Unique Normal Forms in Infinitary Weakly Orthogonal Term Rewriting

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Overview

- Introduction: weakly orthogonal, UN^∞
- Counterexample to UN^∞ for weakly orthogonal TRSs
- Counterexample to UN^∞ for $\lambda^\infty\beta\eta$
- Restoring infinitary confluence
- Diamond and triangle properties for developments

Overview

Weakly orthogonal

Weakly orthogonal (first-/higher-order) systems:

- left-linear
- all critical pairs are trivial.

Examples.

- Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with critical pairs:

$$\mathsf{S}(x) \leftarrow \underline{\mathsf{S}}(\underline{\mathsf{P}}(\overline{\mathsf{S}}(x))) \to \mathsf{S}(x) \qquad \mathsf{P}(x) \leftarrow \underline{\mathsf{P}}(\overline{\mathsf{S}}(\mathsf{P}(x))) \to \mathsf{P}(x)$$

- Parallel-Or TRS ('almost orthogonal'):

$$\mathsf{por}(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{por}(x, \mathsf{true}) \to \mathsf{true}$$
$$\mathsf{por}(\mathsf{false}, \mathsf{false}) \to \mathsf{false}$$

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CR^{∞} en UN^{∞} (definitions). Situation in OTRSs

- CR^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \implies \exists s.\ t_1 \twoheadrightarrow s \twoheadleftarrow t_2$.
- UN^{∞}: $t_1 \twoheadleftarrow t \twoheadrightarrow t_2 \ \wedge\ t_1, t_2$ normal forms $\implies t_1 = t_2$.
- SN^{∞}: all infinite rewrite sequences are progressive (str. conv.)

In orthogonal TRSs (well-known):

- $\mathrm{SN}^{\infty} \implies \mathrm{CR}^{\infty}$, and $\mathrm{CR}^{\infty} \implies \mathrm{UN}^{\infty}$.
- CR^{∞} fails (Kennaway).
  - But for non-collapsing TRSs: CR^{∞} holds.
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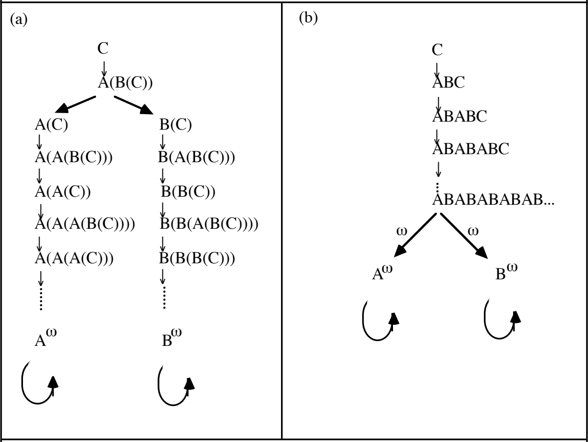
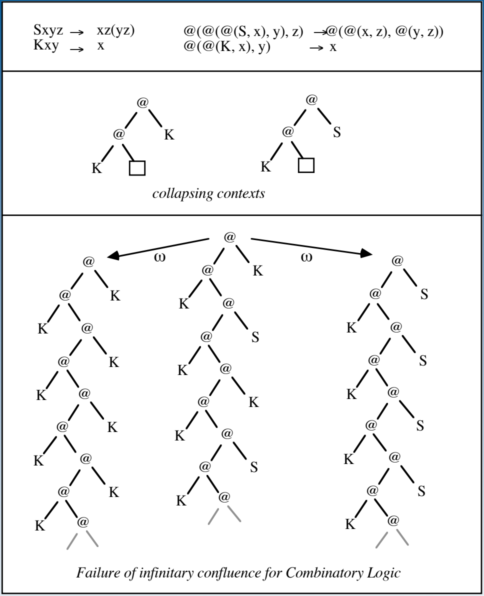
not CR^{∞}

$$
\begin{aligned}
A(x) &\to x \\
B(x) &\to x \\
C &\to A(B(C))
\end{aligned}
$$



Failure of infinitary confluence



collapsing contexts

Failure of infinitary confluence for Combinatory Logic

Overview

Counterexample: UN^{∞} fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

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with the normal forms $\mathsf{S}^{\omega} = \mathsf{SSS}\ldots$ and $\mathsf{P}^{\omega} = \mathsf{PPP}\ldots$ we consider:

$$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\mathsf{S}^4\,\mathsf{P}^5\,\mathsf{S}^6\ldots = \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots$$

