A Duality between derivations and consistency-unfoldings in two different known proof-systems for recursive type equality

(Following an observation by J.W. Klop)

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Overview of the Lecture

- Recursive Types: Definitions and some necessary facts.
  - Tree-unfolding, strong recursive type equivalence $=_\mu$.
  - The known sound and complete axiom- and inference-systems $\mathbf{AC}^{=}$, $\mathbf{HB}^{=}$ and $\mathbf{SM}^{=}$ for $=_\mu$.
- The variant-systems $\mathbf{HB}^{=}_{\mathbf{0}}$ and $\mathbf{SM}^{=}_{\mathbf{0}}$.
- A striking relationship between "checking consistency in $\mathbf{SM}^{=}_{\mathbf{0}}$" and "trying to find a proof in $\mathbf{HB}^{=}_{\mathbf{0}}$" shown by an example.
- Defining a conservative extension $\mathbf{e\text{-}HB}^{=}_{\mathbf{0}}$ of $\mathbf{HB}^{=}_{\mathbf{0}}$.
- Formal definition of necessary notions (finite trees of consequences, . . . , (partial) consistency-unfoldings in $\mathbf{SM}^{=}_{\mathbf{0}}$; *reflection-operations* $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$ between f.t.o.c.'s in $\mathbf{SM}^{=}_{\mathbf{0}}$ and derivations in $\mathbf{e\text{-}HB}^{=}_{\mathbf{0}}$).
- Main Theorem: A *Duality* between the systems $\mathbf{e\text{-}HB}^{=}_{\mathbf{0}}$ and $\mathbf{SM}^{=}_{\mathbf{0}}$ induced by the reflection-operations $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$.
- The image of the duality-operation $\mathcal{C}(\cdot)$ "*developed on fine-grain film*".
- Summary. List of neglected issues.

Main sections of my report on

"*Proof-Theoretic Interconnections between known Axiom- and Inference-Systems concerning the Equality Relation on Recursive Types*":

3. The proof-systems $\mathbf{AC}^{=}$, $\mathbf{HB}^{=}$ and $\mathbf{SM}^{=}$ for recursive type equality $=_\mu$ and the variant-systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and $\mathbf{SM}_{\mathbf{0}}^{=}$.
4. A duality between derivations in an extension $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}_{\mathbf{0}}^{=}$ and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$. [*Topic of this talk*]
5. The rule μ-COMPAT in $\mathbf{AC}^{=}$ is admissible in a very close variant-system $\mathbf{AC}_{\mathbf{1}}^{=}$ of $\mathbf{AC}^{=}$.
6. Transformation of $\mathbf{AC}^{=}$-derivations into $\mathbf{HB}^{=}$-derivations.
7. Transformation of $\mathbf{HB}^{=}$-derivations into $\mathbf{AC}^{=}$-derivations.
8. The conservative extension $\mathbf{e\text{-}HB}^{=}$ of the system $\mathbf{HB}^{=}$ with more general coinductive rules.

Recursive Types

Recursive Types μTp:

$$\tau ::= \alpha \mid \bot \mid \top \mid (\tau_1 \to \tau_2) \mid (\mu\alpha.\, \tau_0) \qquad (\alpha \in TVar).$$

Recursive Types in canonical form $can\text{-}\mu Tp$:

$$\tau ::= \alpha \mid \bot \mid \top \mid (\tau_1 \to \tau_2) \mid (\mu\alpha.\, \underbrace{(\tau_1 \to \tau_2)}_{\text{where } \alpha \in \mathrm{fv}((\tau_1 \to \tau_2))}) \qquad (\alpha \in TVar).$$

Definition 1. (Recursive Type Equality $=_\mu$ (Strong Recursive Type Equivalence))

For all $\tau, \sigma \in \mu Tp$:

$$\tau =_\mu \sigma \;:\Longleftrightarrow\; \mathbf{Tree}(\tau) = \mathbf{Tree}(\sigma)\ .$$

$\boxtimes$

Definition 2. **(Tree-Unfolding of Recursive Types in canonical Form)**

The function **Tree**

$$\begin{aligned} \mathbf{Tree} :\ \textit{can-}\mu Tp &\longrightarrow \text{Labelled Binary Trees} \\ \tau &\longmapsto \mathbf{Tree}(\tau) \end{aligned}$$

which describes the *tree-unfolding* of recursive types in canonical form is defined by:

$$\mathbf{Tree}(\bot) := \begin{array}{c}\downarrow\\ \bot\end{array}\ ; \qquad \mathbf{Tree}(\top) := \begin{array}{c}\downarrow\\ \top\end{array}\ ; \qquad \mathbf{Tree}(\alpha) := \begin{array}{c}\downarrow\\ \alpha\end{array}\ ;$$

$$\mathbf{Tree}(\tau_1 \to \tau_2) := \begin{tikzcd} & \arrow[d] & \\ & \to \arrow[dl] \arrow[dr] & \\ \mathbf{Tree}(\tau_1) & & \mathbf{Tree}(\tau_2) \end{tikzcd} \ ;$$

$$\mathbf{Tree}(\underbrace{\mu\alpha.\,(\tau_1 \to \tau_2)}_{\equiv:\tau}) := \begin{tikzcd} & \arrow[d] & \\ & \to \arrow[dl] \arrow[dr] & \\ \mathbf{Tree}(\underbrace{\tau_1'}_{\equiv_{\mathrm{v}}\tau_1}[\tau/\alpha]) & & \mathbf{Tree}(\underbrace{\tau_2'}_{\equiv_{\mathrm{v}}\tau_2}[\tau/\alpha]) \end{tikzcd}$$

(where τ_1' and τ_2' are appropriate variants of τ_1 and τ_2 respectively to make the above two substitutions possible) .

