? (No)
- Is any LT-property a safety or liveness property? (No)

<u>Lemma.</u> The single LT-property over AP that is <u>both</u> a safety and a liveness property is $(2^{AP})^\omega$.

Proof. Let P be a liveness property over AP. Then $\text{pref}(P) = (2^{AP})^*$. It follows that $\text{closure}(P) = (2^{AP})^\omega$. If P is a safety property, too, then $P = (2^{AP})^\omega$.

<u>Example.</u> "vending machine provides beer infinitely often after initially providing soda three times in a row." $\Big\}$ P

$$P = P_{3\text{-soda}} \cap P_{\infty\text{-beer}}$$

$AP = \{\text{soda}, \text{beer}\}$

$P_{3\text{-soda}} := \{\{\text{soda}\}\{\text{soda}\}\{\text{soda}\} A_3 A_4 \ldots \mid A_j \subseteq AP \text{ for } j \geq 3\}$ safety property

$P_{\infty\text{-beer}} := \{A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \mid \overset{\infty}{\exists} j \geq 0.\ A_j = \{\text{beer}\}\}$ liveness property

$BP = (2^{AP} \setminus \{\text{soda}\})(2^{AP})^*$
$+ \{\text{soda}\}(2^{AP} \setminus \{\text{soda}\})^*$
$+ \{\text{soda}\}\{\text{soda}\}(2^{AP} \setminus \{\text{soda}\})^*$

<u>Theorem.</u> (Decomposition) For every LT-property P over AP there exists a safety property P_{safe} and a liveness property P_{live} such that $P = P_{safe} \cap P_{live}$.

Namely: $P_{safe} := \text{closure}(P)$,
$P_{live} := P \cup ((2^{AP})^\omega \setminus \text{closure}(P))$

Topological characterization:

metric d on $(2^{AP})^\omega$: $d(\sigma_1, \sigma_2) = \begin{cases} 0 & \ldots\ \sigma_1 = \sigma_2 \\ \frac{1}{2^n} & \ldots\ \sigma_1 \neq \sigma_2 \text{ and } n \text{ is the shortest common prefix of } \sigma_1 \text{ and } \sigma_2 \end{cases}$

induces topology $\mathcal{T}_d$

in $\langle (2^{AP})^\omega, \mathcal{T}_d \rangle$: closed sets $\sim$ safety properties
dense sets $\sim$ liveness properties
closure (P) $\sim$ topological closure of P

The decomposition theorem then follows from:

<u>Proposition.</u> $\langle X, \mathcal{T} \rangle$ topological space. For all sets $A \subseteq X$ there exists a dense set $D \subseteq X$ such that $A = \overline{A} \cap D$.

Proof. Let $D := A \cup (X \setminus \overline{A})$.

Fairness

Usually liveness properties cannot be guaranteed without some assumptions about fairness.

Process fairness: A server S for processes $P_1, \dots, P_N$ should answer any continuous request eventually.

Starvation freedom: e.g. mutual exclusion algorithms
"Once access is requested by a process, it is not kept waiting forever."
"Each process is infinitely often in its critical section."

path $\pi = s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3 \longrightarrow \dots$

$A \subseteq Act$

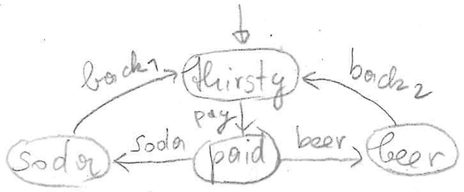
An execution fragment $\rho = s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} s_3 \xrightarrow{\alpha_4} \dots$ is

taken$(s_j) \in$ taken(A)

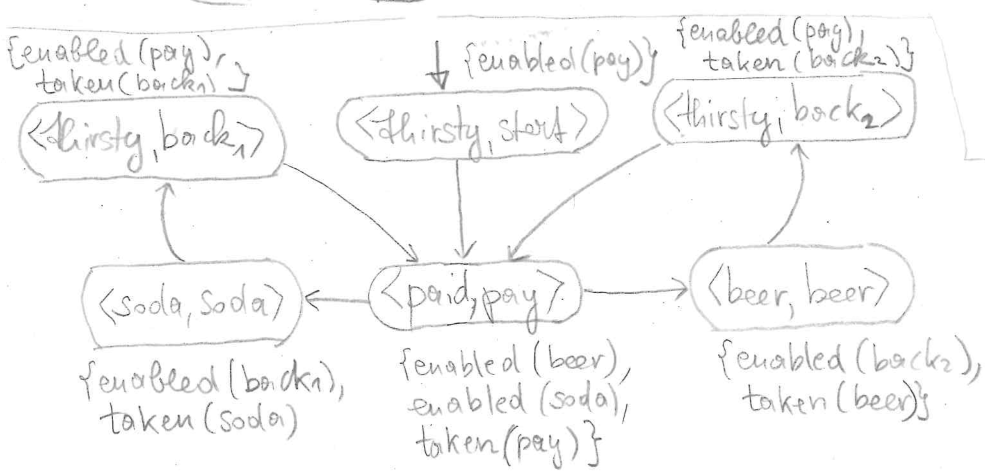
- unconditionally A-fair (impartiality): $\overset{\infty}{\exists} j \geq 0 : \alpha_j \in A$
- strongly A-fair (compassion): $\overset{\infty}{\exists} j \geq 0 : A \cap Act(s_j) \neq \emptyset \Rightarrow \overset{\infty}{\exists} j \geq 0 : \alpha_j \in A$
  enabled$(A) \cap$ enabled$(s_j) \neq \emptyset$ taken$(s_j) \in$ taken(A)
- weakly A-fair (justice): $\overset{\infty}{\forall} j \geq 0 : A \cap Act(s_j) \neq \emptyset \Rightarrow \overset{\infty}{\exists} j \geq 0 : \alpha_j \in A$
  enabled$(A) \cap$ enabled$(s_j) \neq \emptyset$ taken$(s_j) \in$ taken(A)