We find:

$$\begin{aligned}
\psi &= \mathsf{P\;SS\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots\\
&\to \mathsf{S\;PPP\;SSSS\;PPPPP\;SSSSSS}\;\ldots\\
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&\to \mathsf{S}\;\mathsf{PP}\;\mathsf{SSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{SP}\;\mathsf{SS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
&\to \mathsf{S}\;\mathsf{S}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots\\
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&\to \mathsf{SP\,SS\,PPPPP\,SSSSSS}\ldots\\
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And similar:

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Counterexample: UN^∞ fails weakly-ortho iTRS

In the Successor/Predecessor TRS:

$$\mathsf{P}(\mathsf{S}(x)) \to x \qquad\qquad \mathsf{S}(\mathsf{P}(x)) \to x$$

with the normal forms $\mathsf{S}^\omega = \mathsf{SSS}\dots$ and $\mathsf{P}^\omega = \mathsf{PPP}\dots$ we consider:

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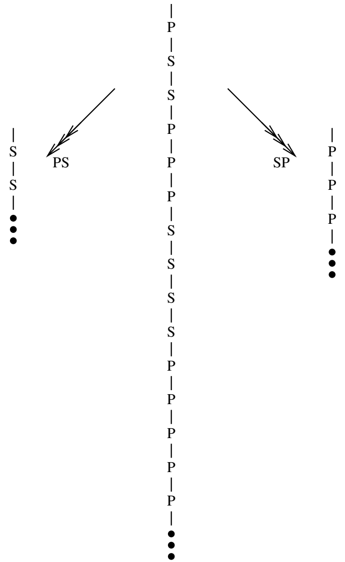

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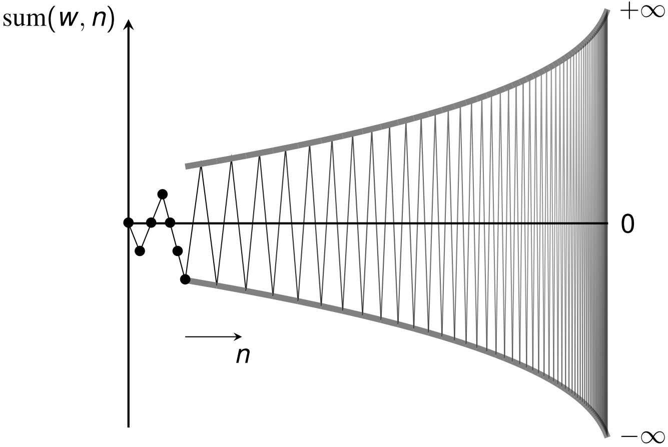

Graph for the oscillating PS-word $\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\ldots$.

Some facts about infinite PS-words

The S-norm $\|w\|_{\mathsf{S}}$ and P-norm $\|w\|_{\mathsf{P}}$ of a PS-word w:

$$\|w\|_{\mathsf{S}} = \sup_{n \in \mathbb{N}} \mathrm{sum}(w, n) \qquad \|w\|_{\mathsf{P}} = \sup_{n \in \mathbb{N}}(-\mathrm{sum}(w, n))$$

1. $w \twoheadrightarrow \mathsf{S}^{\omega}$ if and only if $\|w\|_{\mathsf{S}} = \infty$;
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2. Every PS-word that reduces to both S^{ω} and P^{ω} reduces to every infinite PS-word.
3. A PS-word w is root-active if and only if w is the concatenation of infinitely many finite 'zero-sum-words' $w_1, w_2, w_3, \ldots$.
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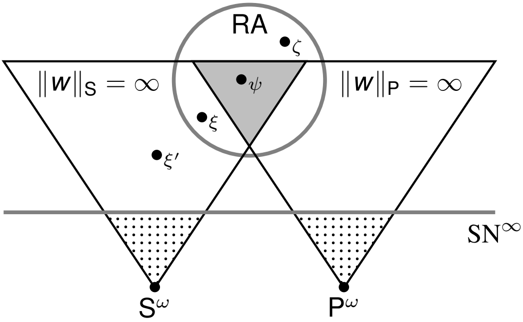
Venn diagram



