Modeling Terms by Graphs with Structure Constraints

(An illustration with background)

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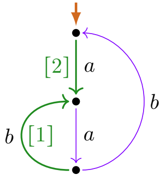

Seminar TCS
Vrije Universiteit Amsterdam
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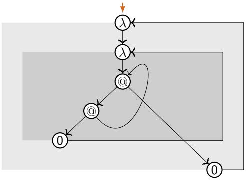

structure constraints (L'Aquila)

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Overview

Illustr.: Process interpretation of regular expressions

- LEE-witnesses: graph labelings based on a loop–condition LEE

Backgr.: Maximal sharing of functional programs

- higher-order λ-term graphs

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- Milner's questions, known results
- structure-constrained process graphs:
  - LEE-witnesses: graph labelings based on a loop–condition LEE
  - preservation under bisimulation collapse
- readback: from graph labelings to regular expressions

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  - λ-NFAs, and λ-DFAs
- minimization / readback / efficiency / Haskell implementation

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Regular expressions under process semantics (bisimilarity $\underline{\leftrightarrow}$)

Given: process graph interpretation $[\![\cdot]\!]_P$, studied under $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, modulo $\underline{\leftrightarrow}$ incomplete

λ-calculus with letrec under unfolding semantics

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Not available: term graph interpretation that is studied under $\underline{\leftrightarrow}$

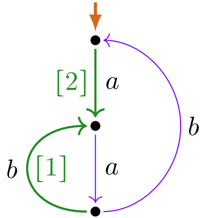
- graph representations used by compilers were not intended for use under $\underline{\leftrightarrow}$

Desired: term graph interpretation that:

- natural correspondence with terms in $\boldsymbol{\lambda}_{\text{letrec}}$
- supports compactification under $\underline{\leftrightarrow}$
- efficient translation and readback

Process interpretation of regular expressions

(current work with Wan Fokkink)



Regular Expressions *(Copi–Elgot–Wright, 1958; based on Kleene, 1951)*

Definition

The set $\text{Reg}(A)$ of regular expressions over alphabet A is defined by the grammar:

$$e, f \ ::= \ 0 \mid 1 \mid a \mid (e + f) \mid (e \cdot f) \mid (e^{\star}) \qquad \text{(for } a \in A\text{).}$$

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Note, here:

- symbol 0 instead of $\varnothing$
- symbol 1 used (often dropped, definable as $0^{\star}$)
- no complementation operation $\overline{e}$
  - is not expressible under language interpretation

Language interpretation $[\![\cdot]\!]_L$ *(Copi–Elgot–Wright, 1958)*

$\mathbf{0} \xrightarrow{[\![\cdot]\!]_L}$ empty language $\varnothing$

$\mathbf{1} \xrightarrow{[\![\cdot]\!]_L} \{\epsilon\}$ (ϵ the empty word)

$a \xrightarrow{[\![\cdot]\!]_L} \{a\}$

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$e + f \xrightarrow{[\![\cdot]\!]_L}$ union of $[\![e]\!]_L$ and $[\![f]\!]_L$

$e \cdot f \xrightarrow{[\![\cdot]\!]_L}$ element-wise concatenation of $[\![e]\!]_L$ and $[\![f]\!]_L$

$e^{*} \xrightarrow{[\![\cdot]\!]_L}$ set of words formed by concatenating words in $[\![e]\!]_L$ plus the empty word ϵ

Process interpretation $[\![\cdot]\!]_P$ (Milner, 1984)

$\mathbf{0} \overset{[\![\cdot]\!]_P}{\longmapsto}$ deadlock $\boldsymbol{\delta}$, no termination

$\mathbf{1} \overset{[\![\cdot]\!]_P}{\longmapsto}$ empty process $\boldsymbol{\epsilon}$, then terminate

$a \overset{[\![\cdot]\!]_P}{\longmapsto}$ atomic action a, then terminate

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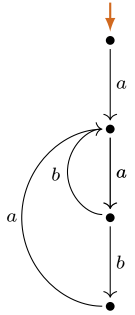
$a \xmapsto{[\![\cdot]\!]_P}$ atomic action a, then terminate

$e + f \xmapsto{[\![\cdot]\!]_P}$ alternative composition of $[\![e]\!]_P$ and $[\![f]\!]_P$

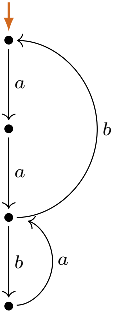
$e \cdot f \xmapsto{[\![\cdot]\!]_P}$ sequential composition of $[\![e]\!]_P$ and $[\![f]\!]_P$

$e^* \xmapsto{[\![\cdot]\!]_P}$ unbounded iteration of $[\![e]\!]_P$, option to terminate

Process interpretation of regular expressions



$a(a(b + ba))^*0$



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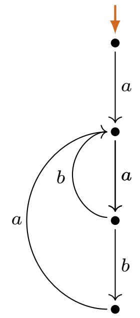
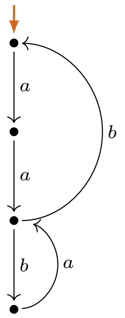


$$a \cdot (a \cdot (b + b \cdot a))^* \cdot 0$$



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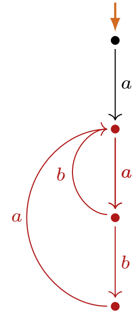

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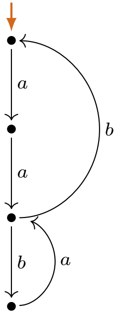

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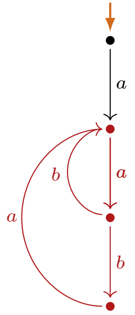

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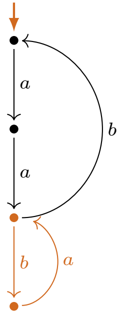

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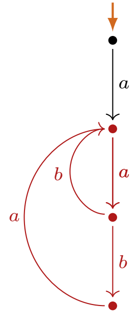

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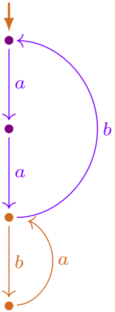

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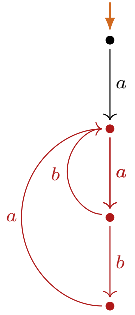
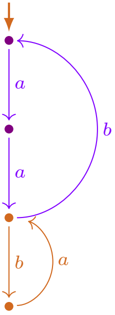


$$[\![a \cdot (a \cdot (b + b \cdot a))^* \cdot 0]\!]_{\boldsymbol{P}}$$



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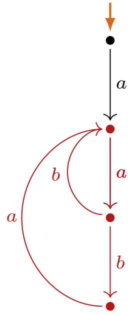

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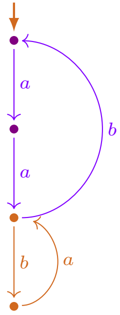

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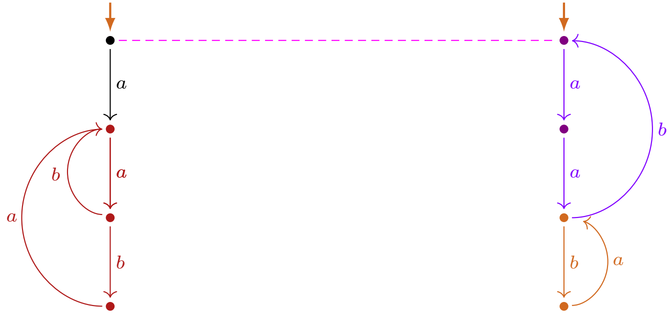

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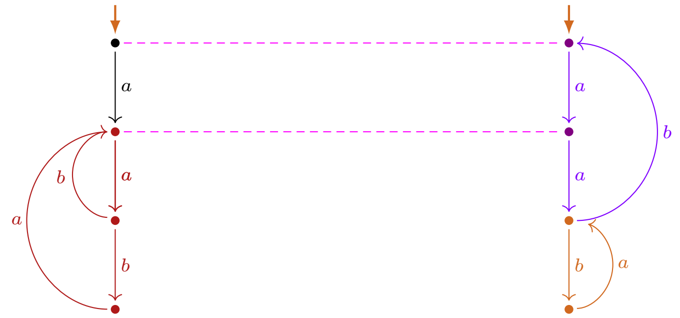

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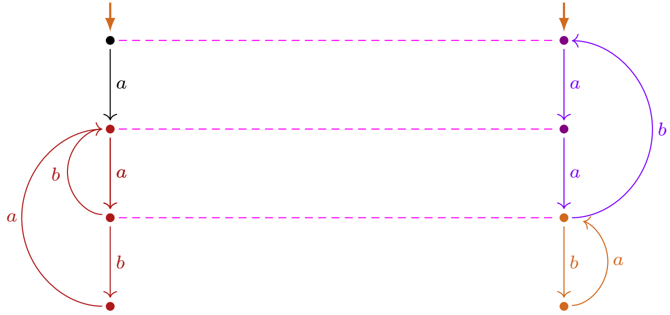

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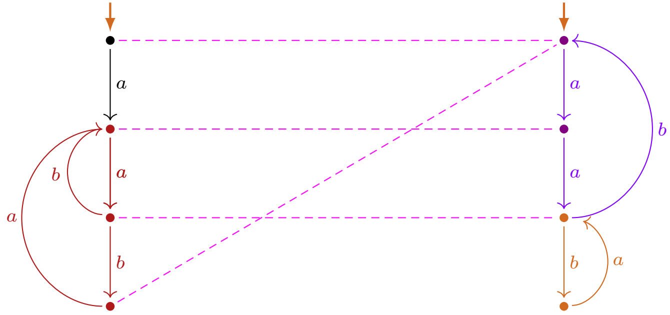

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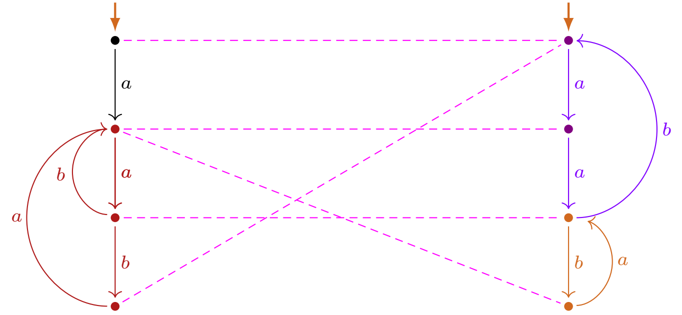

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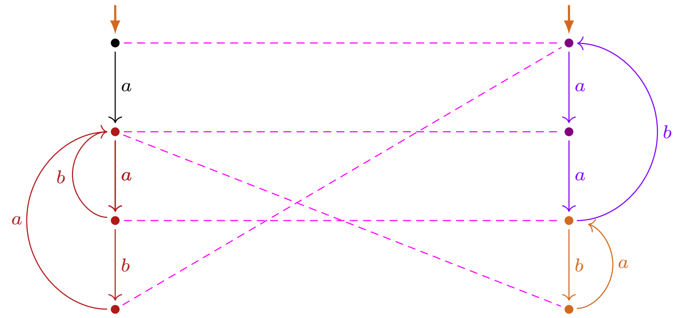

$$[\![a(a(b+ba))^{*}0]\!]_{P} \quad \underline{\leftrightarrow} \quad [\![(aa(ba)^{*}b)^{*}0]\!]_{P}$$

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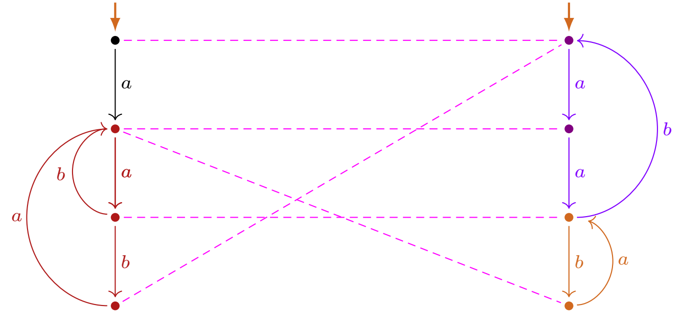

$a(a(b+ba))^{*}0 \quad \underline{\leftrightarrow}_{\boldsymbol{P}} \quad (aa(ba)^{*}b)^{*}0$

Process graphs and NFAs

Definition

A process graph over actions in A is a tuple $G = \langle V, v_s, T, E \rangle$ where:

- V is a set of *vertices*,
- $v_s \in V$ is the *start vertex*,
- $T \subseteq V \times A \times V$ the set of *transitions*,
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that are studied under bisimulation, not under language equivalence.

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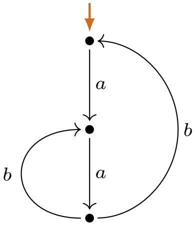
Antimirov (1996): NFA-definition of $[\![\cdot]\!]_{P}$ via partial derivatives.

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



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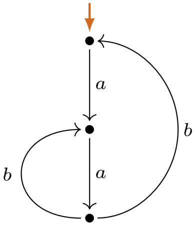

$$\notin im([\![\cdot]\!]_{\boldsymbol{P}})$$

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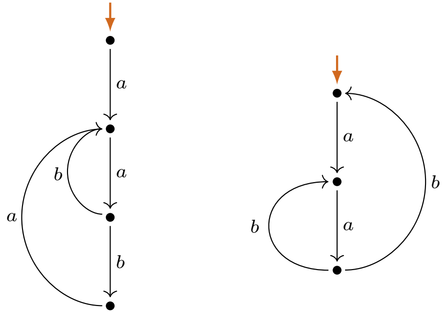

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$[\![\cdot]\!]_{\boldsymbol{P}}$-expressible

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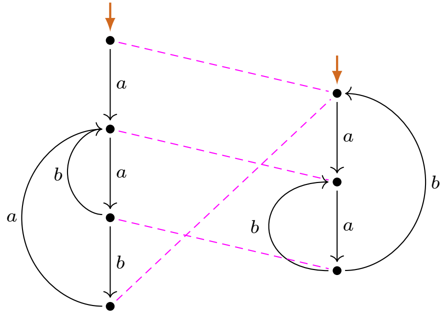

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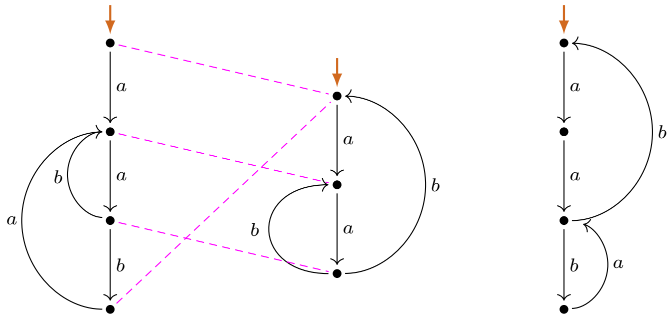

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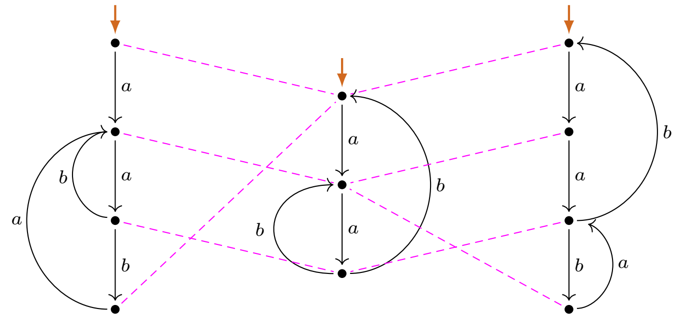

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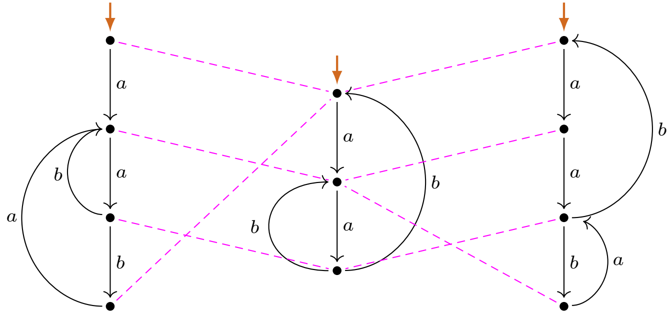

$\in im([\![\cdot]\!]_{\boldsymbol{P}})$ — $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible

$\notin im([\![\cdot]\!]_{\boldsymbol{P}})$ — $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible modulo $\underline{\leftrightarrow}$

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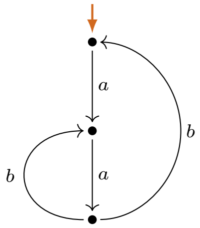
Properties of P

- Not every finite-state process is $[\![\cdot]\!]_P$-expressible.



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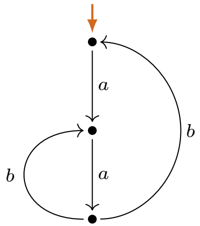

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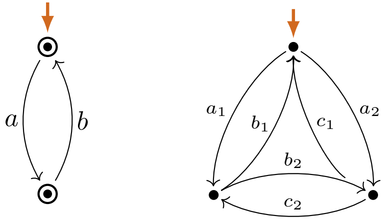

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- Fewer identities hold for $\underline{\leftrightarrow}_P$ than for $=_L$: $\underline{\leftrightarrow}_P \subsetneqq =_L$.

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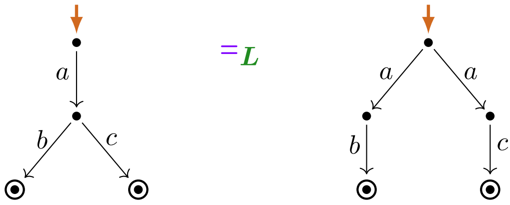

$$a \cdot (b + c) \quad =_{\boldsymbol{L}} \quad a \cdot b + a \cdot c$$

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$$a \cdot (b + c) \quad \not\underline{\leftrightarrow}_{\boldsymbol{P}} \quad a \cdot b + a \cdot c$$

Salomaa's axiomatization of $=_L$ (products commuted)

Axioms :

(B1)	$e + (f + g) = (e + f) + g$	(B7)	$e \cdot 1 = e$
(B2)	$(e \cdot f) \cdot g = e \cdot (f \cdot g)$	(B8)	$e \cdot 0 = 0$
(B3)	$e + f = f + e$	(B9)	$e + 0 = e$
(B4)	$(e + f) \cdot g = e \cdot g + f \cdot g$	(B10)	$e^* = 1 + e \cdot e^*$
(B5)	$e \cdot (f + g) = e \cdot f + e \cdot g$	(B11)	$e^* = (1 + e)^*$
(B6)	$e + e = e$		

Inference rules : equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{FIX} \quad (\text{if } \underbrace{\{\epsilon\} \notin [\![f]\!]_L}_{\textit{non-empty-word property}})$$

Sound and unsound axioms with respect to $\underline{\leftrightarrow}_{P}$

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(B1)	$e + (f + g) = (e + f) + g$	(B7)	$e \cdot 1 = e$
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Milner's adaptation for $\underline{\leftrightarrow}_P$ (Mil = Mil⁻ + RSP*)

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Inference rules : equational logic *plus*

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Q1. Which structural property of finite process graphs characterizes $[\![\cdot]\!]_P$-expressibility modulo $\underline{\leftrightarrow}$?