Checking liveness properties is often done by restricting to fair executions:

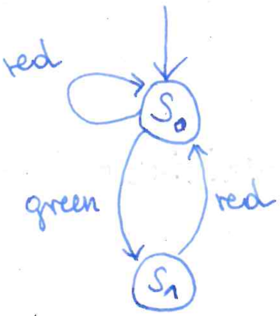
$$TS \models_{fair} P \iff FairTraces(TS) \subseteq P$$



th pay beer b th pay beer ...
(soda, soda)
is not unconditionally {soda}-fair
is not strongly {soda}-fair
is weakly {soda}-fair.

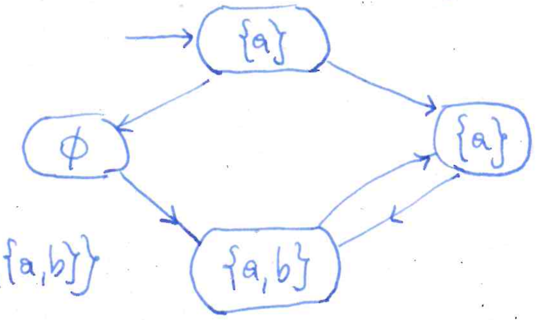


Exercise 3.1,
Exercise 3.5,
Exercise 3.6,



execution
s_0 red s_0 red s_0 red s_0 ...
is <u>not</u> weakly {green}-fair.

Exercise 3.1 Give the traces on the set of atomic propositions $\{a,b\}$ of the following transition system:



$AP = \{a, b\}$

$2^{AP} = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$\{a\}\ \phi\ \{a,b\}\ \{a\}\ \{a,b\}\ \{a\} \dots$

$\{a\}\ \{a\}\ \{a,b\}\ \{a\}\ \{a,b\}\ \{a\} \dots$

Exercise 3.5 $AP = \{x=0, x>1\}$

Formulate as LT-properties and determine whether they are invariant / safety / liveness properties:

(a) false $\quad P_1 = \{\} = \phi = \{A_0A_1A_2 \dots \in (2^{AP})^\omega \,/\, A_i \models false\}$

invariant, safety (bad prefixes: $\{x=0\}, \{x>1\}, \phi, \{x=0, x>1\}$) $= 2^{AP}$

~~liveness~~

(b) initially x is equal to zero

$P_2 = \{\{x=0\}A_1A_2A_3 \dots \in 2^{AP} \,/\, A_i \subseteq AP \text{ for all } i \geq 1\}$

~~invariance~~, safety (bad prefix: $\{x>1\}$), ~~liveness~~

(c) initially x differs from zero

$P_3 = \{\{x>1\}A_1A_2A_3 \dots \in 2^{AP} \,/\, A_i \subseteq AP \text{ for all } i \geq 1\}$

~~invariance~~, safety (bad prefix: $\{x=0\}$), ~~liveness~~

(d) initially x is equal to zero, but at some point exceeds one

$P_4 = \{\{x=0\}A_1A_2A_3 \dots \in 2^{AP} \,/\, A_i \subseteq AP \text{ for all } i \geq 1, \exists j \geq 0.\ A_j = \{x>1\}\}$

~~invariance~~, ~~safety~~, ~~liveness~~, intersection of safety and liveness

$P_4 = \underbrace{\{\{x=0\}A_1A_2A_3 \dots \in 2^{AP} \,/\, A_i \subseteq AP \text{ for all } i \geq 1\}}_{\text{safety prop}} \cap$

$\cap \underbrace{\{A_0A_1A_2 \dots \,/\, A_i \subseteq AP \text{ for all } i \geq 1, \exists j \geq 0 : A_j = \{x>1\}\}}_{\text{liveness prop}}$

(e) x exceeds one only finitely often

$P_5 = \{A_0A_1A_2 \dots \in 2^{AP} \,/\, \forall j\ A_j = \{x=0\}\}$

$\neg \exists j.\ A_j = \{x>1\}$

~~invariance~~, ~~safety~~, liveness (because $pref(P_5) = (2^{AP})^*$).