$\psi = \mathsf{P}^1\,\mathsf{S}^2\,\mathsf{P}^3\,\ldots$ $\qquad$ $\xi = \mathsf{S}\,\mathsf{P}\,\mathsf{S}^2\,\mathsf{P}^2\,\mathsf{S}^3\,\mathsf{P}^3\ldots$

$\zeta = (\mathsf{P}\,\mathsf{S})^\omega$ $\qquad$ $\xi' = \mathsf{S}\,\xi$

Overview

$\lambda^\infty\beta\eta$

Terms of $\lambda^\infty\beta\eta$: the (potentially) infinite λ-terms in $Ter^\infty(\lambda)$

The rewrite rules of $\lambda^\infty\beta\eta$ are:

$$(\lambda x.M)N \xrightarrow{\beta} M[x{:=}N]$$

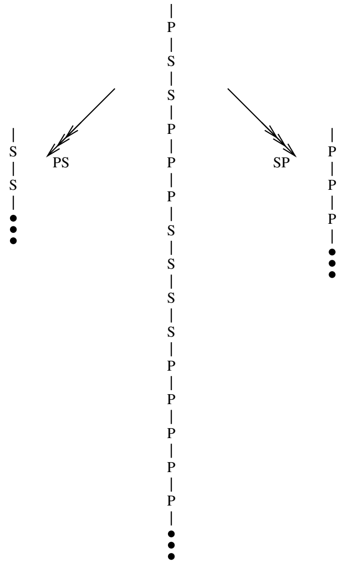
$$\lambda x.Mx \xrightarrow{\eta} M \qquad (x \text{ is not free in } M)$$

$\lambda^\infty\beta\eta$ is weakly orthogonal, since the critical pairs are trivial:

$$Mx \xleftarrow{\beta} (\lambda x.Mx)x \xrightarrow{\eta} Mx \qquad (x \notin fv(M))$$

$$\lambda x.M[y{:=}x] \xleftarrow{\beta} \lambda x.(\lambda y.M)x \xrightarrow{\eta} \lambda y.M \qquad (x \notin fv(M))$$

Translating the S-P-example to $\lambda^{\infty}\beta\eta$



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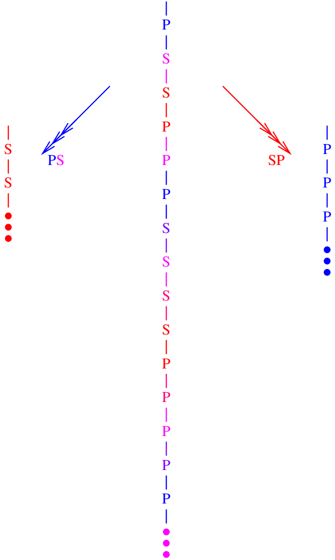

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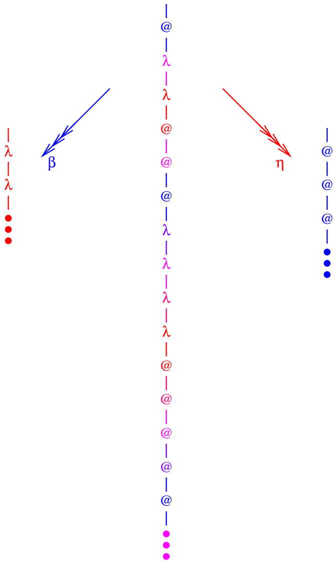