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Milner's questions, and partial results

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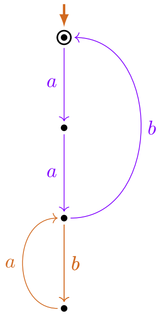
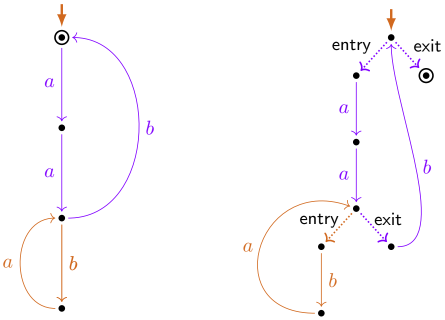
- Mil is complete when restricted to 1-return-less expressions *(Corradini, De Nicola, Labella, 2002)*
- Mil⁻ + one of two stronger rules (than RSP*) is complete *(G, 2006)*
  - with a coinductive rule (based on *Antimirov's* partial derivatives)
  - with a unique solvability principle USP

Well-behaved form, looping palm trees



$[\![(aa(ba)^* b)^*]\!]_{\boldsymbol{P}}$

Well-behaved form, looping palm trees



well-behaved form

(Corradini, Baeten)

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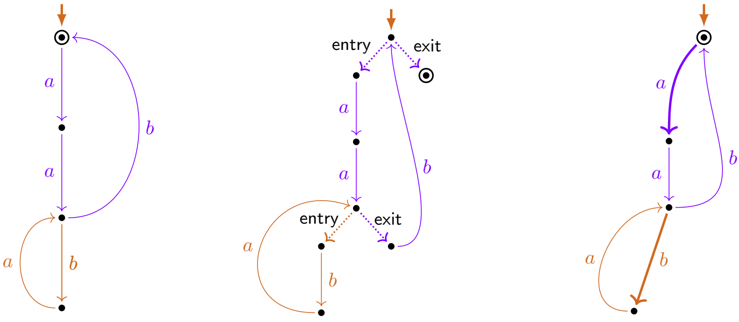

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Loop chart

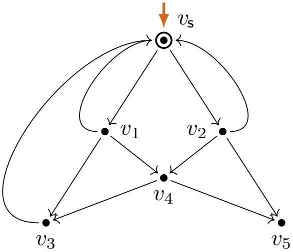
Definition

A process graph is a loop chart if:

L-1.

L-2.

L-3.



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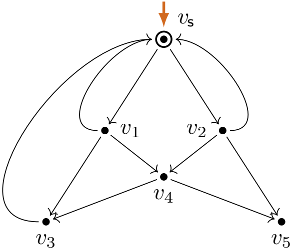

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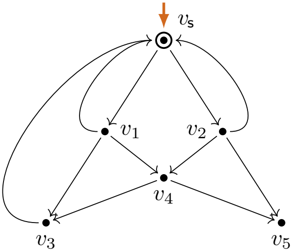

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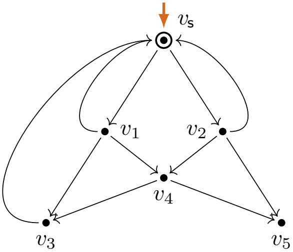

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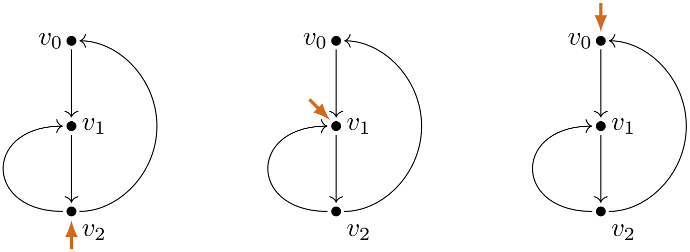

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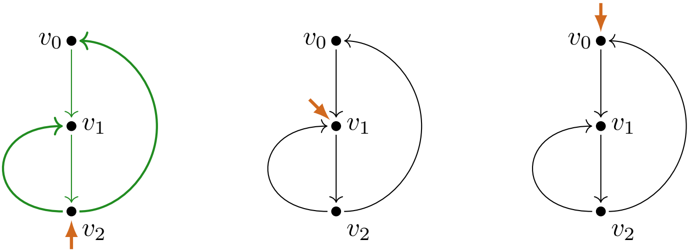

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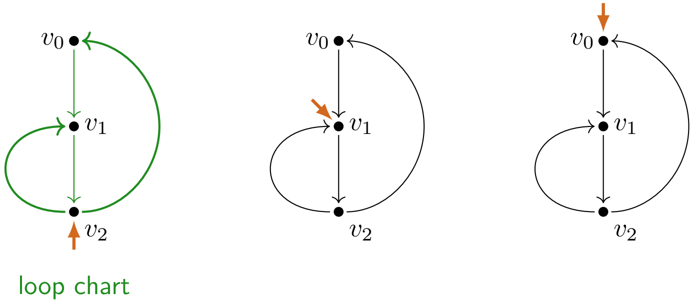

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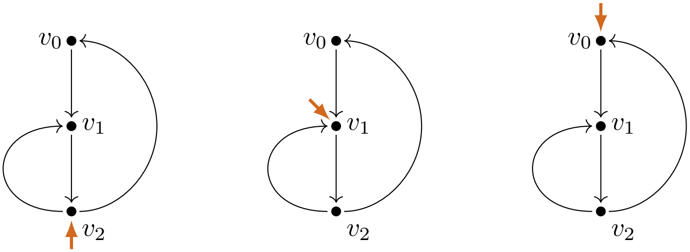

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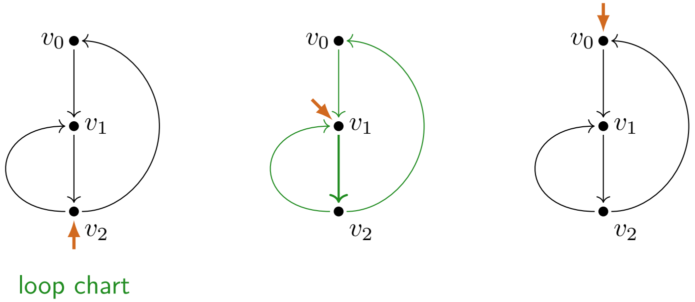

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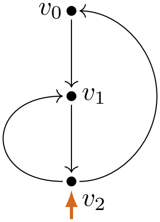

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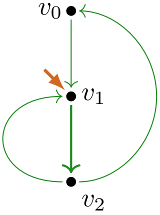

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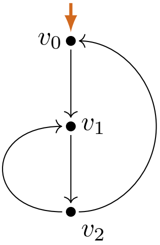

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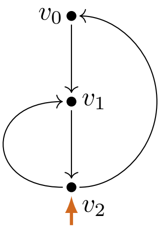

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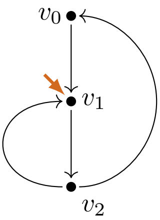

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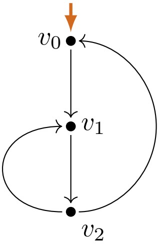

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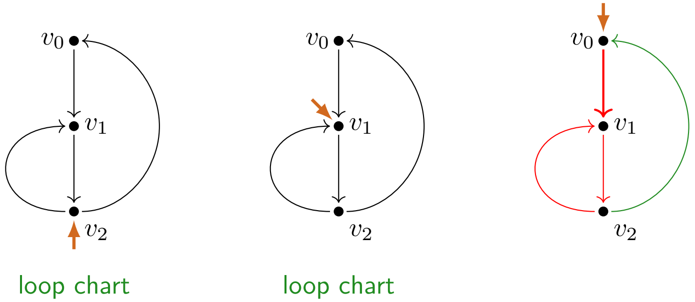

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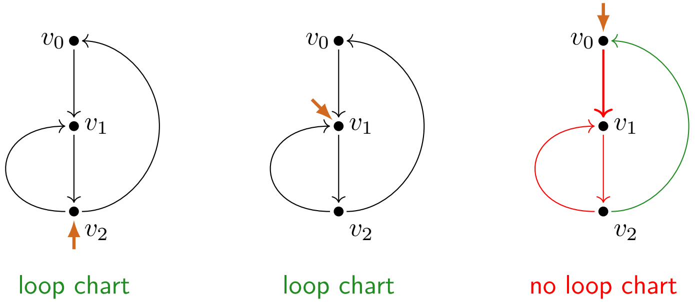

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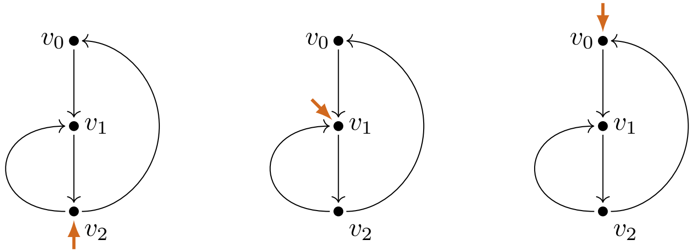
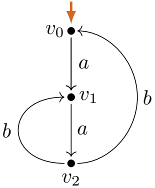


loop chart | loop chart | no loop chart

Loop elimination



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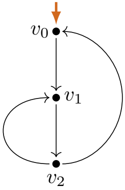

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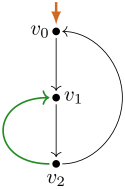

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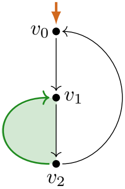

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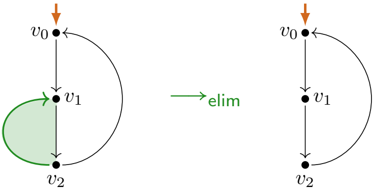

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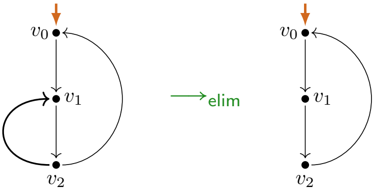

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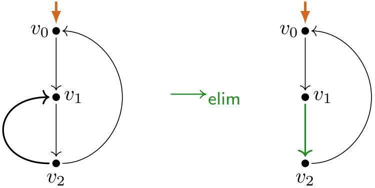

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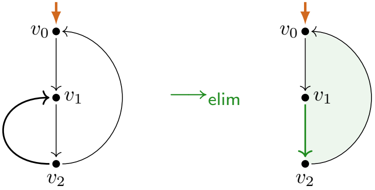

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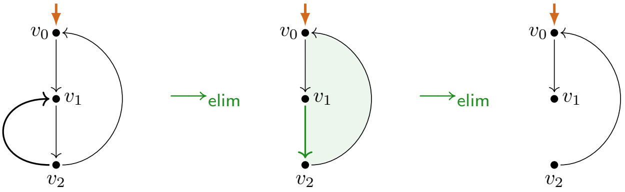

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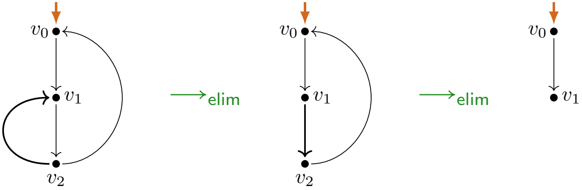

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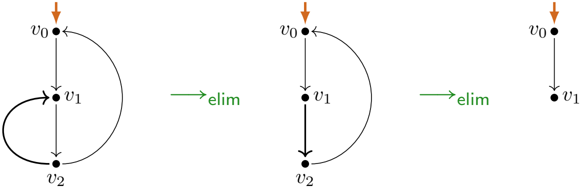

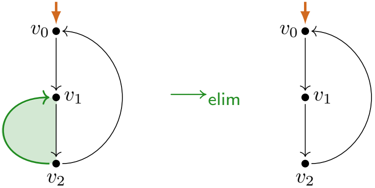

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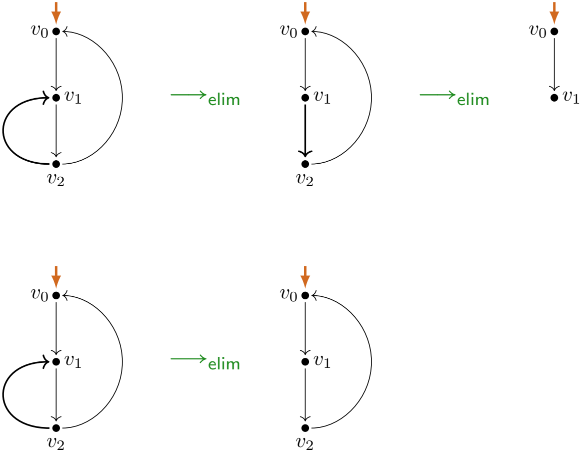

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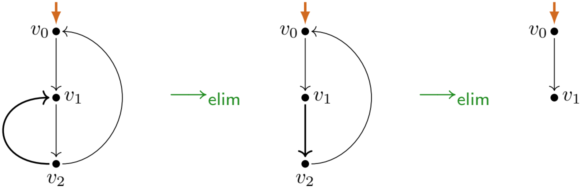

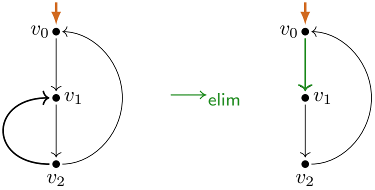

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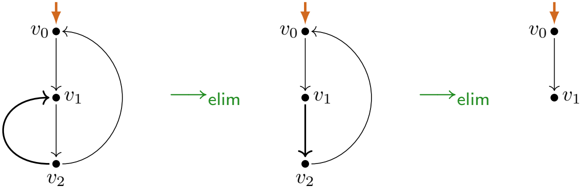

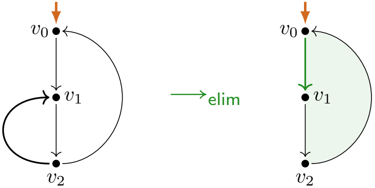

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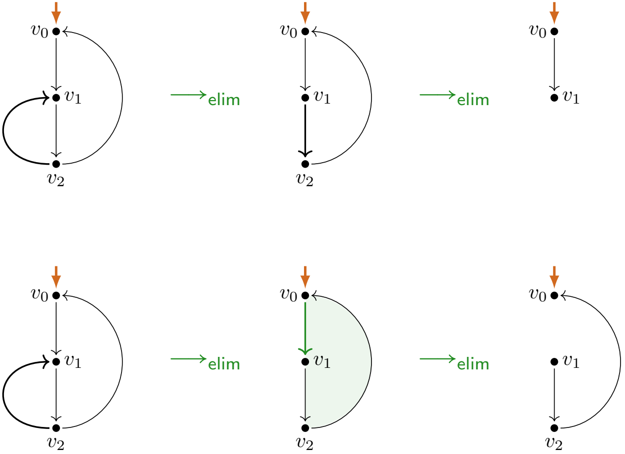
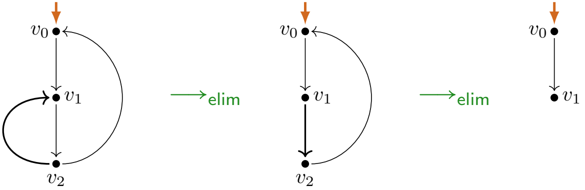
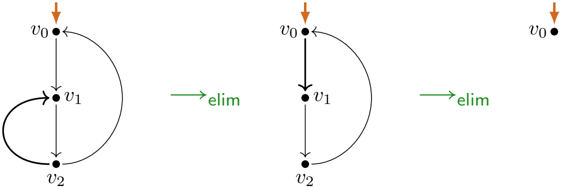
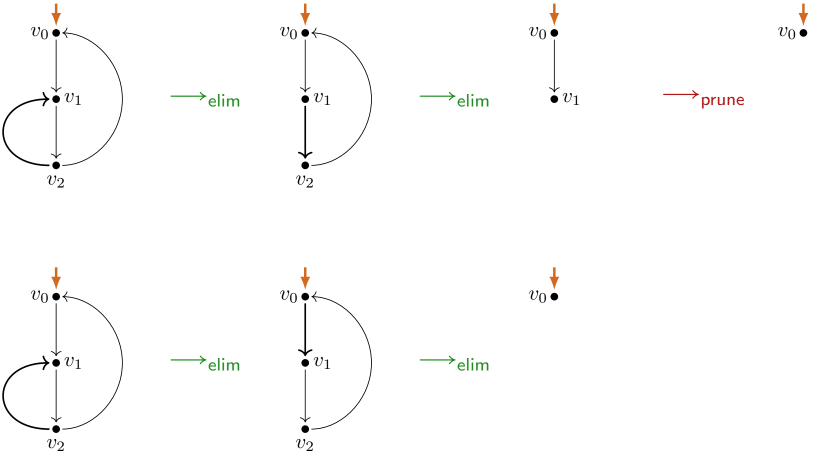


Loop elimination





Loop elimination



Loop elimination, and properties

$\longrightarrow_{\text{elim}}$: eliminate a transition-induced loop by:

- removing the loop-entry transition(s)
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$\longrightarrow_{\text{prune}}$: remove a transition to a deadlocking state

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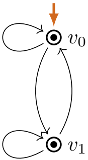
Lemma

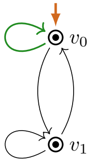
(i) $\longrightarrow_{\text{elim}}$ *is terminating.*

(ii) $\longrightarrow_{\text{elim}} \cup \longrightarrow_{\text{prune}}$ *is terminating and confluent.*

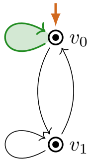
Loop elimination



Loop elimination



Loop elimination



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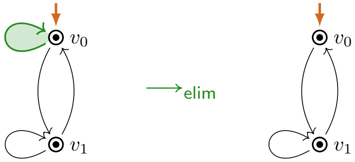
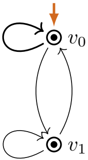
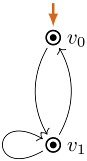


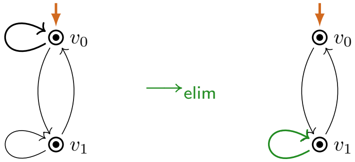
Loop elimination



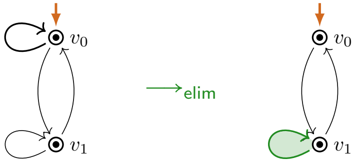
$\longrightarrow_{\text{elim}}$



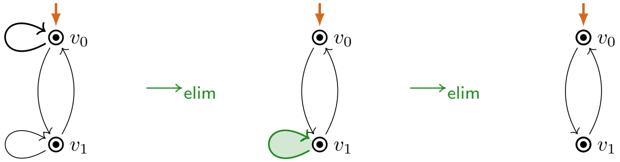
Loop elimination



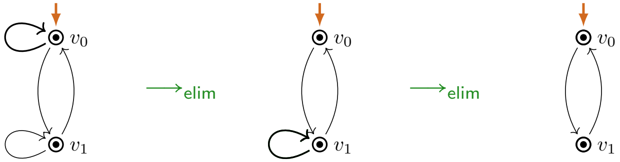
Loop elimination



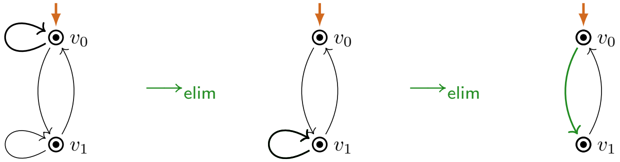
Loop elimination



Loop elimination



Loop elimination



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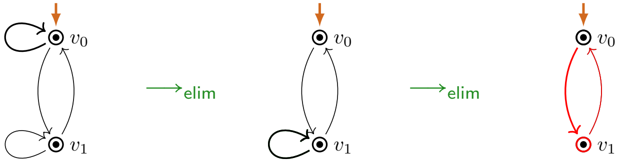

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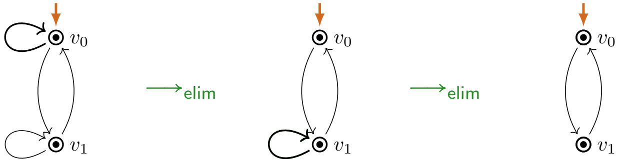

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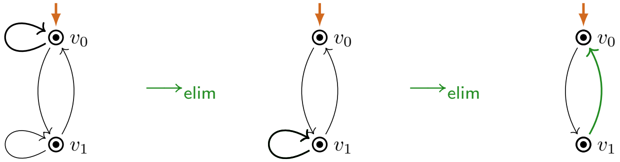

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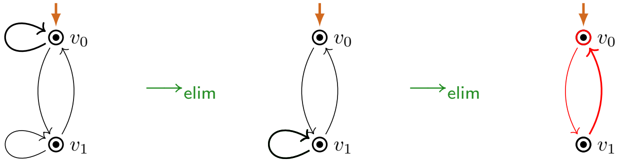

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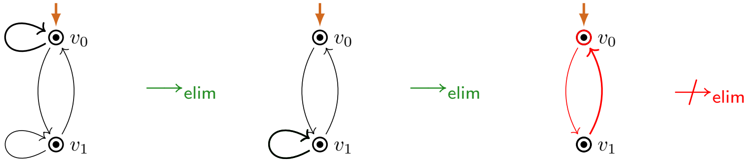