(f) x exceeds one infinitely often

$P_6 = \{A_0A_1A_2\ldots \in (2^{AP})^\omega \mid \overset{\infty}{\exists} j \geq 0 : A_j = \{x > 1\}$

$\exists j_0 \geq 0\ \forall j \geq j_0 : A_j = \{x>1\}\}$

~~invariance~~, ~~safety~~, liveness since $pref(P_6) = (2^{AP})^*$

(g) the value of x alternates between zero and two:

$P_7 \approx \{A_0A_1A_2\ldots \in (2^{AP})^\omega \mid A_i \subseteq AP$ for all $i \geq 0$,

$(A_i = \{x=0\} \Leftrightarrow A_{i+1} = \{x>1\})$ for all $i \geq 0\}$

because $\{x=2\}, \{x\neq 2\} \notin AP$

~~invariance~~, safety (bad prefixes: e.g. $\{x=0\}\{x=0\}, \{x>1\}\{x>1\}$,

~~liveness~~ in general: $(\{x=0\}\{x>1\})^*\{x=0\}\{x=0\}$

$(\{x=0\}\{x>1\})^+\{x>1\}$

and the same with roles inversed

(h) ~~true~~

$P_8 = (2^{AP})^\omega = \{A_0A_1A_2A_3\ldots \in (2^{AP})^\omega \mid A_i \models true\}$

invariant, safety, liveness

(see above $BP(P_8) = \phi$ $Pref((2^{AP})^\omega) = (2^{AP})^*$

$\boxed{\text{Exercise 3.6}}$ Consider $AP = \{a, b\}$. Formulate as AP-properties and characterize as invariance, safety, or liveness properties:

(a) a should never occur:

$P_1 = \{A_0A_1A_2\ldots \in (2^{AP})^\omega \mid \overbrace{\underbrace{a \notin A_i}_{A_i \in \{\phi, \{b\}\}}}^{A_i \models \neg a}$ for all $i \geq 0\}$

invariant, safety, ~~liveness~~

(b) ~~a~~ should occur exactly once

$P_2 = \{A_0A_1A_2\ldots \in (2^{AP})^\omega \mid \exists! j \geq 0 : a \in A_j\}$

~~invariant~~, ~~safety~~, ~~liveness~~

$= \underbrace{\{\ldots \mid |\{j \geq 0 \mid a \in A_j\}| \leq 1\}}_{\text{safety}} \cap \underbrace{\{\ldots \mid |\{j \geq 0 \mid a \in A_j\}| \geq 1\}}_{\text{liveness}}$

$BP = \{A_0\ldots A_nA_{n+1} \in (2^{AP})^\omega \mid |\{j \in \{0,\ldots,n\} \mid a \in A_j\}| = 1, a \in A_{n+1}\}$

(c) a and b alternate infinitely often, $AP = \{a, b\}$

first attempt $P_{30} = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, \forall i \geq 0 \,(A_i = \{a\} \Rightarrow \exists j > i.\ A_j = \{b\})$
and
$\forall i \geq 0 \,(A_i = \{b\} \Rightarrow \exists j > i\ A_j = \{a\}) \}$

$= \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, (\overset{\infty}{\exists} i \geq 0.\ A_i = \{a\}) \,\&\, (\overset{\infty}{\exists} i \geq 0.\ A_i = \{b\}) \}$

~~invariant~~, ~~safety~~, liveness

$P_3 = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, \forall i \geq 0 \,(A_i = \{a\} \text{ or } A_i = \{b\})$
and
$\forall i \geq 0 \,(A_i = \{a\} \Rightarrow \exists j > i.\ A_j = \{b\})$
and
$\forall i \geq 0 \,(A_i = \{b\} \Rightarrow \exists j > i.\ A_j = \{a\}) \}$

~~invariant~~, ~~safety~~, ~~liveness~~

$= \underbrace{\{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, \forall i \geq 0 \,(A_i = \{a\} \text{ or } A_i = \{b\}) \}}_{\text{safety}} \cap \underbrace{P_{30}}_{\text{liveness}}$

(d) (every) a should eventually be followed by b

$P_4 = \{ A_0 A_1 A_2 \ldots \in (2^{AP})^\omega \,/\, \forall i \geq 0 \,(a \in A_i \Rightarrow \exists j > i.\ b \in A_j) \}$

liveness, ~~invariant~~, ~~safety~~