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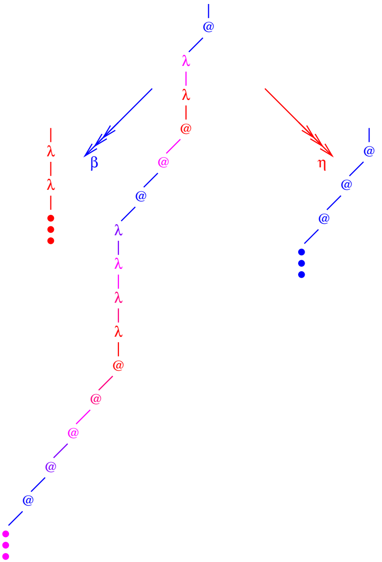

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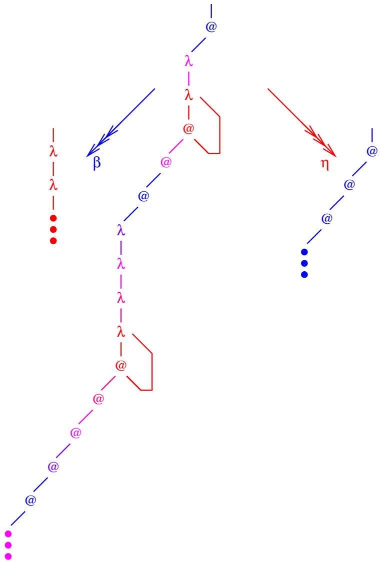

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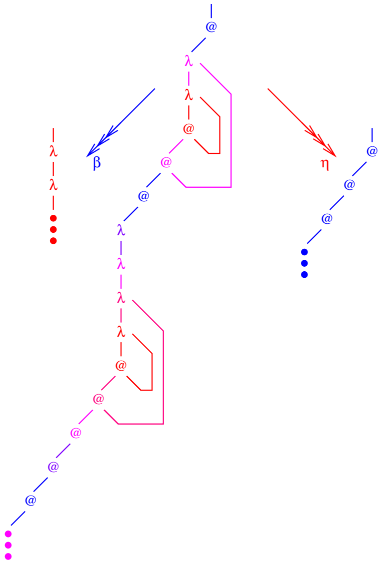

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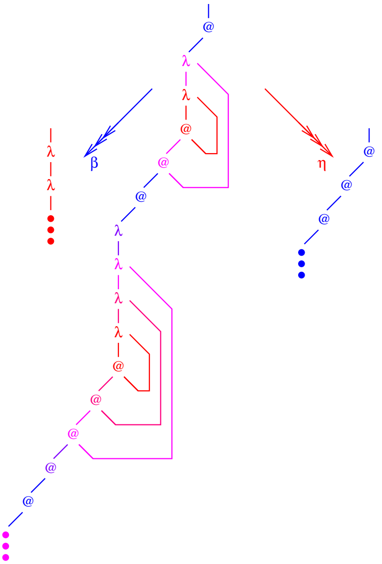

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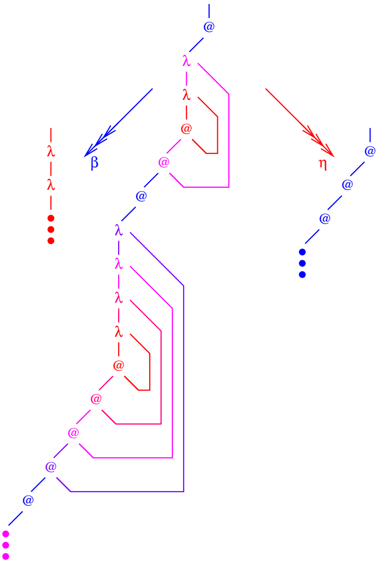

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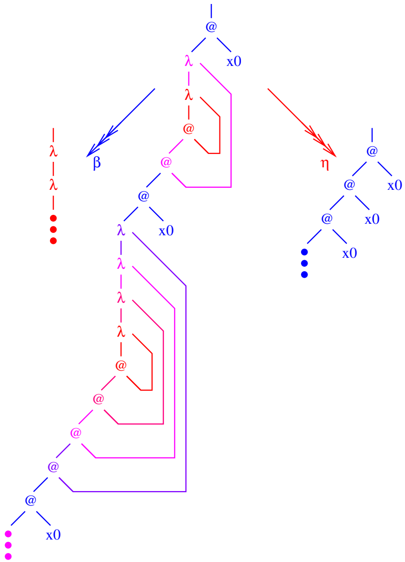