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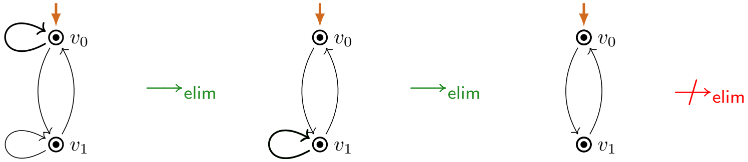

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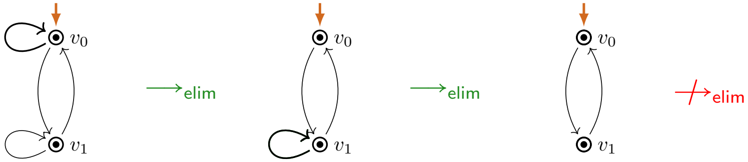

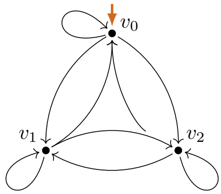

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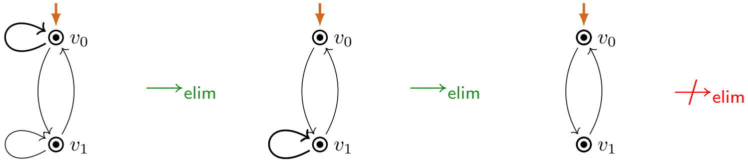
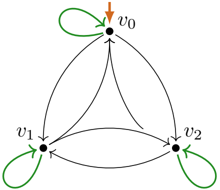
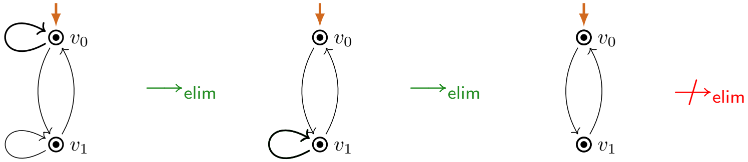
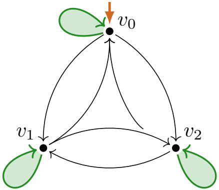


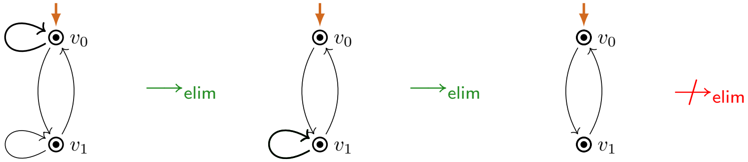


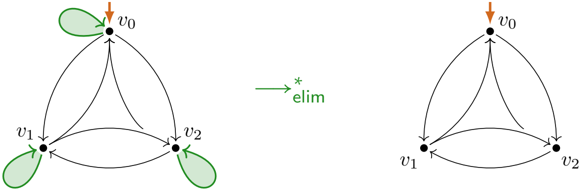
Loop elimination





Loop elimination





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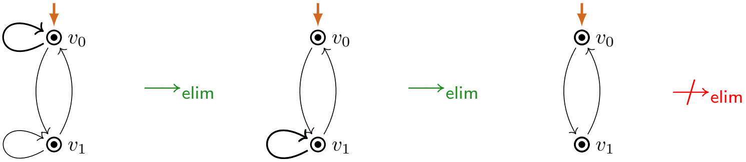

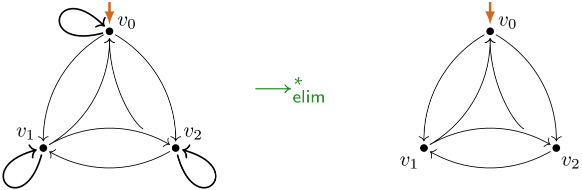

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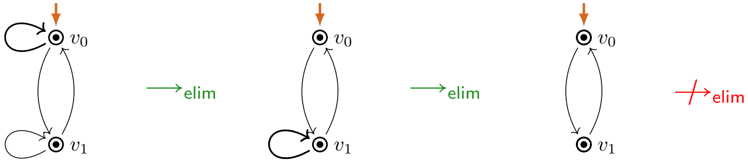
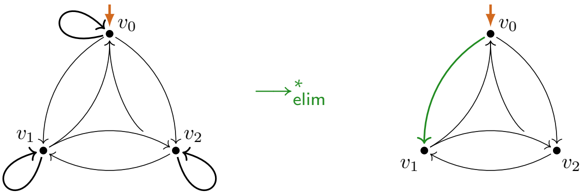
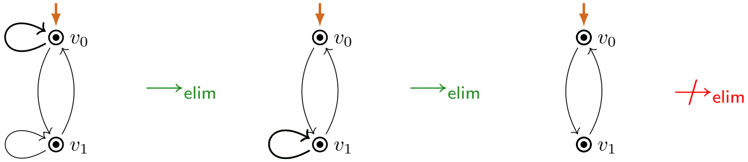
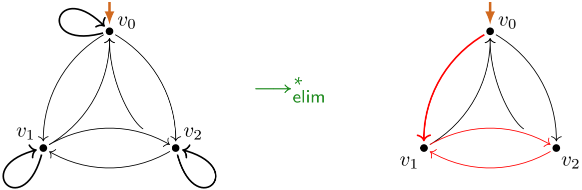


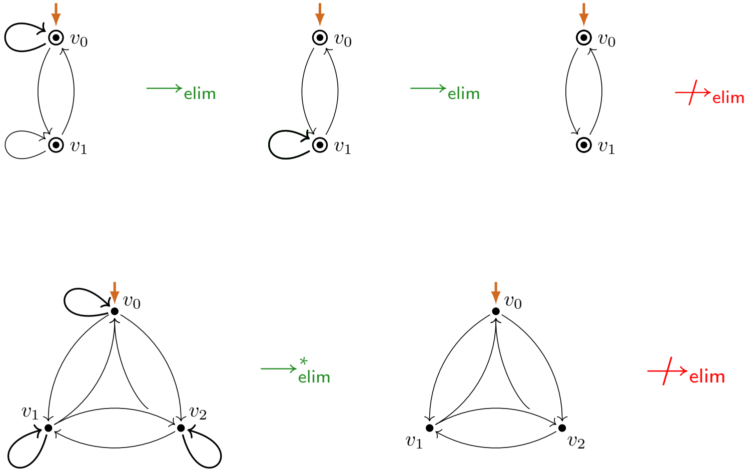


Loop elimination





Loop elimination



Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

$$\exists G_0 \,\big(\, G \longrightarrow^{*}_{\text{elim}} G_0 \not\longrightarrow_{\text{elim}} \;\wedge\; G_0 \text{ has no infinite trace} \,\big)\,.$$

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Lemma (by using confluence properties)

For every process graph G the following are equivalent:

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(iv) *Every $\longrightarrow_{\mathsf{elim}}$ normal form is without an infinite trace.*

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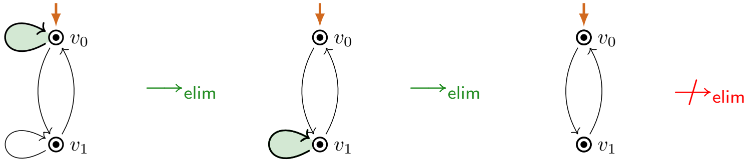
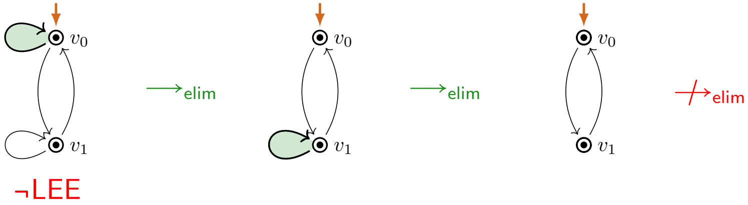
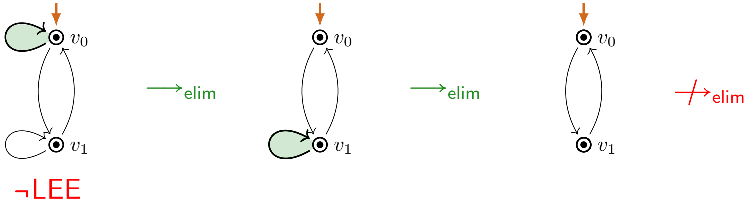
LEE fails


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

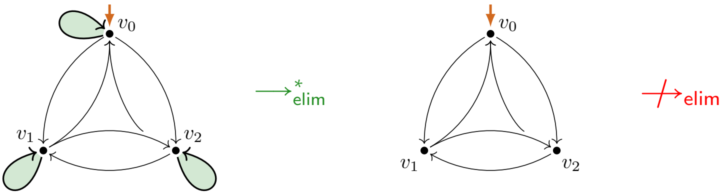
LEE fails


v_0
v_1
$\neg$LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

LEE fails

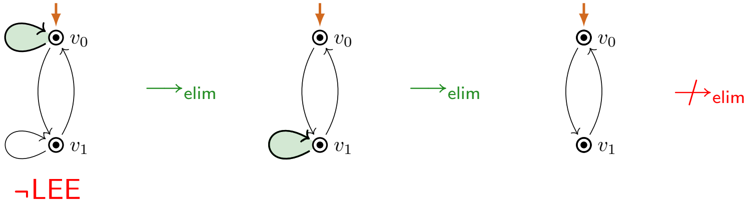


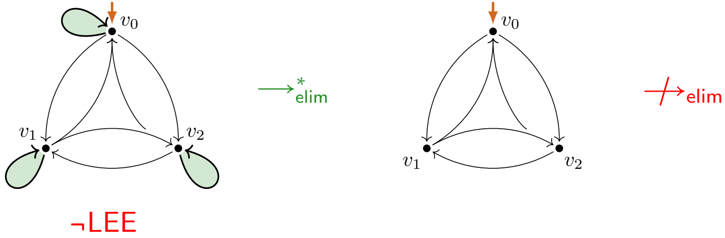
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$
$\neg$LEE



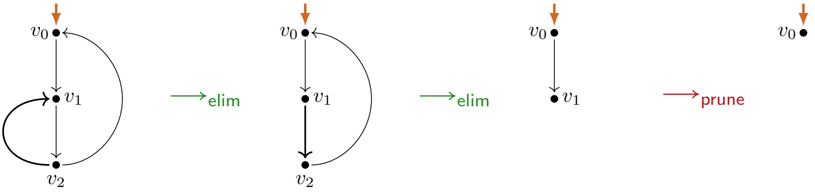
v_0
v_1
v_2
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

LEE fails


v_0
v_1
¬LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

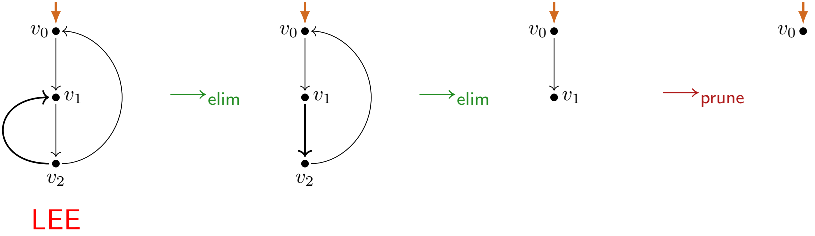

v_0
v_1
v_2
¬LEE
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

LEE holds



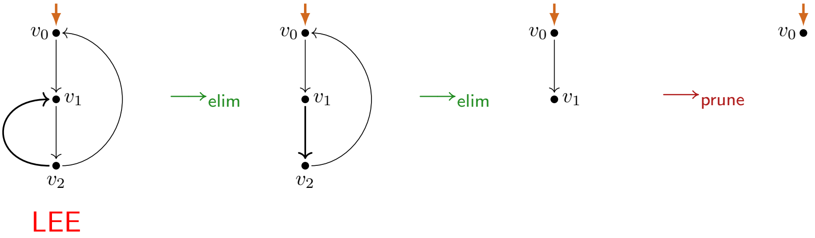
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

LEE holds

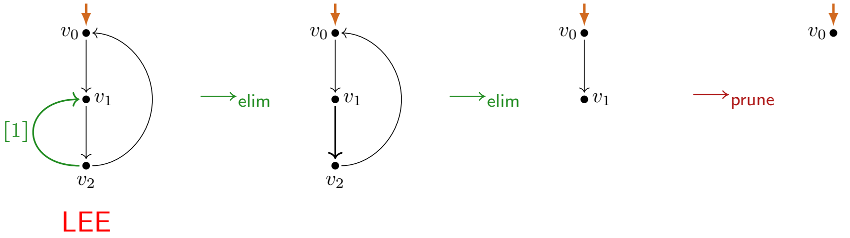


v_0
v_1
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

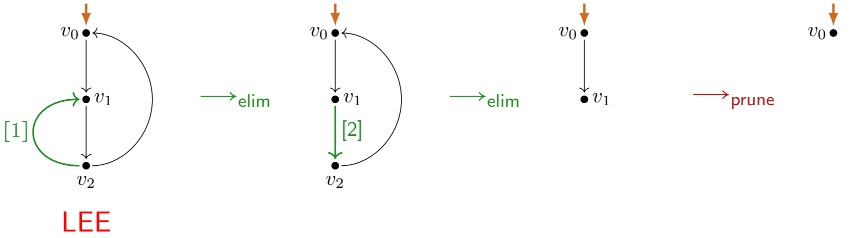
LEE holds / Recording loop elimination



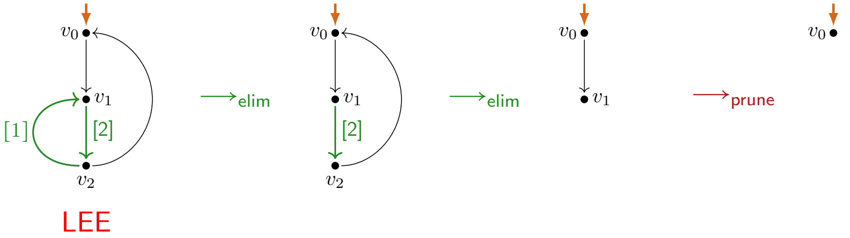
LEE holds / Recording loop elimination



LEE holds / Recording loop elimination



LEE holds / Recording loop elimination



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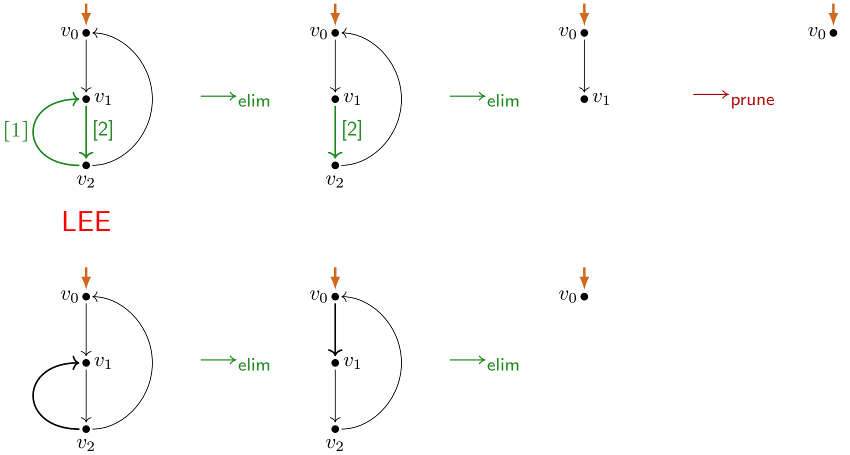
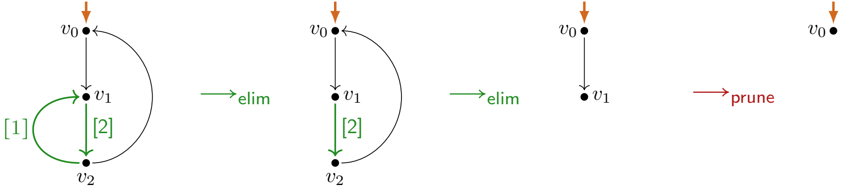
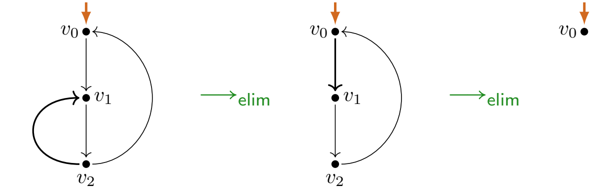


LEE holds / Recording loop elimination

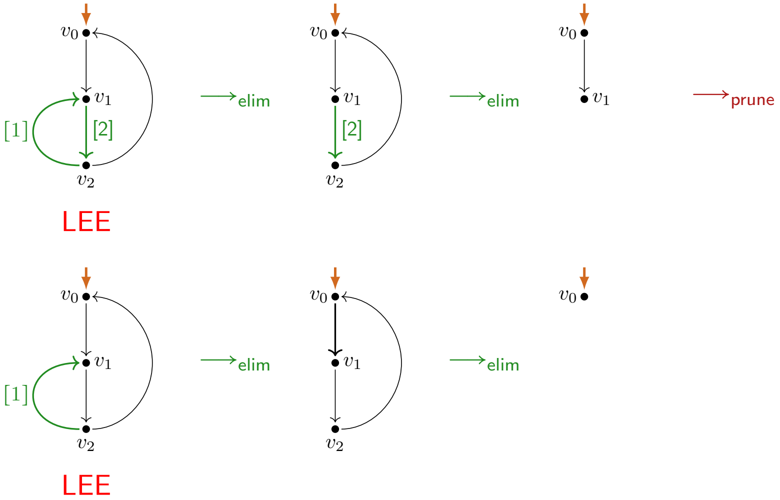


LEE



LEE

LEE holds / Recording loop elimination



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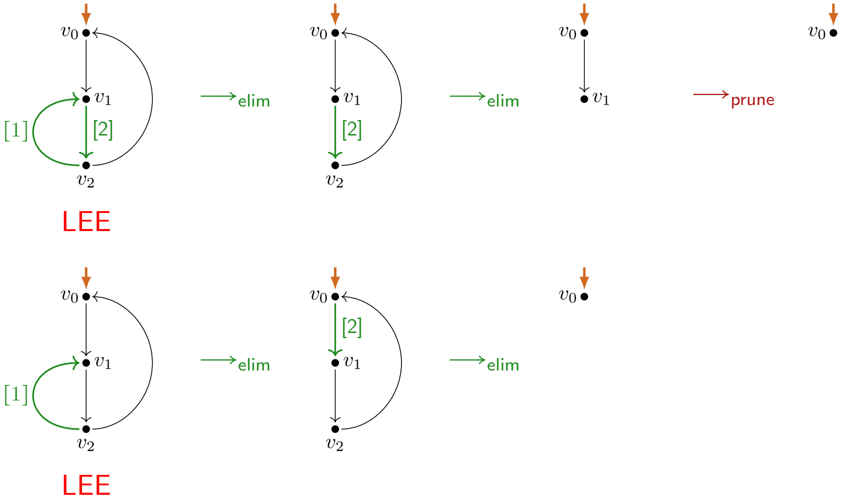

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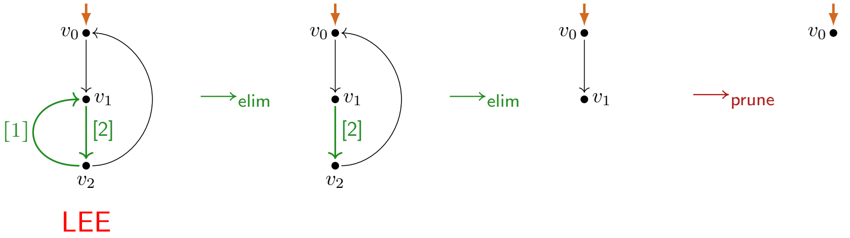
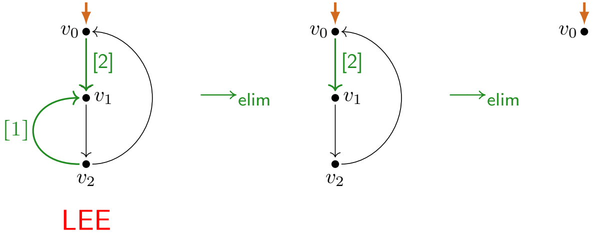
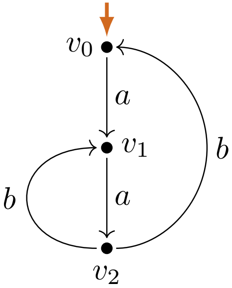


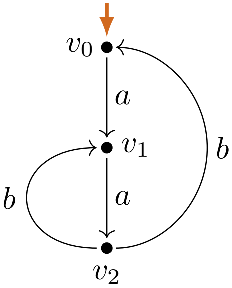


LEE-witness



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:



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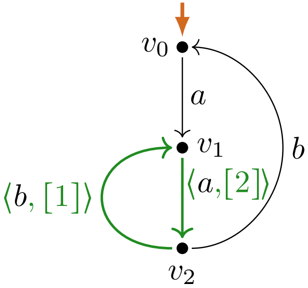

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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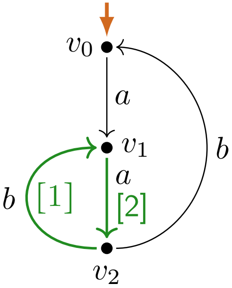
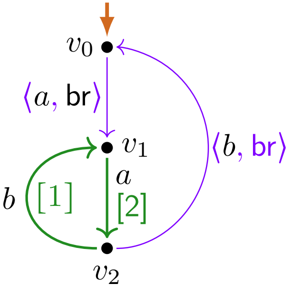


LEE-witness



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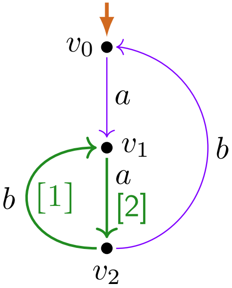
LEE-witness



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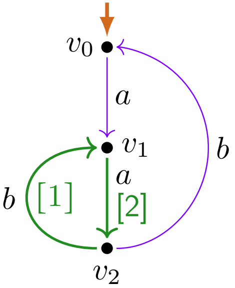
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

Definition

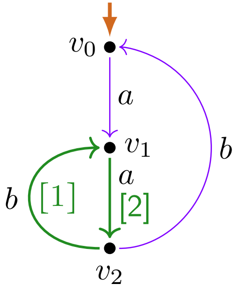
A loop–branch labeling is a LEE-witness, if:

L1.

L2.

L3.

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

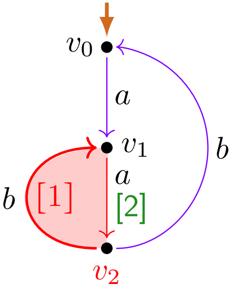
L1.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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A loop–branch labeling is a LEE-witness, if:

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followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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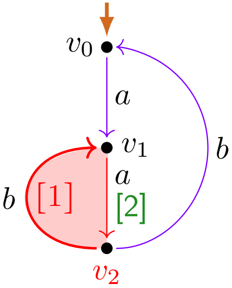

$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\text{br},[>1]})$
is loop subchart

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
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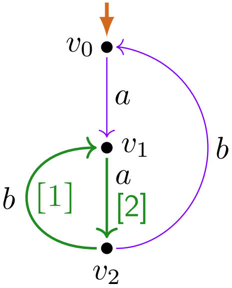
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

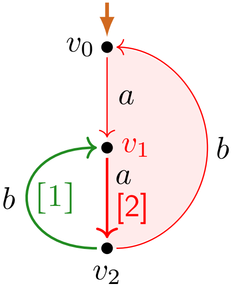
L1.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



$\mathcal{L}(v_1, \rightarrow_{[2]}, \rightarrow_{\mathsf{br},[>2]})$

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced by entry steps $\rightarrow_{[n]}$ from v followed by branch steps $\rightarrow_{\mathsf{br}}$ or entry steps $\rightarrow_{[m]}$ with $m > n$, until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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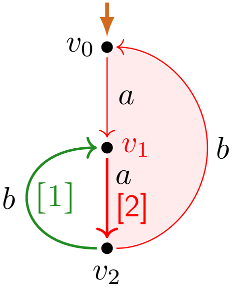


$\mathcal{L}(v_1, \to_{[2]}, \to_{\mathsf{br},[>2]})$
is loop subchart

Definition

A loop–branch labeling is a LEE-witness, if:

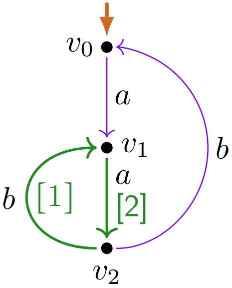
L1. $\forall n \in \mathbb{N} \forall v \in V \Big(v \to_{[n]} \Rightarrow \mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br},[>n]})$ is a loop subchart $\Big)$.

L2.

L3.

$\mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br},[>n]}) :=$ subchart induced
by entry steps $\to_{[n]}$ from v
followed by branch steps $\to_{\mathsf{br}}$
or entry steps $\to_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(\begin{array}{r} v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]}) \\ \text{is a loop subchart} \end{array} \right)$.

L2.

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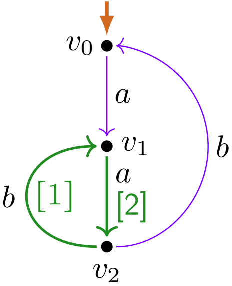
until v is reached again

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

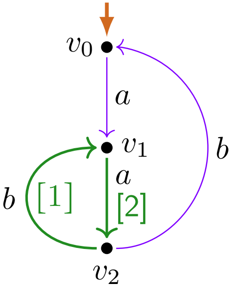
L1. $\forall n \in \mathbb{N} \forall v \in V \left(\begin{array}{c} v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]}) \\ \text{is a loop subchart} \end{array} \right)$.

L2. No infinite $\rightarrow_{\mathsf{br}}$ path from the start vertex.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

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A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \Big(v \rightarrow_{[n]} \;\Rightarrow\; \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$ is a loop subchart $\Big)$.

L2. No infinite $\rightarrow_{\mathsf{br}}$ path from the start vertex.

L3. Overlapping/touching loop subcharts gen. from different vertices have different entry-step levels.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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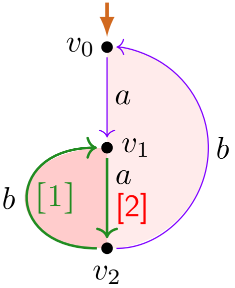

$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

$\mathcal{L}(v_1, \rightarrow_{[2]}, \rightarrow_{\mathsf{br},[>2]})$

Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]}) \text{ is a loop subchart} \right)$.

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$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
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or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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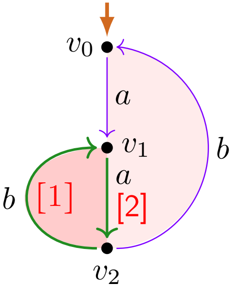


$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\text{br},[>1]})$

$\mathcal{L}(v_1, \rightarrow_{[2]}, \rightarrow_{\text{br},[>2]})$

Definition

A loop–branch labeling is a LEE-witness, if:

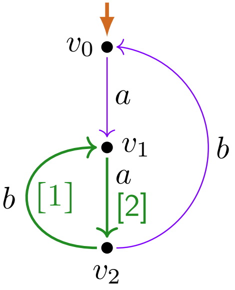
L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]}) \text{ is a loop subchart} \right)$.

L2. No infinite $\rightarrow_{\text{br}}$ path from the start vertex.

L3. $\mathcal{L}(w_i, \rightarrow_{[n_i]}, \rightarrow_{\text{br},[>n_i]})$ for $i \in \{1,2\}$ loop charts $\wedge\, w_1 \neq w_2 \,\wedge\, w_1 \in \mathcal{L}(w_2, \ldots, \ldots) \Longrightarrow n_1 \neq n_2$.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \Big(v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$ is a loop subchart $\Big)$.

L2. No infinite $\rightarrow_{\mathsf{br}}$ path from the start vertex.

L3. $\mathcal{L}(w_i, \rightarrow_{[n_i]}, \rightarrow_{\mathsf{br},[>n_i]})$ for $i \in \{1,2\}$ loop charts $\wedge\, w_1 \neq w_2 \,\wedge\, w_1 \in \mathcal{L}(w_2, \ldots, \ldots) \implies n_1 \neq n_2$.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
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or entry steps $\rightarrow_{[m]}$ with $m > n$,
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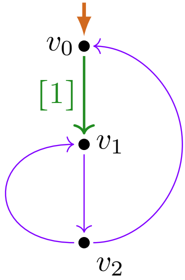
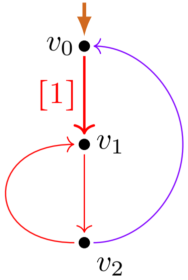
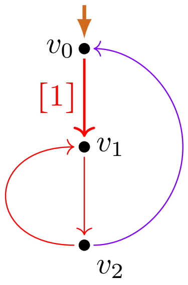
LEE-witness ?



LEE-witness ?



LEE-witness ?



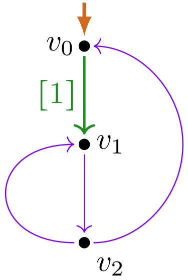
no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

not a loop chart

LEE-witness ?

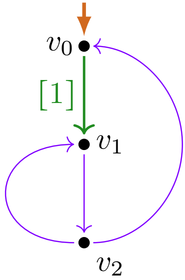


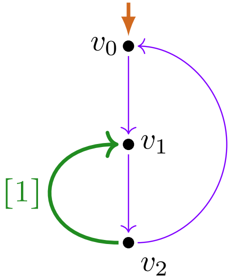
no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\text{br},[>1]})$
not a loop chart

LEE-witness ?



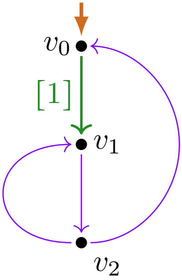


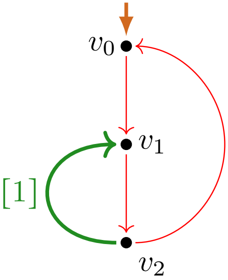
no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
not a loop chart

LEE-witness ?



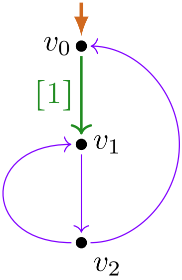


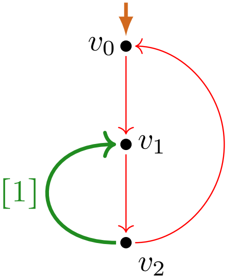
no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
not a loop chart

LEE-witness ?





no!

(L1.) violated:

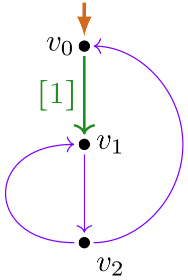
$\mathcal{L}(v_0, \to_{[1]}, \to_{\mathsf{br},[>1]})$
not a loop chart

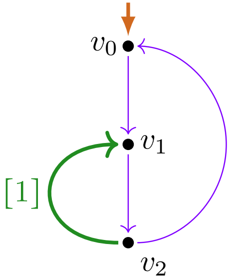
no!

(L2.) violated:

infinite $\to_{\mathsf{br}}$ path
from start vertex

LEE-witness ?





no!

(L1.) violated:

$\mathcal{L}(v_0, \to_{[1]}, \to_{\mathsf{br},[>1]})$

not a loop chart

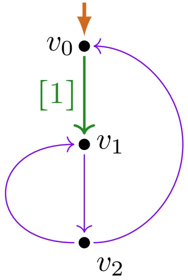
no!

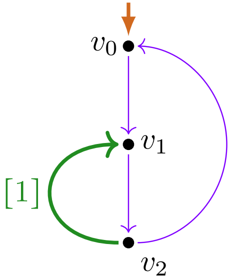
(L2.) violated:

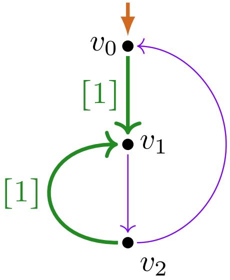
infinite $\to_{\mathsf{br}}$ path

from start vertex

LEE-witness ?







no!

(L1.) violated:

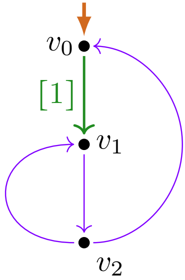
$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
not a loop chart

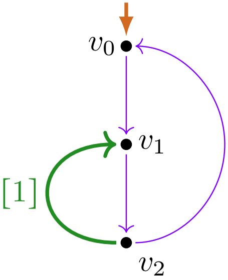
no!

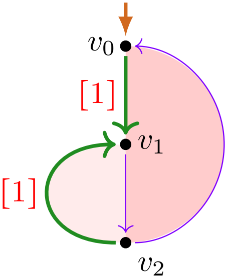
(L2.) violated:

infinite $\rightarrow_{\mathsf{br}}$ path
from start vertex

LEE-witness ?







no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

not a loop chart

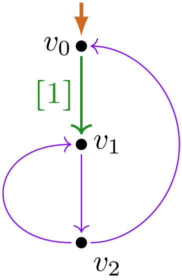
no!

(L2.) violated:

infinite $\rightarrow_{\mathsf{br}}$ path

from start vertex

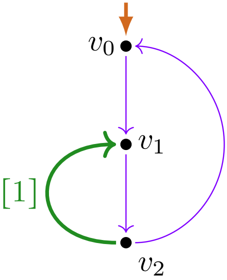
LEE-witness ?



no!

(L1.) violated:

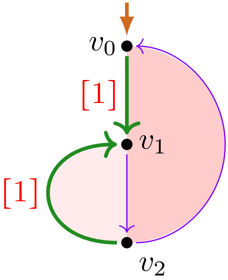
$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
not a loop chart



no!

(L2.) violated:

infinite $\rightarrow_{\mathsf{br}}$ path
from start vertex

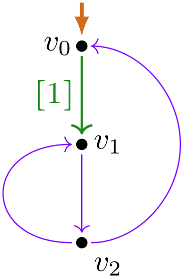


no!

(L3.) violated:

overlapping loop charts
have same level

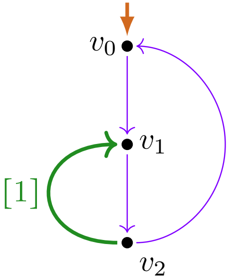
LEE-witness ?



no!

(L1.) violated:

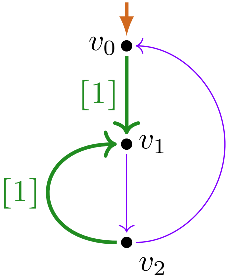
$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
not a loop chart



no!

(L2.) violated:

infinite $\rightarrow_{\mathsf{br}}$ path
from start vertex

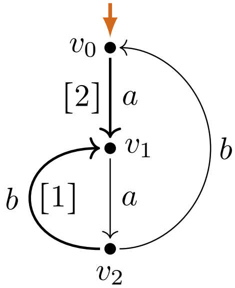


no!

(L3.) violated:

overlapping loop charts
have same level

LEE-witness ?



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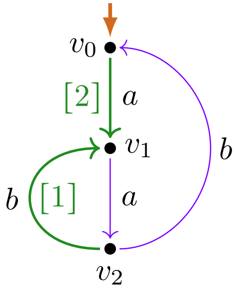

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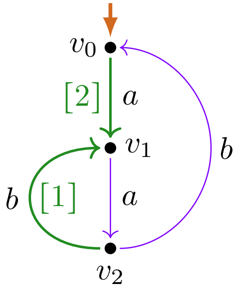

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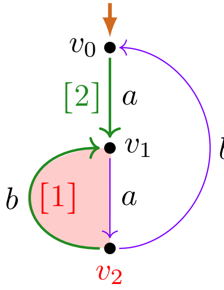

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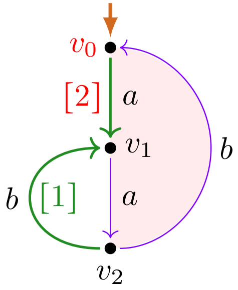

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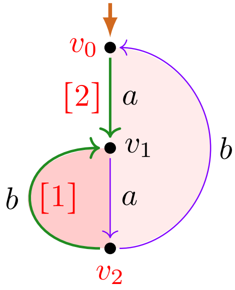

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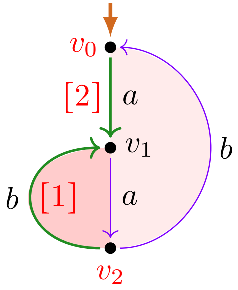

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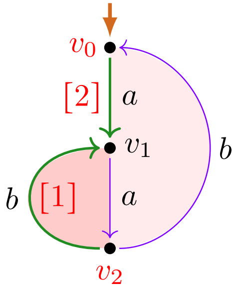

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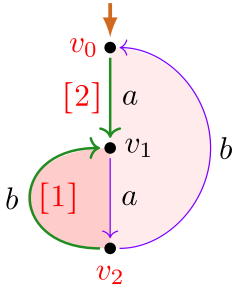
Layered LEE-witness



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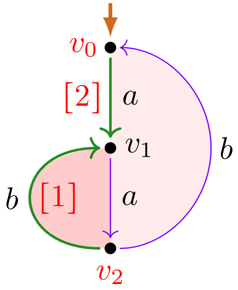

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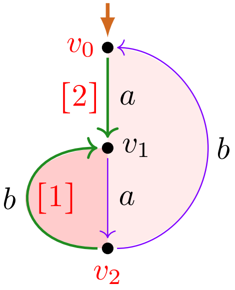

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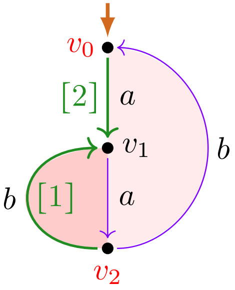

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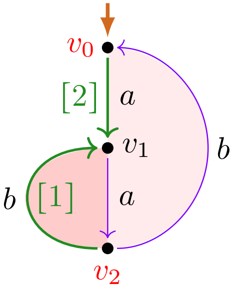

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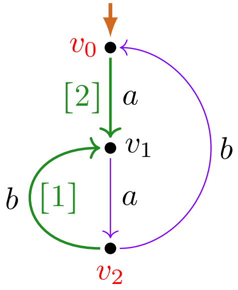

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LEE versus LEE-witness

Theorem

For every process graph G :

$$\mathsf{LEE}(G) \iff G \text{ has a } \mathsf{LEE}\text{-witness.}$$

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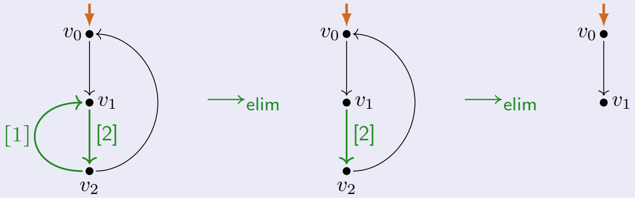
Proof (Idea).

$\Rightarrow$: record loop elimination

$\Leftarrow$: carry out loop-elimination as indicated in the LEE-witness, in *inside–out* direction, e.g.:



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For every process graph G the following are equivalent:

(i) LEE(G).