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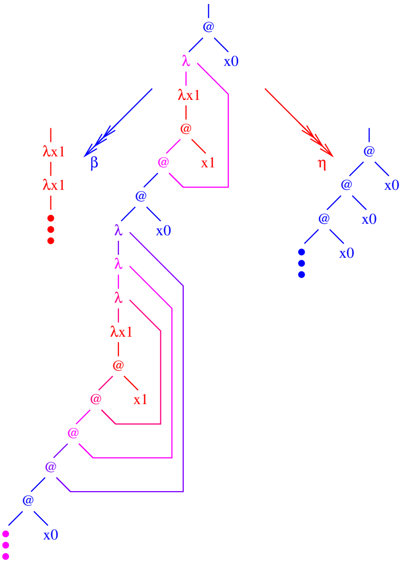

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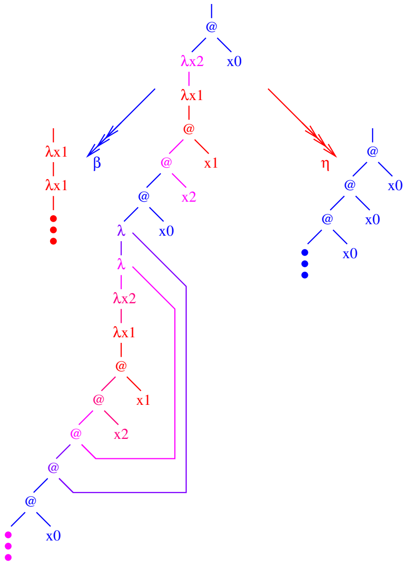

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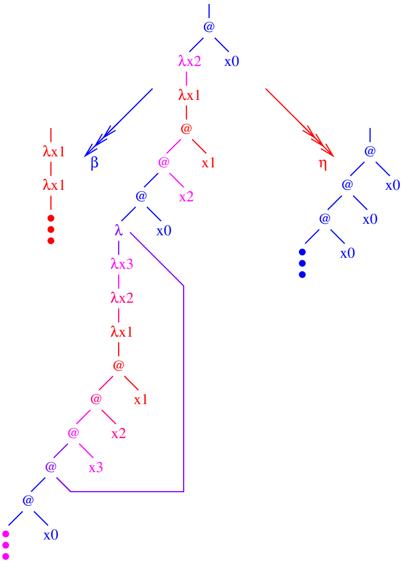
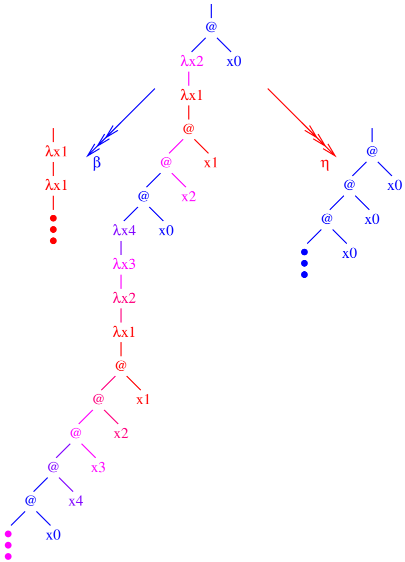


Counterexample: UN^∞ fails in $\lambda^\infty\beta\eta$



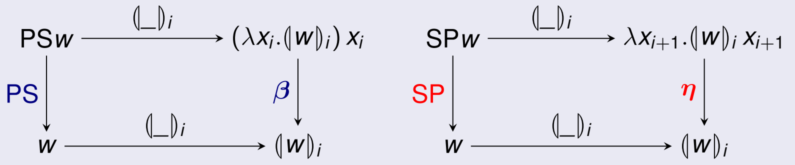
Translating the S-P-example to $\lambda^\infty\beta\eta$

$(\!|\_|\!) : \{\mathsf{P},\mathsf{S}\}^\omega \to Ter^\infty(\lambda)$ defined by:

- $(\!|w|\!) = (\!|w|\!)_0$;
- for all $w \in \{\mathsf{P},\mathsf{S}\}^\omega$, and $i \in \mathbb{Z}$:

$$(\!|\mathsf{P}w|\!)_i = (\!|w|\!)_{i-1}\, x_i \qquad\qquad (\!|\mathsf{S}w|\!)_i = \lambda x_{i+1}.(\!|w|\!)_{i+1}$$

Lemma



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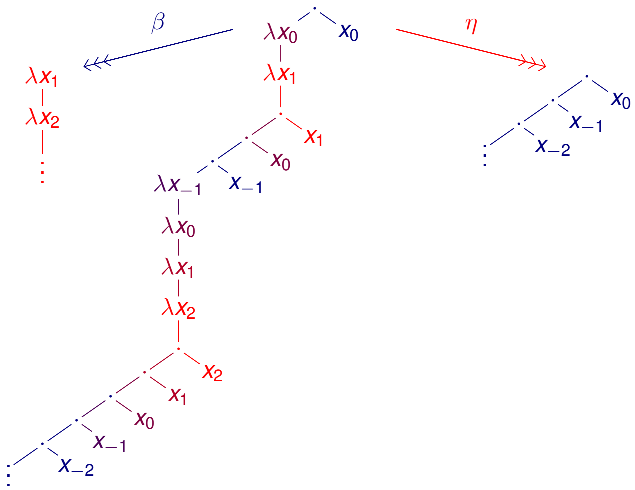

Contrast with $\lambda^\infty\beta$

We saw for $\lambda^\infty\beta\eta$:

- UN^∞ **fails**
- Consequently: CR^∞ **fails**

However for $\lambda^\infty\beta$ it holds:

- CR^∞ **fails**
- But: UN^∞ **holds!**

  Due to this, $\lambda^\infty\beta$ is important for the model theory of λ-calculus: for several models equality is captured by $\lambda^\infty\beta$-convertibility:

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Overview

Refined compression

Lemma (Refined Compression Lemma)

Let R be a left-linear iTRS.
Let $\kappa : s \to_R^{\alpha} t$ be a rewrite sequence, let d the min. depth of a step, and n the number of steps at depth d.
Then there exists a rewrite sequence $\kappa' : s \to_R^{\leq\omega} t$ in which all steps take place at depth $\geq$ d, and precisely n steps at depth d.

Theorem

Let R be a left-linear iTRS.
For every divergent rewrite sequence $\kappa : s \to_R^{\alpha}$ (length α)
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Orthogonalization (of parallel steps)

Proposition

For parallel steps $\phi : s \twoheadrightarrow t_1$ and $\psi : s \twoheadrightarrow t_2$ in a w-o TRS there exists orthogonal steps ϕ' and ψ' such that $\phi' : s \twoheadrightarrow t_1$ and $\psi' : s \twoheadrightarrow t_2$ (the pair $\langle \phi', \psi' \rangle$ is an orthogonalization of ϕ and ψ).

Proof.

In case of overlaps, we replace the outer redex with the inner one.

(by weak orthogonality overlapping redexes have the same effect) □

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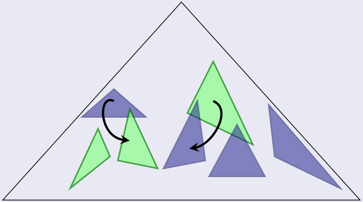

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Let R be a weakly orthogonal TRS. Let $\phi : s \twoheadrightarrow\!\!\!\!\!\!\!\parallel\ \ t_1$, $\psi : s \twoheadrightarrow\!\!\!\!\!\!\!\parallel\ \ t_2$ be parallel steps in a weakly orthogonal TRS R.

Definition

Let the weakly orthogonal projections ϕ/ψ and ψ/ϕ be the orthogonal projections ϕ'/ψ' and ψ'/ϕ' of the orthogonalization $\langle\phi', \psi'\rangle$ of $\langle\phi, \psi\rangle$.

Lemma

Let d_ϕ and d_ψ be the minimal depth of a step in ϕ and ψ.