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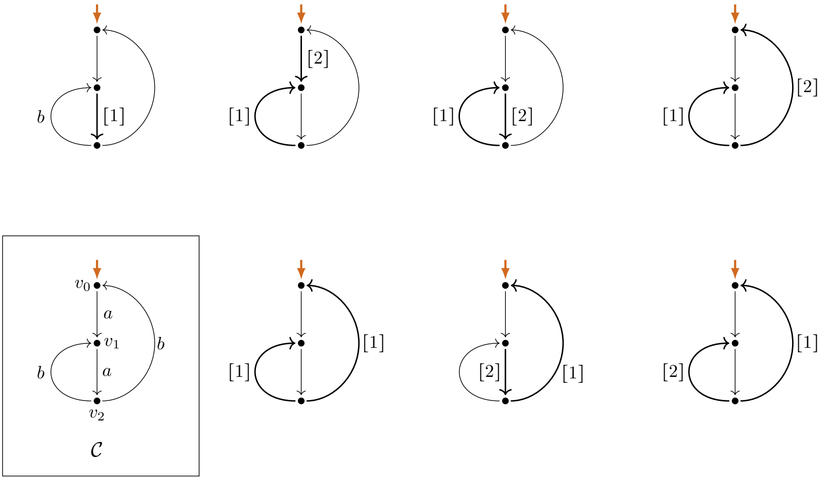
7 LEE-witnesses



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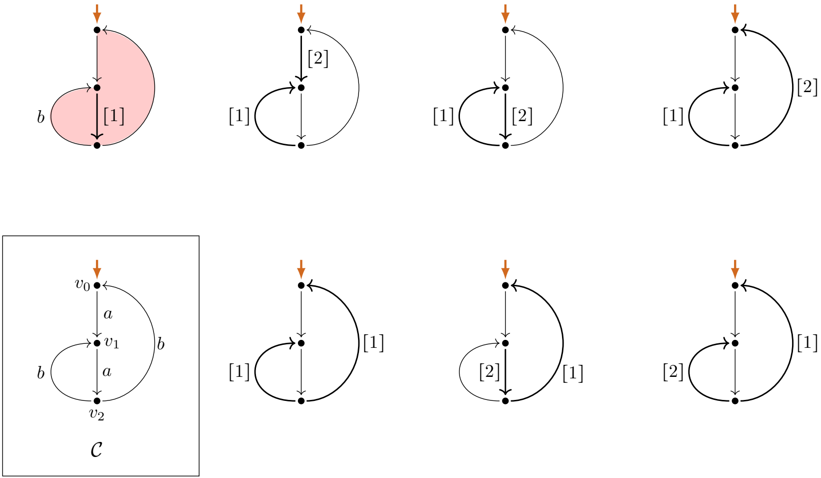

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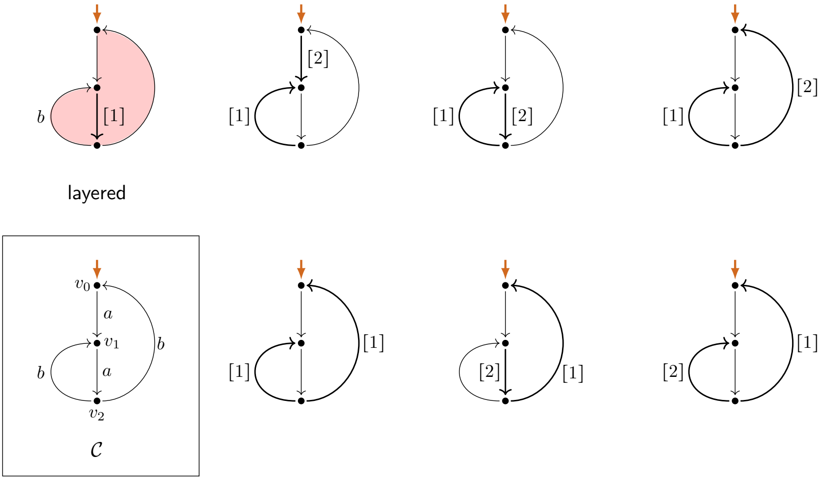

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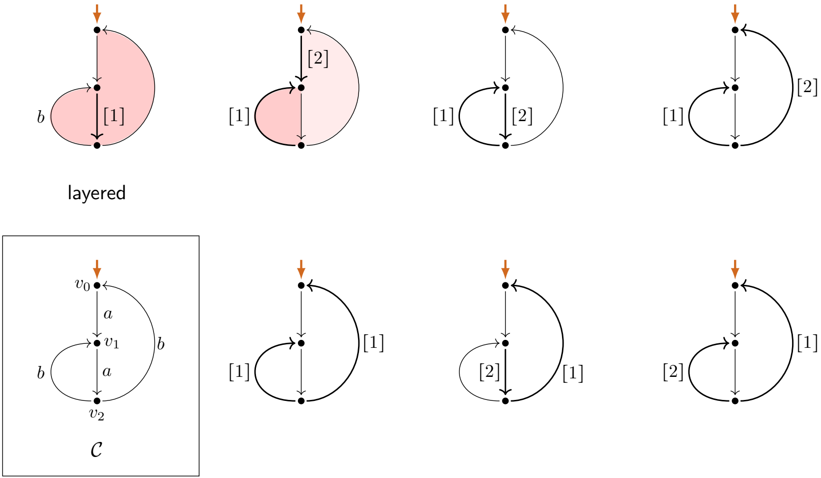

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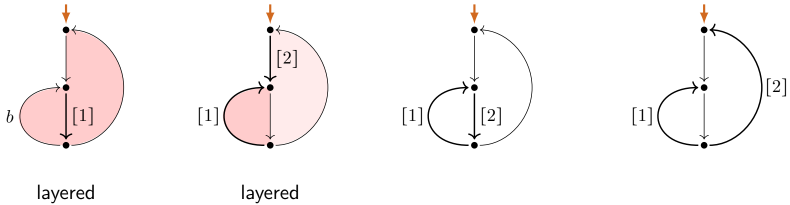

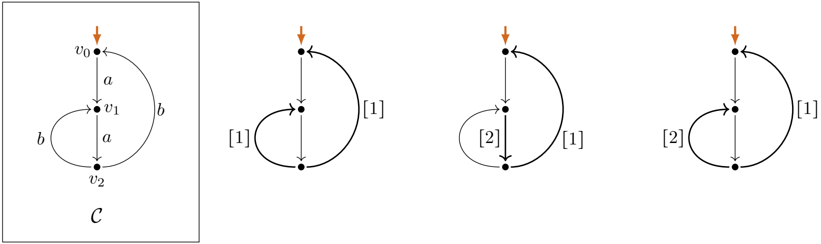

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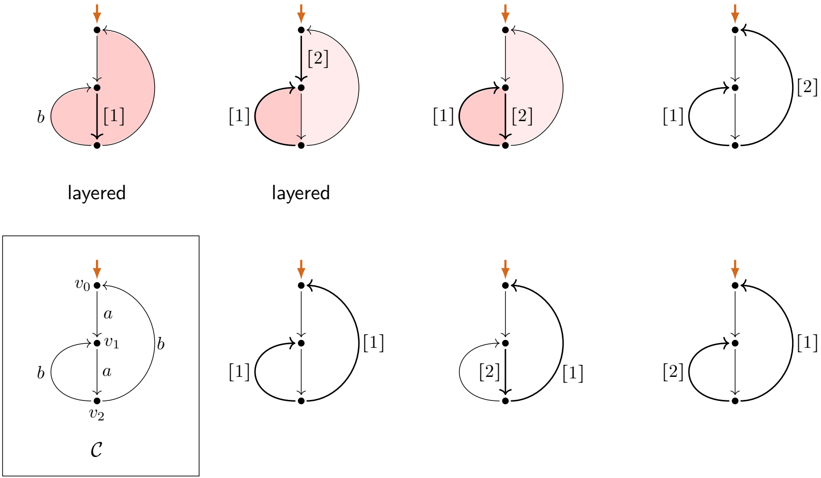

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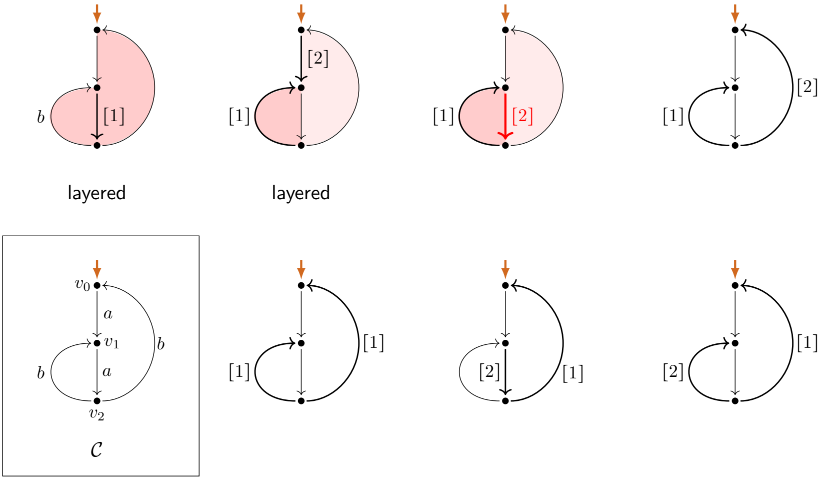

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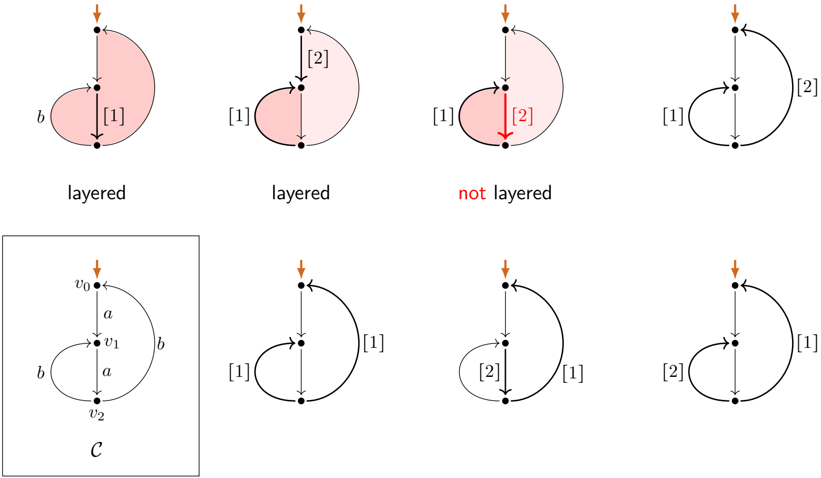

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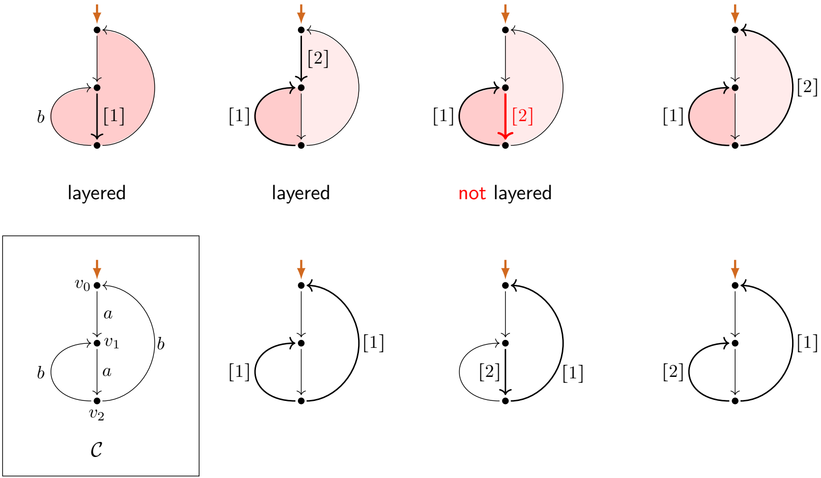

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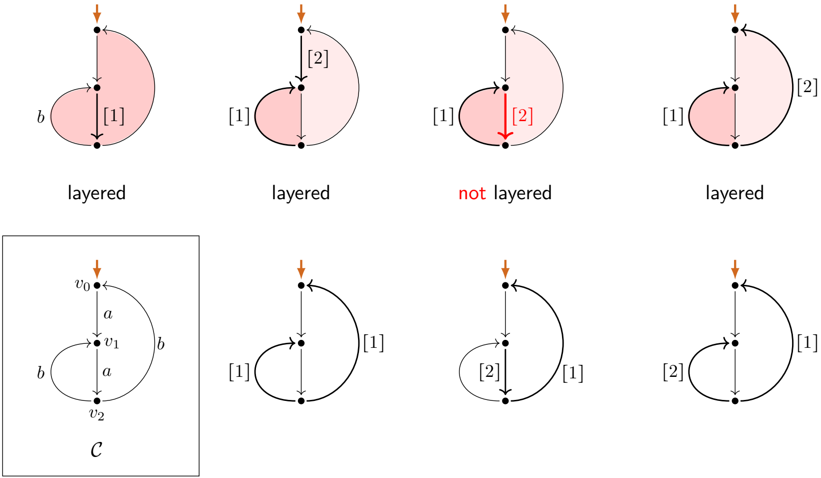

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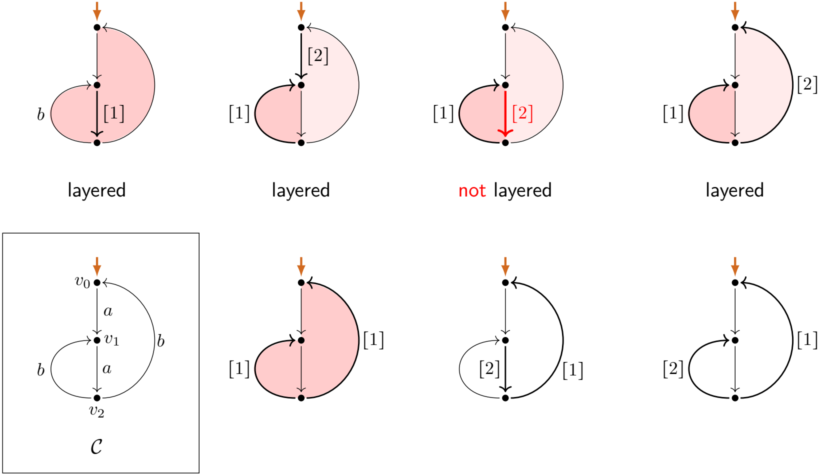

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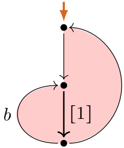
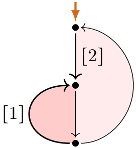


layered



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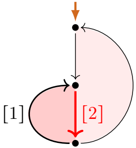

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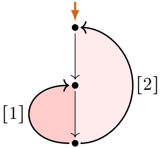

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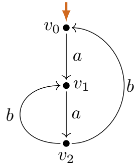
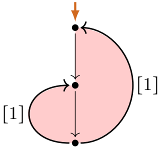


$\mathcal{C}$



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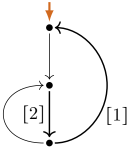
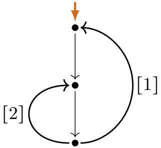
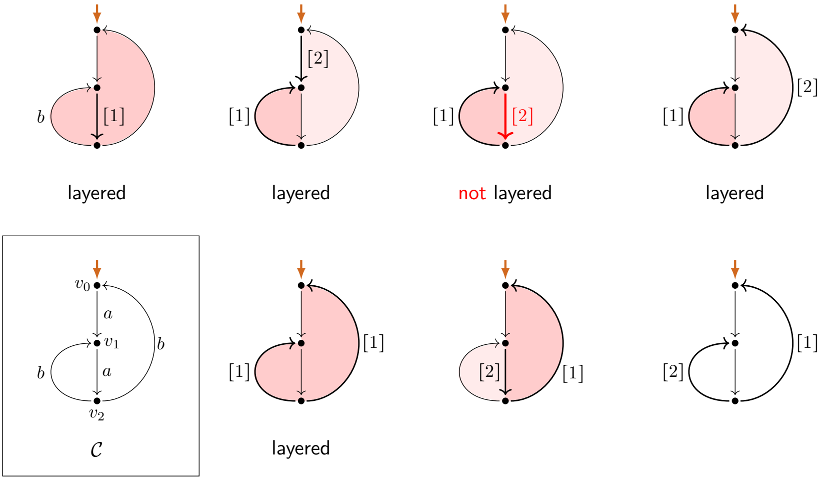
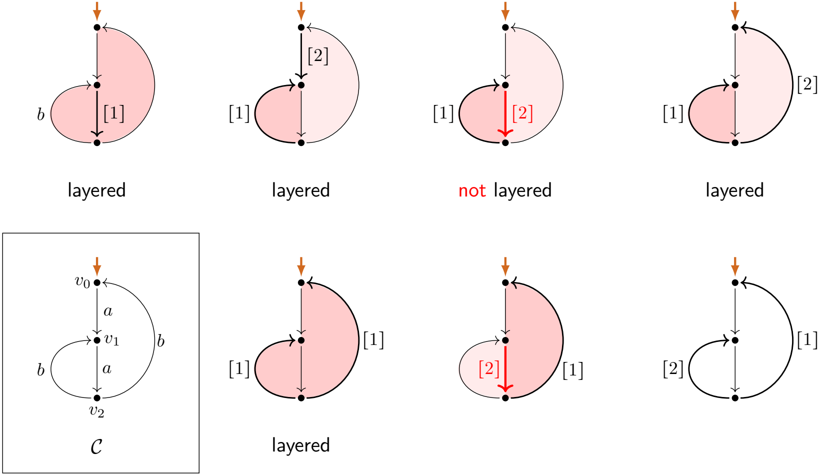




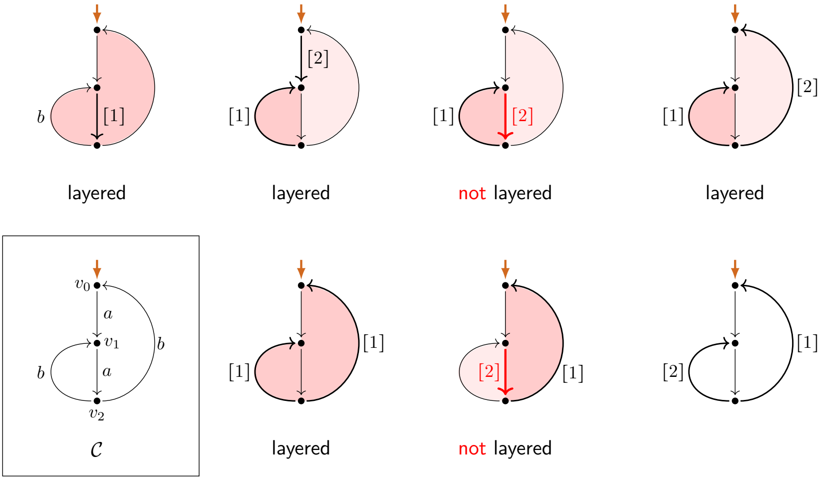
7 LEE-witnesses



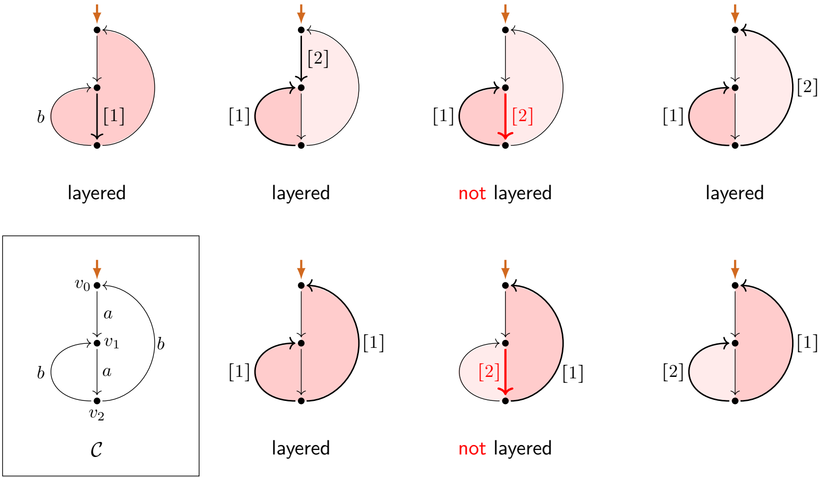
7 LEE-witnesses



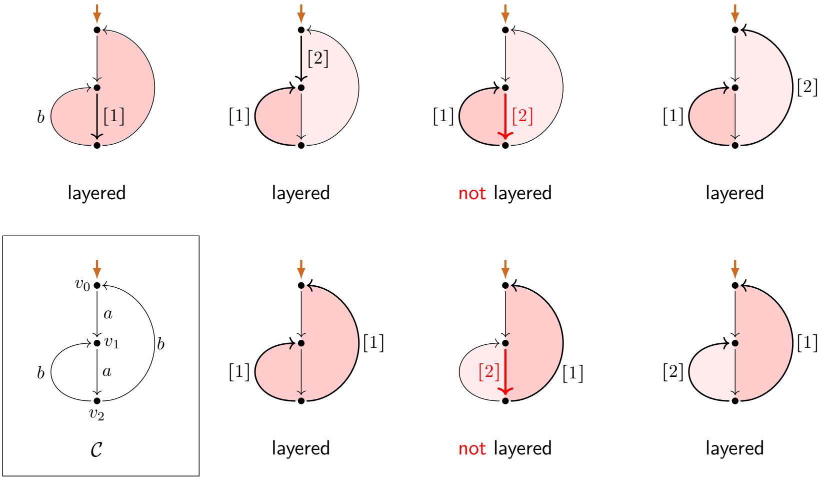
7 LEE-witnesses



7 LEE-witnesses



7 LEE-witnesses



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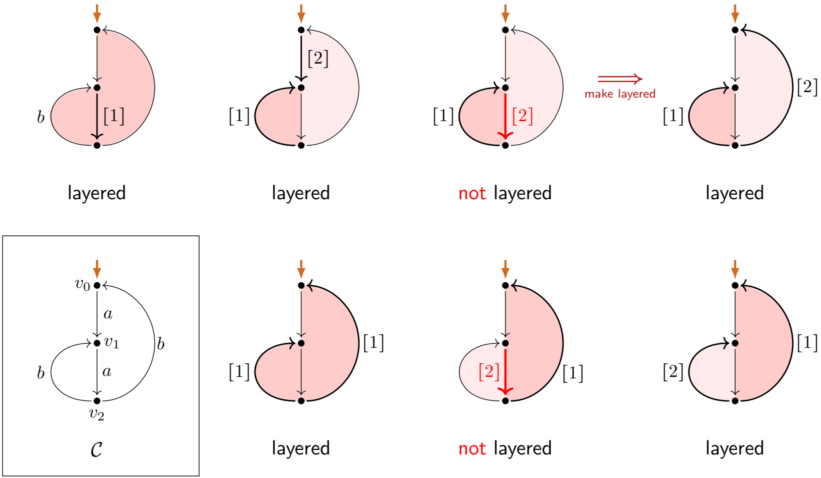
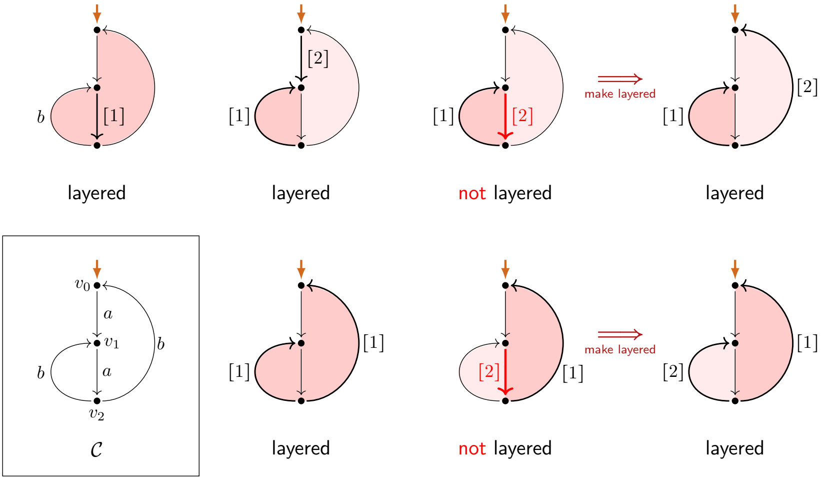
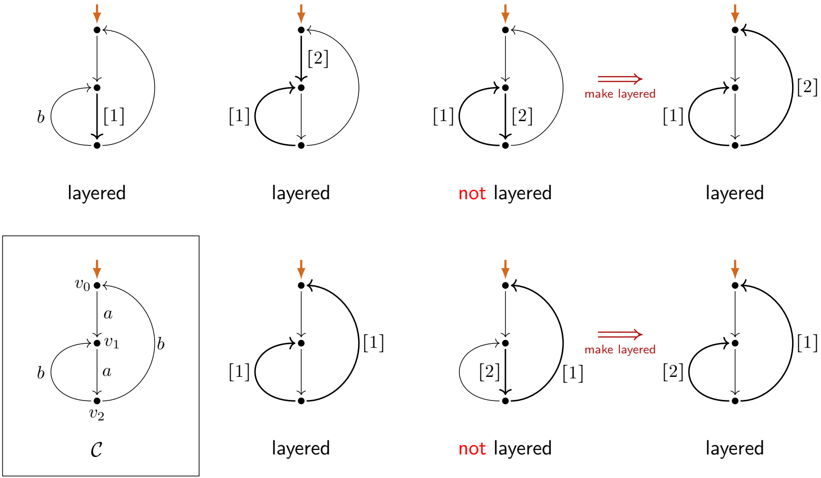


7 LEE-witnesses



7 LEE-witnesses



LEE under bisimulation?

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.

LEE under bisimulation

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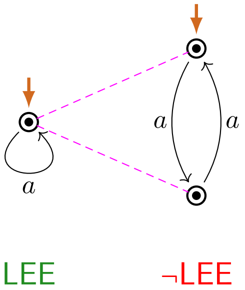
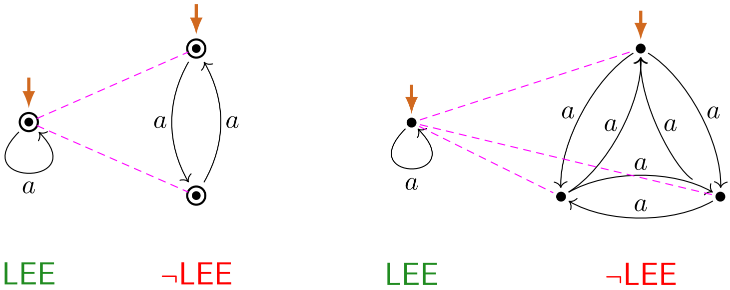


LEE under bisimulation

Observation

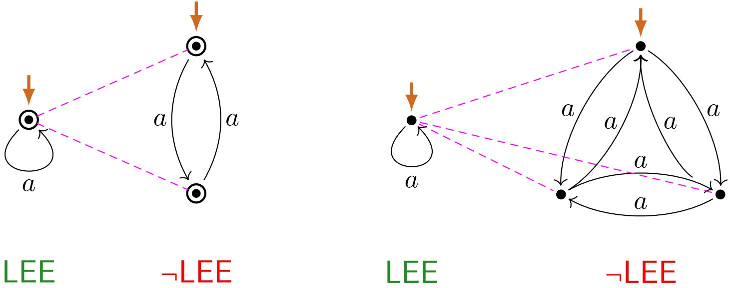
- LEE is not invariant under bisimulation.



LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.
- LEE is not preserved by converse functional bisimulation.



LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \rightarrow\!\!\!\!\!\underline{}\; G_2 \implies \mathsf{LEE}(G_2)\ .$$

LEE under functional bisimulation

Lemma

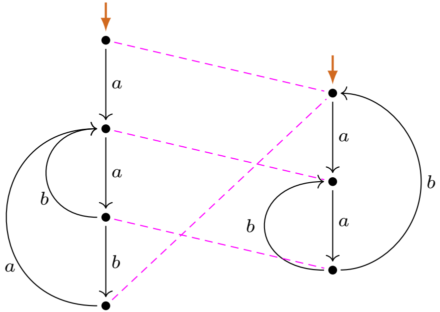
(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

Proof (Idea).

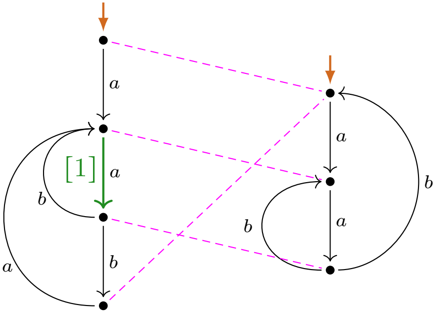
Use loop elimination in G_1 to carry out loop elimination in G_2.

Collapsing LEE-witnesses



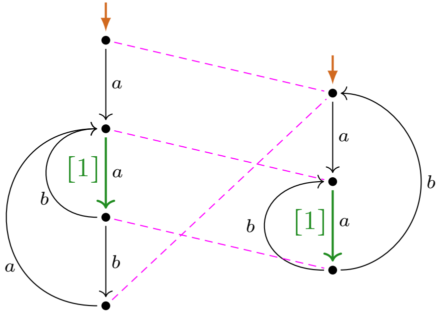
$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses



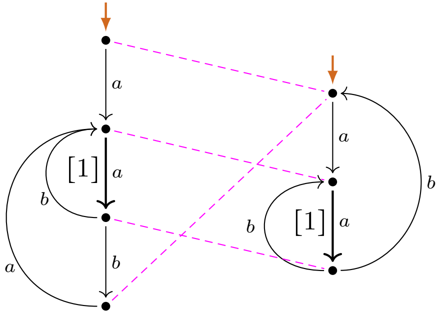
$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses



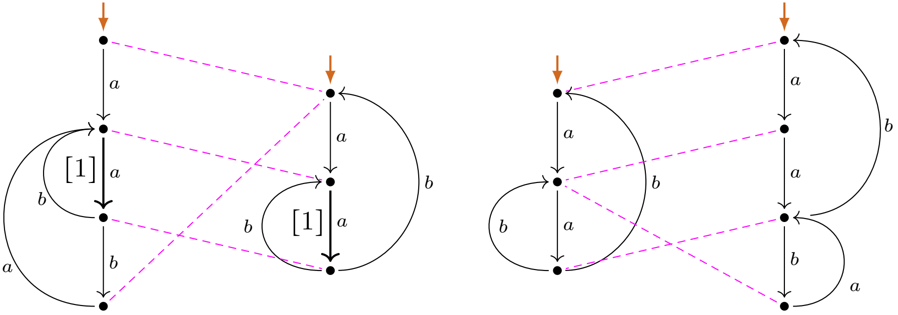
$$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses



$[\![a(a(b+ba))^* 0]\!]_{\boldsymbol{P}}$

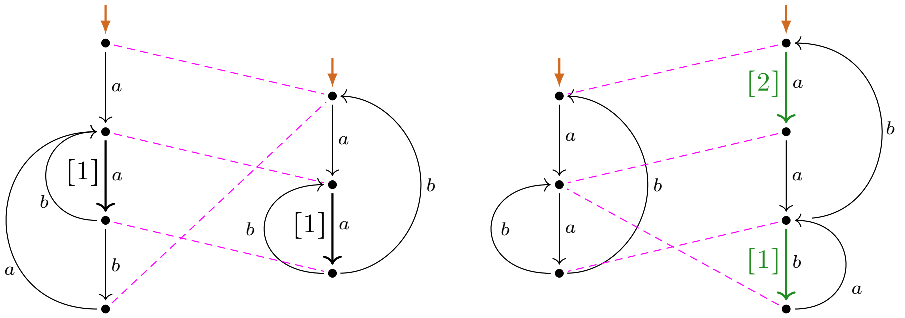
Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

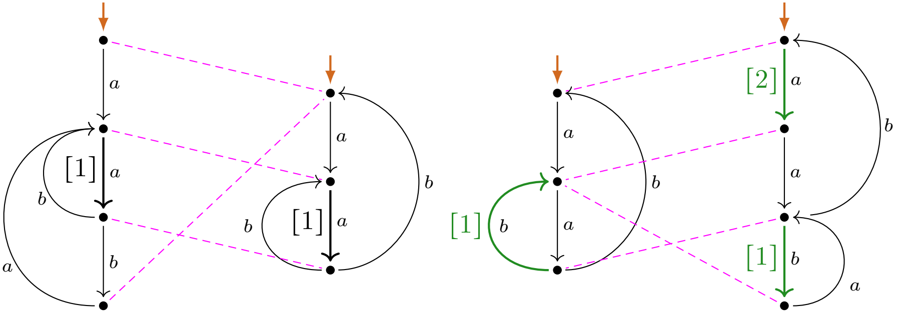
Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

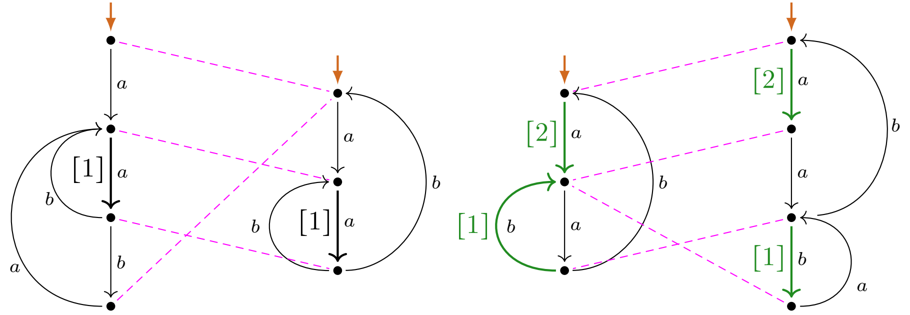
$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

Collapsing LEE-witnesses



$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$ $[\![(aa(ba)^{*}b)^{*}0]\!]_{\boldsymbol{P}}$

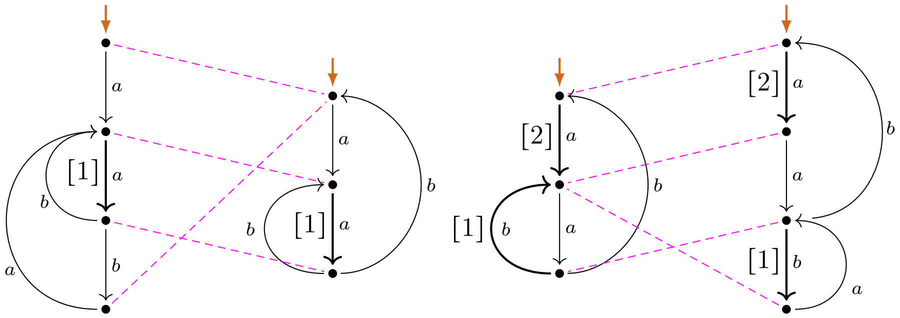
Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2) \; .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

LEE under functional bisimulation / bisimulation collapse

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\text{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \text{LEE}(G_2)\ .$$

(ii) LEE *is preserved from a process graph to its bisimulation collapse:*

$$\text{LEE}(G) \wedge C \text{ is bisimulation collapse of } G \implies \text{LEE}(C)\ .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

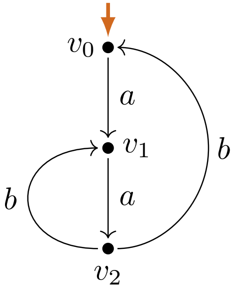
Readback

Lemma

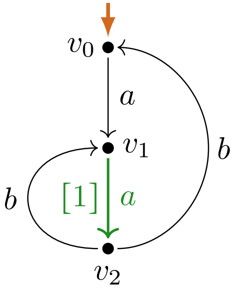
Process graphs with LEE are $[\![\cdot]\!]_{P}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathsf{Reg}(A) \left(G \leftrightarrow [\![e]\!]_{P} \right) .$$

Readback from layered LEE-witness (example)

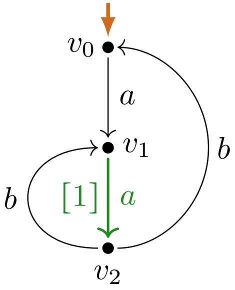


Readback from layered LEE-witness (example)



layered
LEE-witness

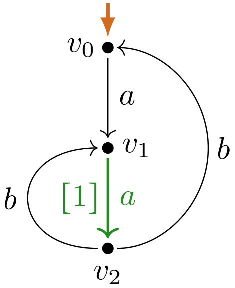
Readback from layered LEE-witness (example)



layered
LEE-witness

$$
\begin{aligned}
s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

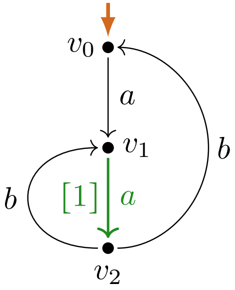
Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

Readback from layered LEE-witness (example)

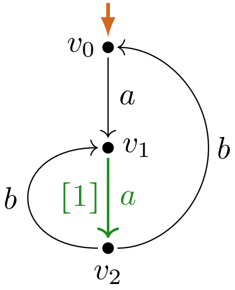


layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

Readback from layered LEE-witness (example)



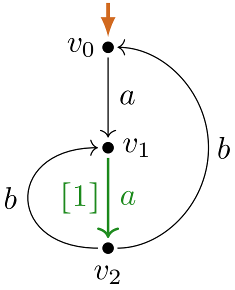
layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$s(v_2, v_1) = \quad 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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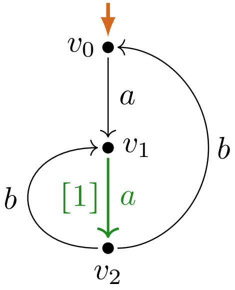
$$s(v_1, v_1) = 1$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

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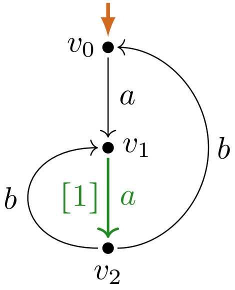


layered LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

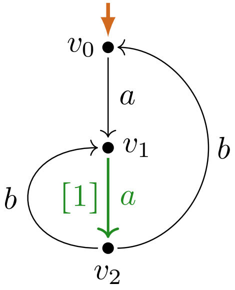
$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

$$s(v_1, v_1) = 1$$

$$s(v_0, v_1) = 0^* \cdot a \cdot s(v_1, v_1)$$

$$= 0^* \cdot a \cdot 1$$

Readback from layered LEE-witness (example)



layered LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

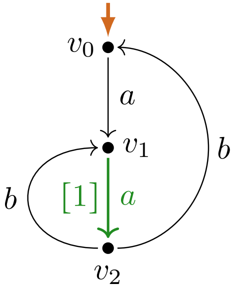
$$s(v_1, v_1) = 1$$

$$s(v_0, v_1) = 0^* \cdot a \cdot s(v_1, v_1)$$

$$= 0^* \cdot a \cdot 1$$

$$=_{\mathsf{Mil}^-} a$$

Readback from layered LEE-witness (example)



layered
LEE-witness

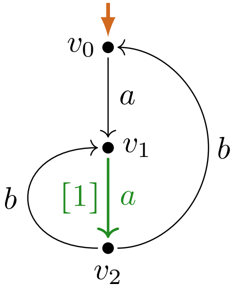
$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$\begin{aligned} s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\ &=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \end{aligned}$$

$$\begin{aligned} s(v_1, v_1) &= 1 \\ s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\ &= 0^* \cdot a \cdot 1 \\ &=_{\mathsf{Mil}^-} a \end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

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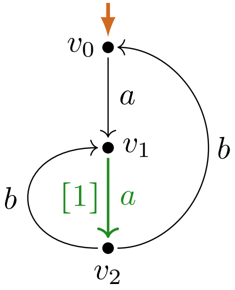
\end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$\begin{aligned}
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
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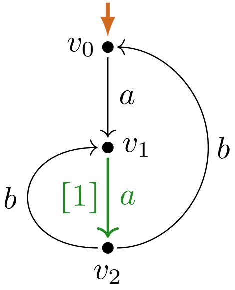
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}$$

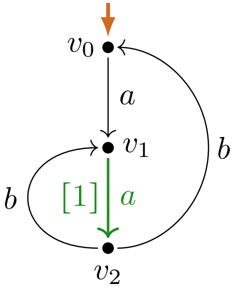
Readback from layered LEE-witness (example)



layered
LEE-witness

$$
\begin{aligned}
s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\[1em]
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

Readback from layered LEE-witness (example)



layered
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$$
\begin{aligned}
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&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathrm{Reg}(A) \left(G \leftrightarrow [\![e]\!]_{P} \right) .$$

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- $(a \cdot (1 + b))^*$

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Characterization of expressibility$^{\mathbf{1r}\backslash\star}$ modulo $\underline{\leftrightarrow}$

Theorem

For every process graph G with bisimulation collapse C the following are equivalent:

(i) *G is $[\![\cdot]\!]_{P}^{\mathbf{1r}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$.*

(ii) LEE(C).