Then the minimal depth of the w-o projections ϕ/ψ and ψ/ϕ is:

- *in general:* $\geq \min(d_\phi, d_\psi)$.
- *if R is non-collapsing:* $\geq \min(d_\phi, d_\psi + 1)$ *and* $\geq \min(d_\psi, d_\phi + 1)$, *respectively.*

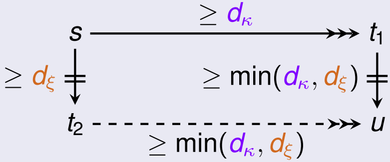
Proof.

The orthogonalization does not decrease the height as we always replace the outer by the inner redex. Orthogonal projection cannot lift redexes above the the depth of variables in the right-hand sides. □

Infinitary Parallel Moves Lemma PML$^{\infty}$

Lemma

Let R be a weakly orthogonal TRS. In general it holds:



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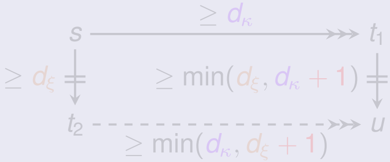

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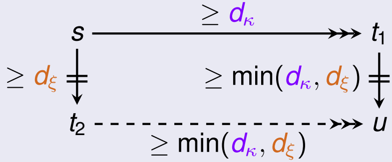
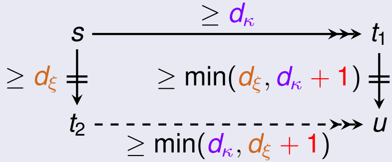


If R is non-collapsing, then also:



Restoring infinitary confluence

Theorem

Weakly orthogonal TRSs without collapsing rules are inf. confluent.

Proof.

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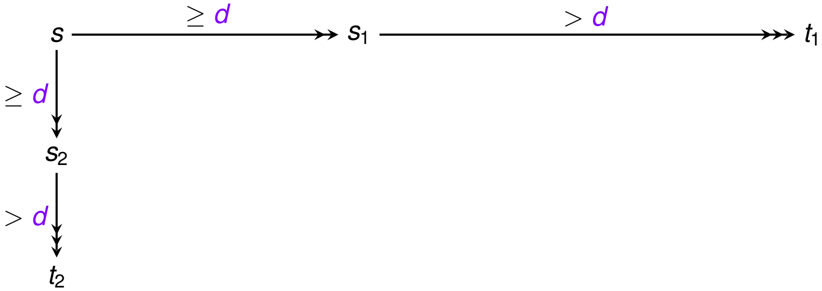

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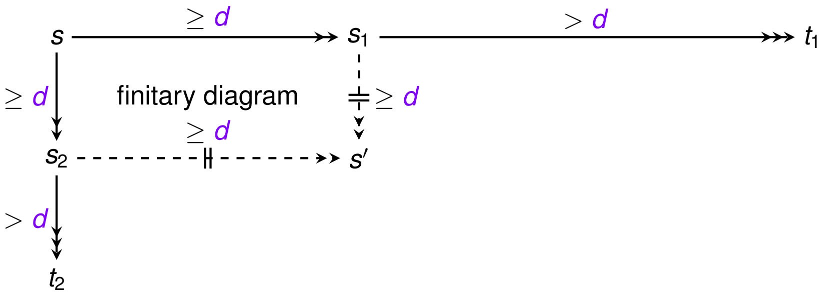

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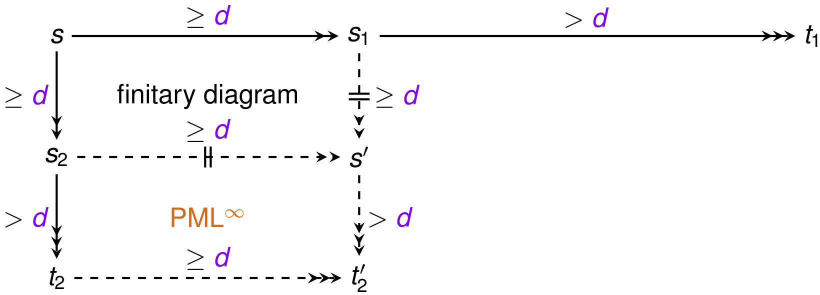