(iii) *C has a* LEE*-witness.*

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Milners characterization question:

Q1. Which structural property of finite process graphs characterizes $[\![\cdot]\!]_{P}$-expressibility modulo $\underline{\leftrightarrow}$?

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Milners characterization question restricted:

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Milners characterization question restricted, and adapted:

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Answering Milners characterization question restricted, and adapted:

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- The loop-existence and elimination property LEE.

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Answering Milners characterization question restricted, and adapted:

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- The loop-existence and elimination property LEE.

Also yields: efficient decision method of $\llbracket\cdot\rrbracket_{P}^{\mathbf{1r}\backslash\star}$-expressibility modulo $\underline{\leftrightarrow}$.

Structure constrained finite process graphs

graphs with LEE / a (layered) LEE-witness

Benefits of the class of process graphs with LEE:

- is closed under $\Rightarrow$
- forth-/back-correspondence with 1-return-less regular expressions

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$\subsetneq$ graphs whose collapse satisfies LEE

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loop–exit palm trees $\subsetneqq$ $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressible graphs
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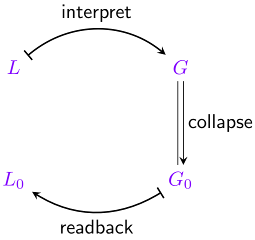
Application to Milner's questions yields partial results:

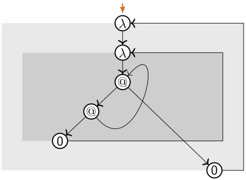
Q1: characterization/efficient decision of $[\![\cdot]\!]_{P}^{\not{1r}\backslash\star}$-expressibility modulo $\underline{\leftrightarrow}$

Q2: alternative compl. proof of Mil on 1-return-less expressions (C/DN/L)

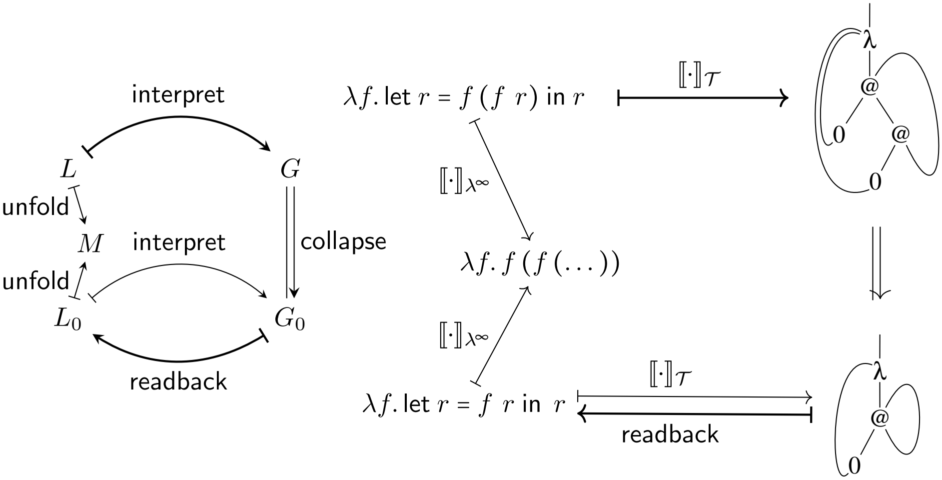
Maximal sharing of functional programs

(joint work with Jan Rochel)





maximal sharing: example (fix)



maximal sharing: the method

$$L \xrightarrow{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G}$$

1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L as:
   a. higher-order term graph
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$

maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G} \longmapsto G$$

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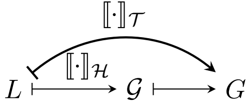

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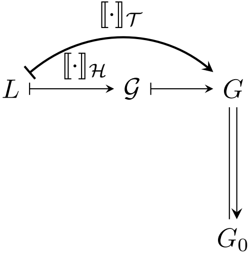

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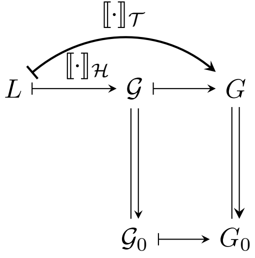

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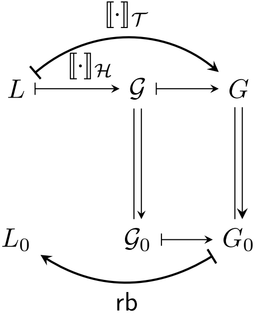

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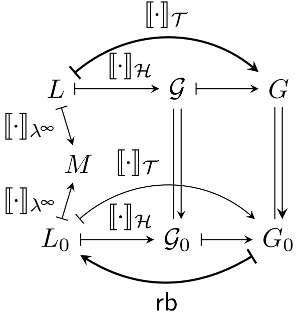
3. readback rb
   of f-o term graph G_0
   yielding program $L_0 = \text{rb}(G_0)$.

maximal sharing: the method



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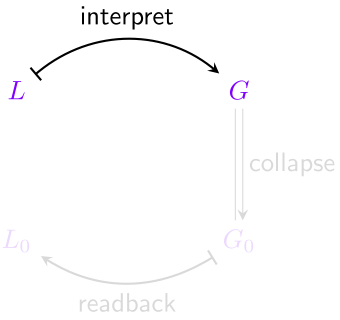
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$
   b. first-order term graph $G = [\![L]\!]_{\mathcal{T}}$
2. bisimulation collapse $\downarrow\!\!\downarrow$
   of f-o term graph G into G_0
3. readback rb
   of f-o term graph G_0
   yielding program $L_0 = \mathsf{rb}(G_0)$.

interpretation



running example

instead of:

$\lambda f.\,\mathsf{let}\ r = f\,(f\,r)\ \mathsf{in}\ r \quad\longmapsto_{\text{max-sharing}}\quad \lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r$

we use:

$\lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,(f\,r\,x)\,x\ \mathsf{in}\ r \quad\longmapsto_{\text{max-sharing}}\quad \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$

$L \quad\longmapsto_{\text{max-sharing}}\quad L_0$

graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$

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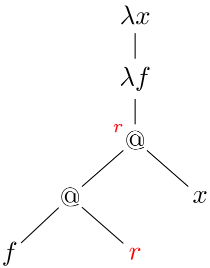
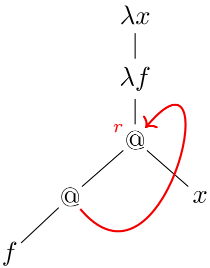
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



syntax tree

graph interpretation (example 1)

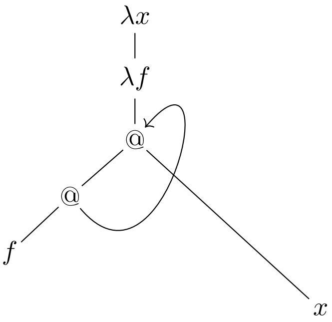
$L_0 = \lambda x. \lambda f.$ let $r = f\, r\, x$ in r



syntax tree (+ recursive backlink)

graph interpretation (example 1)

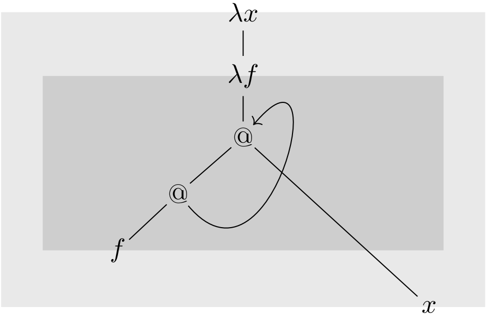
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



syntax tree (+ recursive backlink)

graph interpretation (example 1)

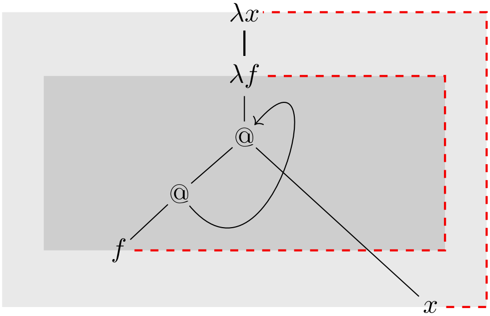
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



syntax tree (+ recursive backlink, + scopes)

graph interpretation (example 1)

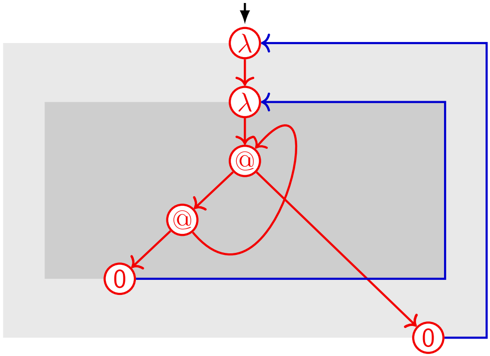
$L_0 = \lambda x. \lambda f. \text{let } r = f\,r\,x \text{ in } r$



syntax tree (+ recursive backlink, + scopes, + binding links)

graph interpretation (example 1)

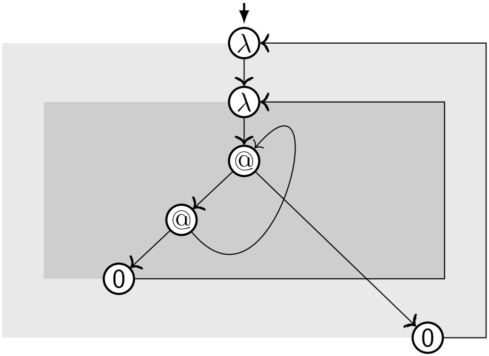
$L_0 = \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

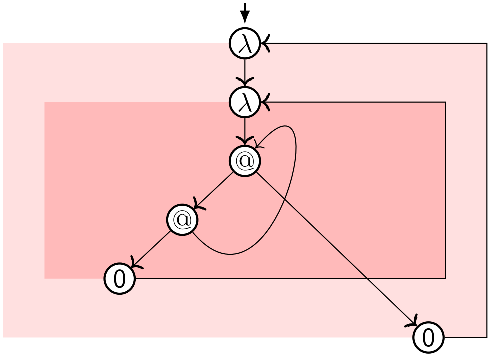
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

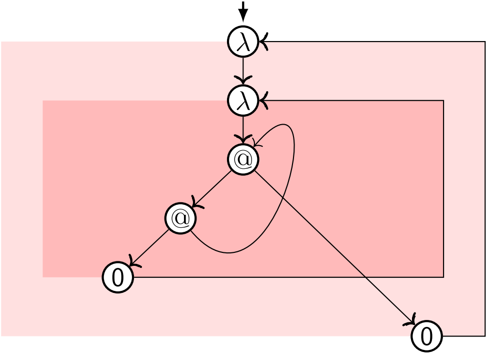
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph (+ scope sets)

graph interpretation (example 1)

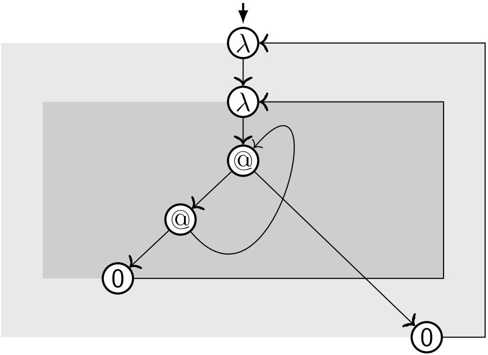
$L_0 = \lambda x.\,\lambda f.\,\text{let } r = f\,r\,x \text{ in } r$



higher-order term graph (with scope sets, Blom [2003])

graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



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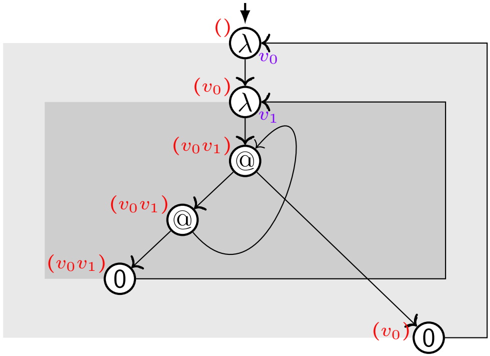
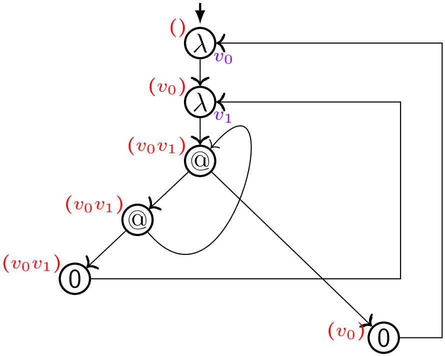


higher-order term graph (with scope sets, + abstraction-prefix function)

graph interpretation (example 1)

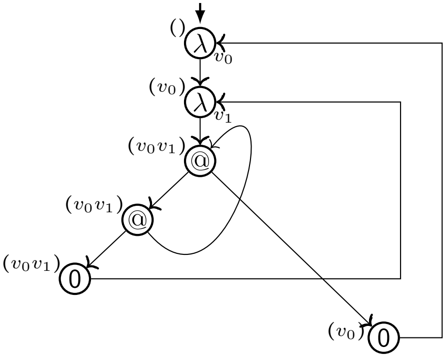
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



higher-order term graph (with abstraction-prefix function)

graph interpretation (example 1)

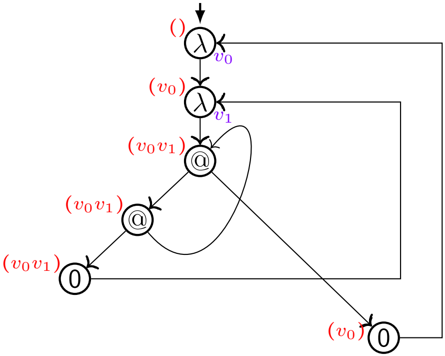
$L_0 = \lambda x.\,\lambda f.\,\text{let } r = f\,r\,x \text{ in } r$



λ-higher-order-term-graph $[\![L_0]\!]_{\mathcal{H}}$

graph interpretation (example 1)

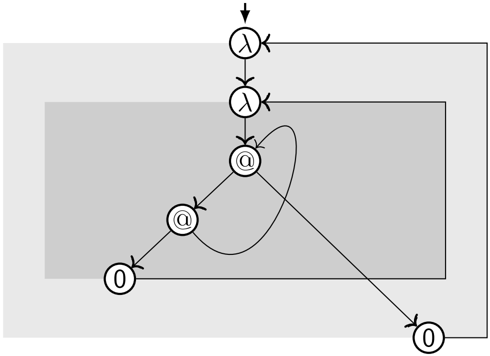
$L_0 = \lambda x. \lambda f. \,\mathsf{let}\; r = f\, r\, x \;\mathsf{in}\; r$



first-order term graph (+ abstraction-prefix function)

graph interpretation (example 1)

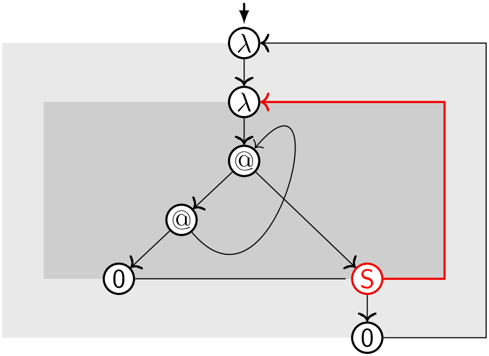
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

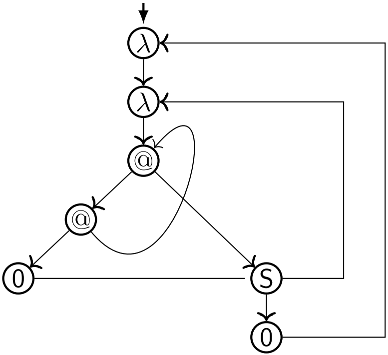
$L_0 = \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\ r\ x\ \mathsf{in}\ r$



first-order term graph with scope vertices with backlinks (+ scope sets)

graph interpretation (example 1)

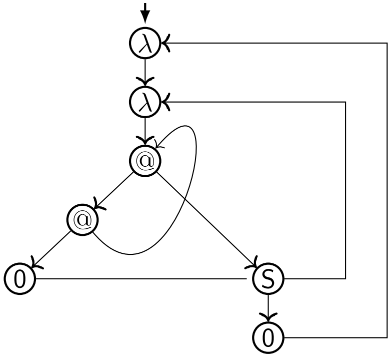
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with scope vertices with backlinks

graph interpretation (example 1)

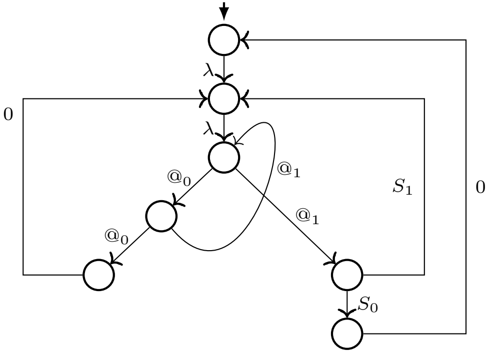
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$

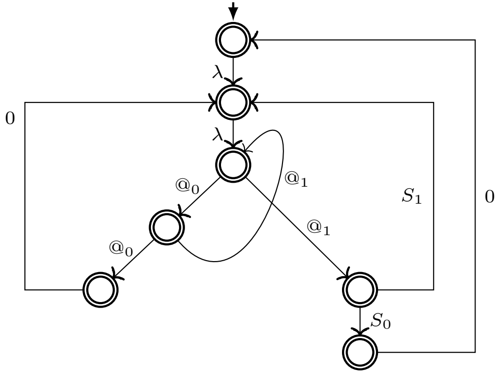
graph interpretation (example 1)

$L_0 = \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,r\,x\ \mathsf{in}\ r$



graph interpretation (example 1)

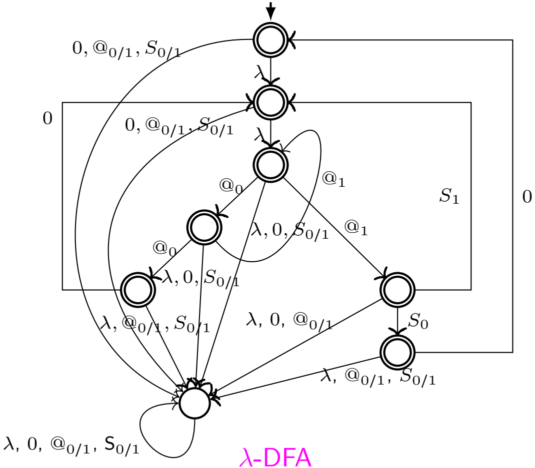
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



λ-NFA

graph interpretation (example 1)

$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



λ-DFA

graph interpretation (example 2)

$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$

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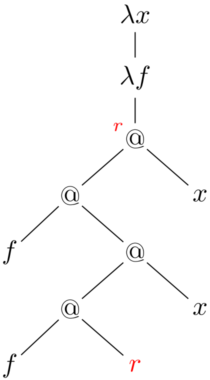
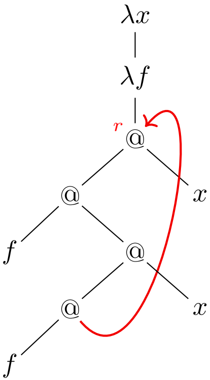


syntax tree

graph interpretation (example 2)

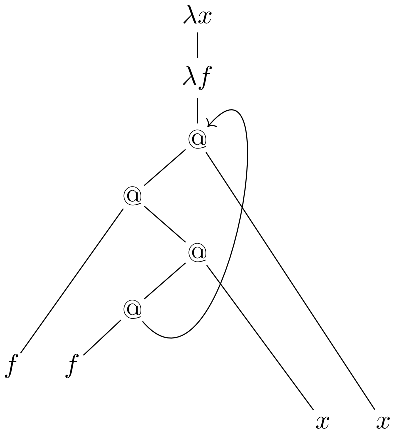
$L = \lambda x.\,\lambda f.\,\text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink)

graph interpretation (example 2)