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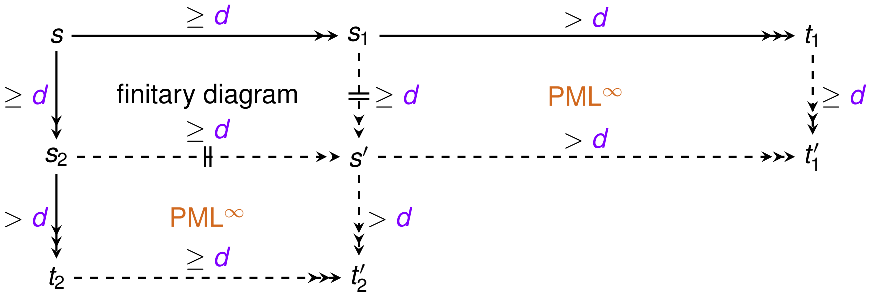

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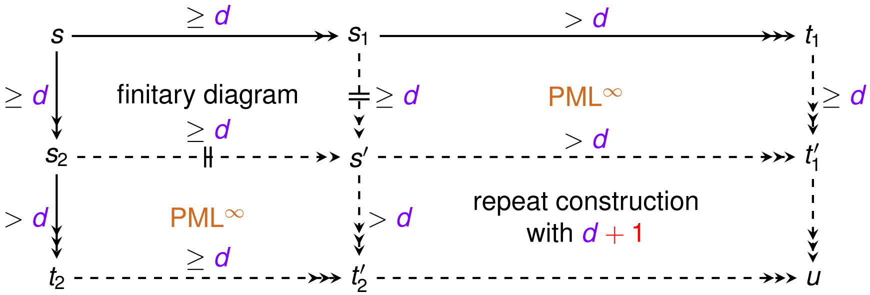

Overview

1. Introduction
2. Counterexample to UN^{∞} for weakly orthogonal iTRSs
3. Counterexample to UN^{∞} in $\lambda^{\infty}\beta\eta$
4. Restoring infinitary confluence
5. Diamond and triangle properties for developments
6. Conclusion

Diamond and triangle properties for developments

A binary relation $\to$ on A is said to have:

- the diamond property if: $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow$;
- the triangle property if:

$$\forall a \in A.\, \exists a' \in A.\, a \to a' \wedge (\forall b \in A.\, a \to b \Rightarrow b \to a').$$

Theorem

For every weakly orthogonal TRS without collapsing rules, (infinite) multi-steps have:

1. *the diamond property;*
2. *the triangle property.*

Our proof proceeds by:

- refining an earlier cluster analysis (I-clusters and Y-clusters) from the finite case;
- a top-down orthogonalization algorithm.

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Overview

- Counterexample to $\mathrm{UN}^{\infty}/\mathrm{CR}^{\infty}$ for weakly orthogonal TRSs
- By translation: counterexample to $\mathrm{UN}^{\infty}/\mathrm{CR}^{\infty}$ for $\lambda^{\infty}\beta\eta$
- Restoring CR^{∞} for non-collapsing w-o TRSs
- Diamond and triangle properties for developments in non-collapsing w-o TRSs

Summary

		finitary				*infinitary*			
		PML	CR	UN	NF	PML$^{\infty}$	CR$^{\infty}$	UN$^{\infty}$	NF$^{\infty}$
first-order	OTRS	yes	yes	yes	yes	yes	no	yes	yes
	WOTRS	yes	yes	yes	yes	**yes**	no	**no**	**no**
	nc-WOTRS	yes	yes	yes	yes	**yes**	**yes**	**yes**	**yes**
	1c-WOTRS	yes	yes	yes	yes	**yes**	no	**?**	**?**
higher-order	$\lambda\beta$	yes	yes	yes	yes	no	no	yes	yes
	fe-OCRS	yes	yes	yes	yes	no	no	yes	yes
	$\lambda\beta\eta$	yes	yes	yes	yes	no	no	**no**	**no**
	WOCRS	yes	yes	yes	yes	no	no	**no**	**no**