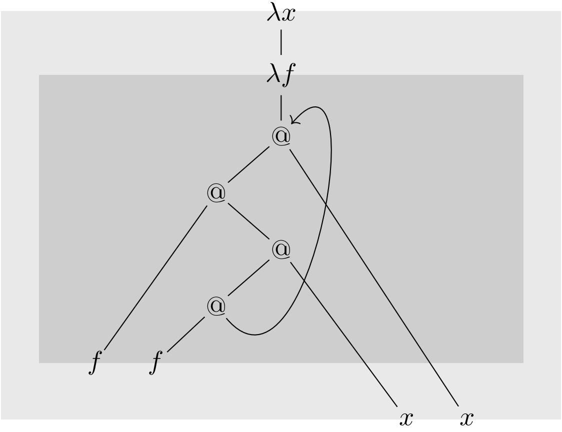
$L = \lambda x.\, \lambda f.\, \mathsf{let}\ r = f\,(f\,r\,x)\,x\ \mathsf{in}\ r$



syntax tree (+ recursive backlink)

graph interpretation (example 2)

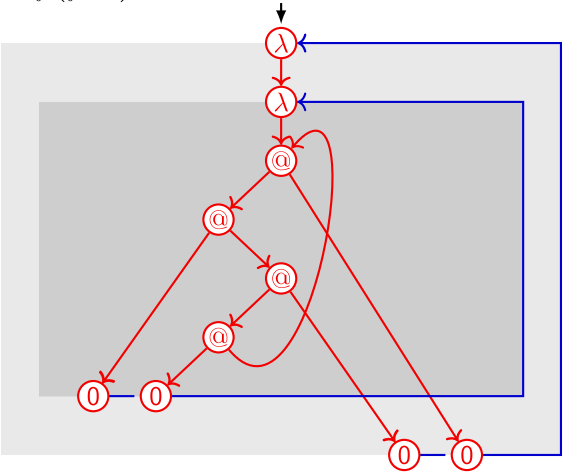
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink, + scopes)

graph interpretation (example 2)

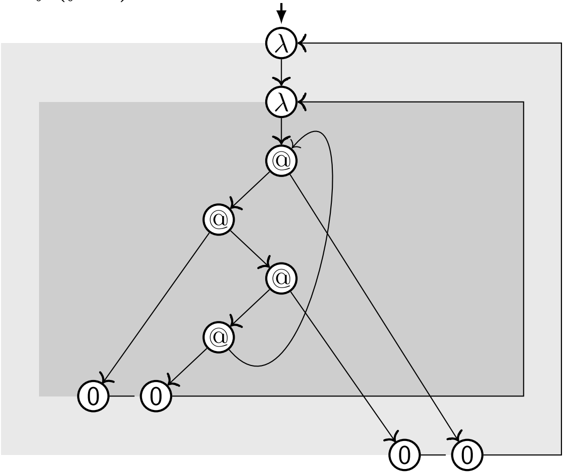
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 2)

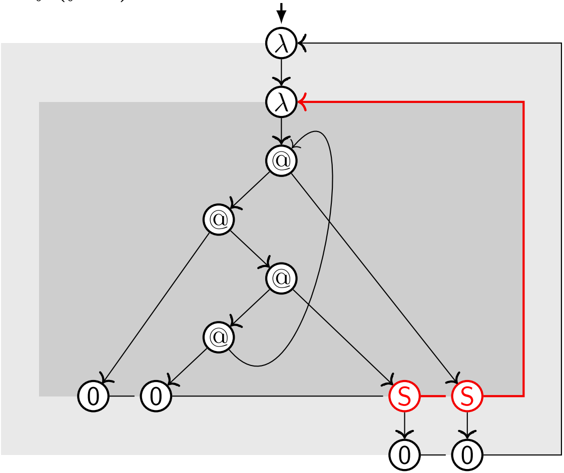
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



λ-higher-order-term-graph $[\![L]\!]_{\mathcal{H}}$

graph interpretation (example 2)

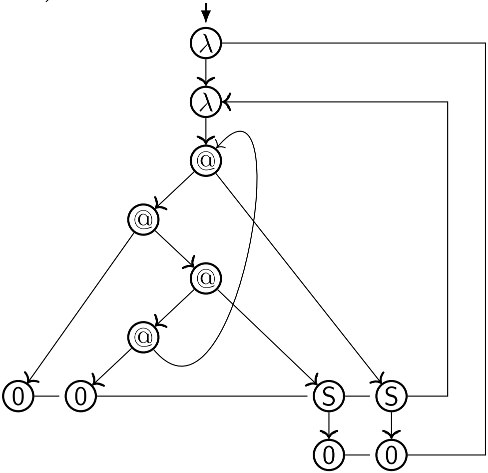
$L = \lambda x.\,\lambda f.\,\mathsf{let}\ r = f\,(f\,r\,x)\,x\ \mathsf{in}\ r$



first-order term graph with scope vertices with backlinks (+ scope sets)

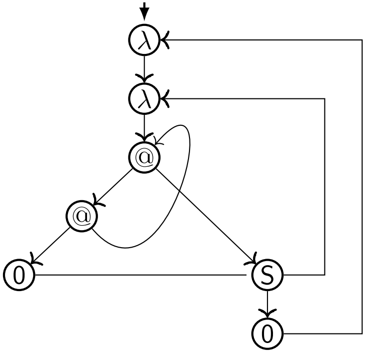
graph interpretation (example 2)

$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$

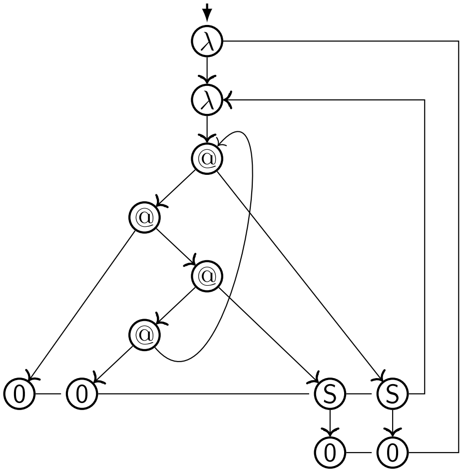


λ-term-graph $[\![L]\!]_{\mathcal{T}}$

graph interpretation (examples 1 and 2)



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto \lambda$-term-graph $[\![L]\!]_{\mathcal{T}}$

- defined by induction on structure of L
- similar analysis as fully-lazy lambda-lifting
- yields eager-scope λ-term-graphs: $\sim$ minimal scopes

Theorem

For $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 it holds: Equality of infinite unfolding coincides with bisimilarity of λ-term-graph interpretations:

$$[\![L_1]\!]_{\lambda^\infty} = [\![L_2]\!]_{\lambda^\infty} \iff [\![L_1]\!]_{\mathcal{T}} \leftrightarrows [\![L_2]\!]_{\mathcal{T}}$$

interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto$ λ-term-graph $[\![L]\!]_{\mathcal{T}}$

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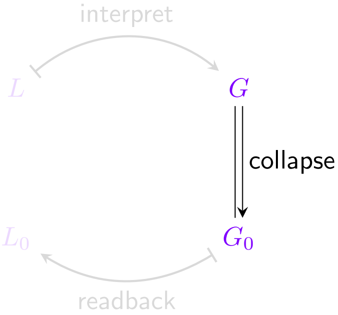
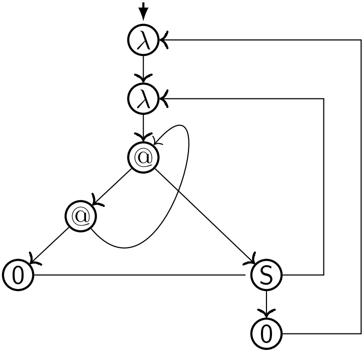
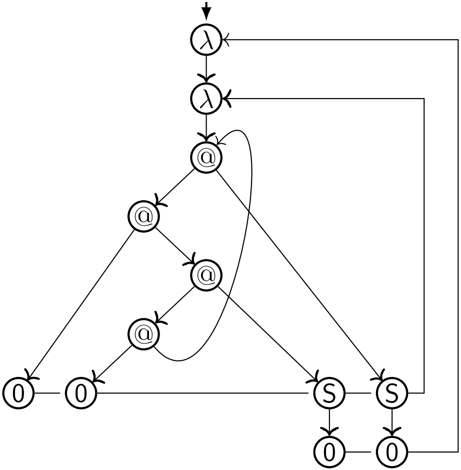
collapse



bisimulation check between λ-term-graphs

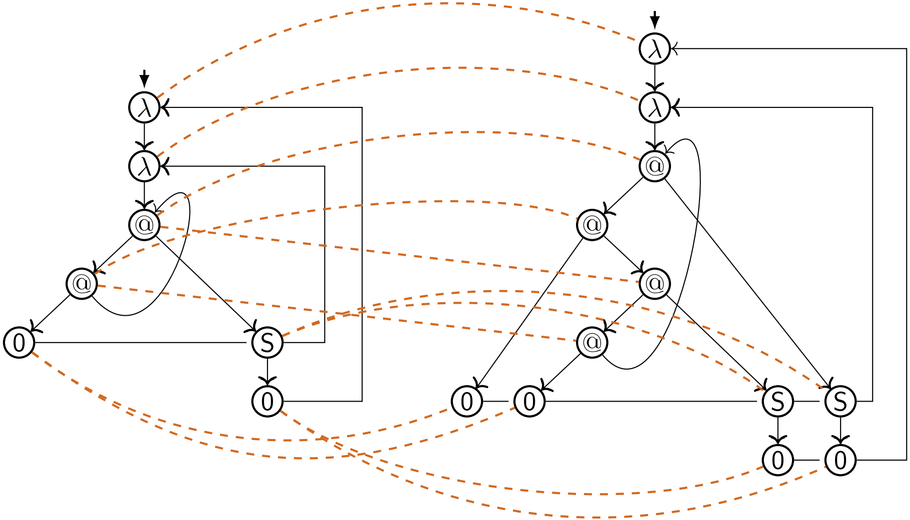


$[\![L_0]\!]_{\mathcal{T}}$



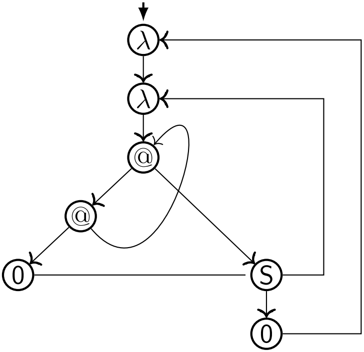
$[\![L]\!]_{\mathcal{T}}$

bisimulation between λ-term-graphs

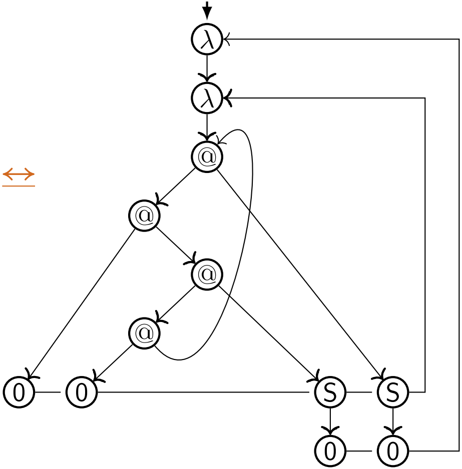


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

bisimilarity between λ-term-graphs

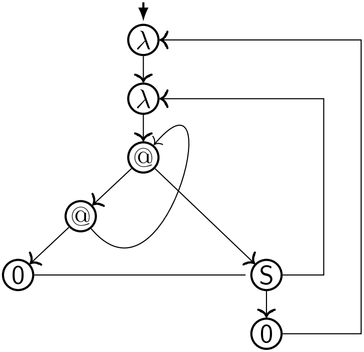


$\underline{\leftrightarrow}$

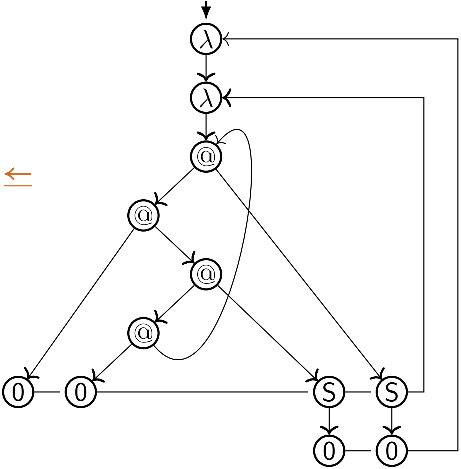


$[\![L_0]\!]_{\mathcal{T}} \quad \underline{\leftrightarrow} \quad [\![L]\!]_{\mathcal{T}}$

functional bisimilarity and bisimulation collapse



$\Leftarrow$



$[\![L_0]\!]_{\mathcal{T}}$ $\Leftarrow$ $[\![L]\!]_{\mathcal{T}}$

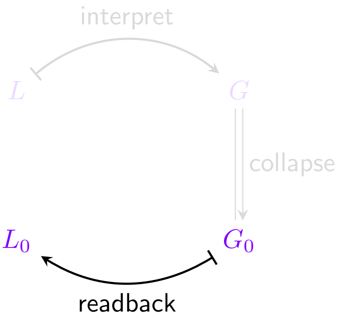
bisimulation collapse: property

Theorem

The class of eager-scope λ-term-graphs
is closed under functional bisimilarity $\rightarrowtail$.

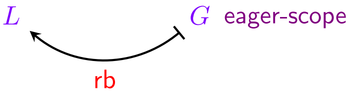
$\Longrightarrow$ For a $\boldsymbol{\lambda}_{\text{letrec}}$-term L
the bisimulation collapse of $[\![L]\!]_{\mathcal{T}}$ is again an eager-scope λ-term-graph.

readback



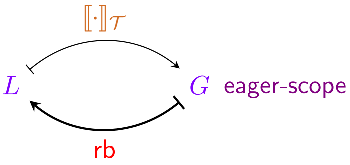
readback

defined with property:



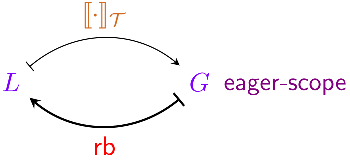
readback

defined with property:



readback

defined with property:



Theorem

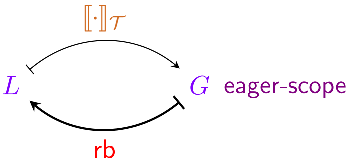
For all eager-scope λ-term-graphs G:

$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $[\![\cdot]\!]_{\mathcal{T}}$ modulo isomorphism $\simeq$.

readback

defined with property:



Theorem

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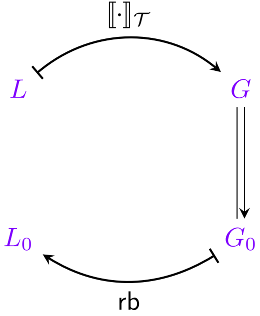
$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $[\![\cdot]\!]_{\mathcal{T}}$ modulo isomorphism $\simeq$.

idea:

1. construct a spanning tree T of G
2. using local rules, in a bottom-up traversal of T synthesize $L = \mathsf{rb}(G)$

maximal sharing: complexity



1. interpretation
   - of $\boldsymbol{\lambda}_{\text{letrec}}$-term L with $|L| = n$
   - as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$
   - ▶ in time $O(n^2)$, size $|G| \in O(n^2)$.
2. bisimulation collapse $|\!\downarrow$
   - of f-o term graph G into G_0
   - ▶ in time $O(|G| \log |G|) = O(n^2 \log n)$
3. readback rb
   - of f-o term graph G_0
   - yielding $\boldsymbol{\lambda}_{\text{letrec}}$-term $L_0 = \text{rb}(G_0)$.
   - ▶ in time $O(|G| \log |G|) = O(n^2 \log n)$

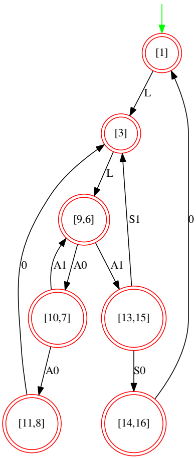
Theorem

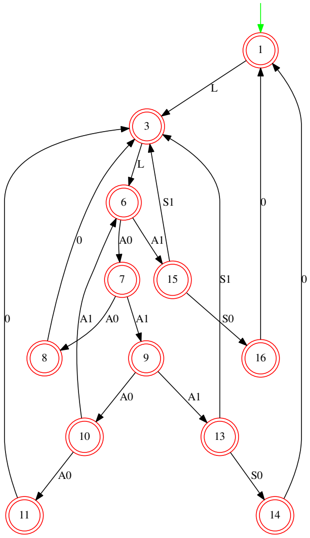
Computing a maximally compact form $L_0 = (\text{rb} \circ |\!\downarrow \circ [\![\cdot]\!]_{\mathcal{T}})(L)$ *of* L *for a* $\boldsymbol{\lambda}_{\text{letrec}}$*-term* L *requires time* $O(n^2 \log n)$, *where* $|L| = n$.

Demo: console output

```
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> maxsharing running.l
λ-letrec-term:
λx. λf. let r = f (f r x) x in r

derivation:
                ---------- 0                            ------- 0
                (x f[r]) f     (x f[r]) r               (x) x
                ------------------------ @              ---------- S
                (x f[r]) f r                            (x f[r]) x
------------ 0  ---------------------------------------- @    ------- 0
(x f[r]) f      (x f[r]) f r x                                (x) x
---------------------------------------------------------- @  ---------- S
(x f[r]) f (f r x)                                            (x f[r]) x
--------------------------------------------------------------------------- @
(x f[r]) f (f r x) x                                                           (x f[r]) r
------------------------------------------------------------------------------------------ let
(x f) let r = f (f r x) x in r
---------------------------------------------------------------------------------------------- λ
(x) λf. let r = f (f r x) x in r
------------------------------------------------------------------------------------------------ λ
() λx. λf. let r = f (f r x) x in r

writing DFA to file: running-dfa.pdf

readback of DFA:
λx. λy. let F = y (y F x) x in F

writing minimised DFA to file: running-mindfa.pdf

readback of minimised DFA:
λx. λy. let F = y F x in F
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014>
```

Demo: generated λ-NFAs





Resources (maximal sharing)

- tool maxsharing on `hackage.haskell.org`
- papers and reports
  - Maximal Sharing in the Lambda Calculus with Letrec
    - ICFP 2014 paper
    - accompanying report arXiv:1401.1460
  - Term Graph Representations for Cyclic Lambda Terms
    - TERMGRAPH 2013 proceedings
    - extended report arXiv:1308.1034
  - Vincent van Oostrom, CG: Nested Term Graphs
    - TERMGRAPH 2014 post-proceedings in EPTCS 183
- thesis Jan Rochel
  - Unfolding Semantics of the Untyped λ-Calculus with letrec
    - Ph.D. Thesis, Utrecht University, 2016

Comparison results: structure-constrained graphs

Regular expressions under $\underline{\leftrightarrow}_{P}$

Given: graph interpretation $[\![\cdot]\!]_{P}$, studied under bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, incomplete under $\underline{\leftrightarrow}$

λ-calculus with letrec under $=_{\lambda^\infty}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Comparison results: structure-constrained graphs

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Defined: class of process graphs with LEE / (layered) LEE-witness

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G $\iff G$ is $[\![\cdot]\!]_{\boldsymbol{P}}^{\not{1r}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$

λ-calculus with letrec under $=_{\boldsymbol{\lambda}^{\infty}}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Comparison results: structure-constrained graphs

Regular expressions under $\underline{\leftrightarrow}_{P}$

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- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G $\Longleftrightarrow$ G is $[\![\cdot]\!]_{P}^{\not{1r}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$

λ-calculus with letrec under $=_{\lambda^\infty}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Defined: int's $[\![\cdot]\!]_{\mathcal{H}}/[\![\cdot]\!]_{\mathcal{T}}$ as higher-order/first-order λ-term graphs

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with λ-calculus with letrec
  - efficient translation and readback
  - translation is inverse of readback

L'Aquila (from Monte Castelvecchia la Crocetta)

Corno Grande, Gran Sasso (from close to GSSI, L'Aquila)