$\boxtimes$

Definition 3 (Tree-Unfolding of Recursive Types).

The function

$$\begin{aligned}\mathbf{Tree} : \mu Tp &\longrightarrow TpTrees \\ \tau &\longmapsto \mathbf{Tree}(\tau) : \{0,1\}^* \to \{\bot, \top, \to\} \cup TVar \\ &\qquad\qquad p \longmapsto \mathbf{Tree}(\tau)(p)\end{aligned}$$

which to every recursive type $\tau \in \mu Tp$ assigns its *tree-unfolding* $\mathbf{Tree}(\tau)$ is defined as follows:

For every $\tau \in \mu Tp$ and every (path) $p \in \{0,1\}^*$ the symbol $\mathbf{Tree}(\tau)(p)$ is defined by induction on $|p|$ together with a subinduction on the number of leading μ-bindings in τ according to the following clauses:

(1) If $\tau \equiv \bot$ or $\tau \equiv \top$ or $\tau \equiv \alpha$ (for some $\alpha \in TVar$) , then

$$\mathbf{Tree}(\tau)(\epsilon) := \tau\,, \quad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(2) If $\tau \equiv \tau_1 \to \tau_2$ and $p = i\,p_0$ (for $i = 0, 1$), then

$$\mathbf{Tree}(\tau)(\epsilon) := \to\,, \quad \mathbf{Tree}(\tau)(p) := \mathbf{Tree}(\tau_i)(p_0) \text{ (for } i = 0, 1).$$

(3) If $\tau \equiv \mu\alpha_1 \ldots \alpha_n.\,\alpha$ with $n \in \mathbb{N}$, $\alpha \not\equiv \alpha_1, \ldots, \alpha_n$, then

$$\mathbf{Tree}(\tau)(\epsilon) := \alpha\,, \quad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(4) If $\tau \equiv \mu\alpha_1 \ldots \alpha_n.\,\alpha_i$ for $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, then

$$\mathbf{Tree}(\tau)(\epsilon) := \bot\,, \quad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(5) If $\tau \equiv \mu\alpha_1.\,\tau_0 \equiv \mu\alpha_1 \ldots \alpha_n.\,(\rho_1 \to \rho_2)$ with $n \in \mathbb{N}$, then

$$\mathbf{Tree}(\tau)(p) := \mathbf{Tree}(\tau_0'[\tau/\alpha_0])(p)$$

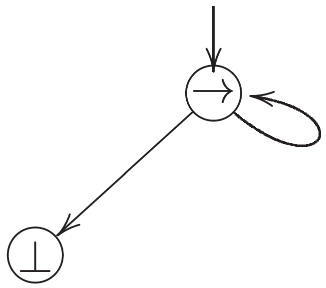
(for some variant τ_0' of τ_0). ⊠

Example 1 (Tree-Unfolding).

The recursive type

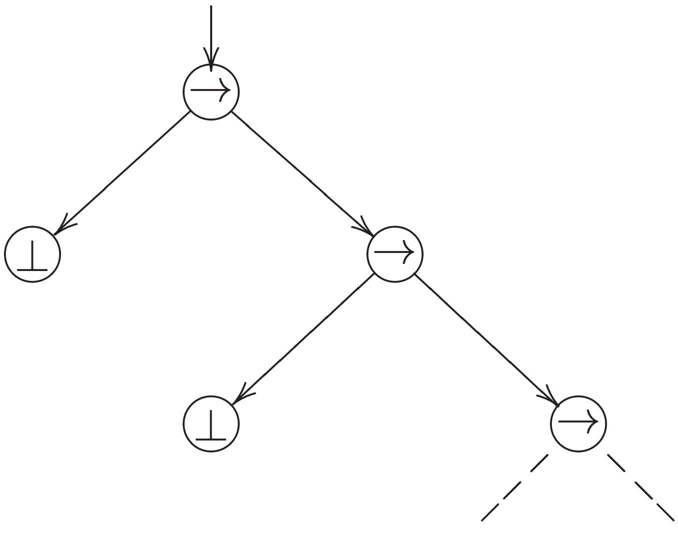
$$\tau := \mu\alpha.\,(\bot \to \alpha)$$

corresponds to the *cyclic term-graph*



and has the tree-unfolding

Tree(τ)=



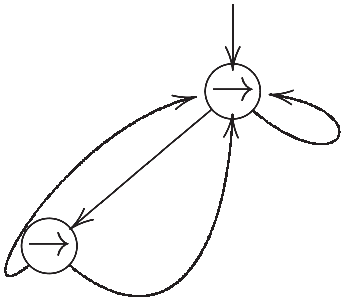
∎

Example 2 (Tree-Unfolding).

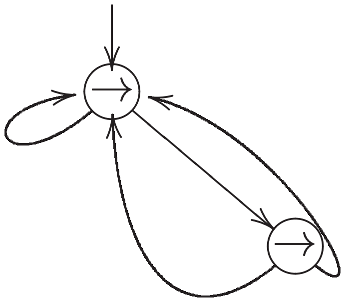
The recursive types

$$\tau := \mu\alpha.\,((\alpha \to \alpha) \to \alpha) \qquad \text{and}$$
$$\sigma := \mu\alpha.\,(\alpha \to (\alpha \to \alpha))$$

correspond respectively to the different cyclic term-graphs

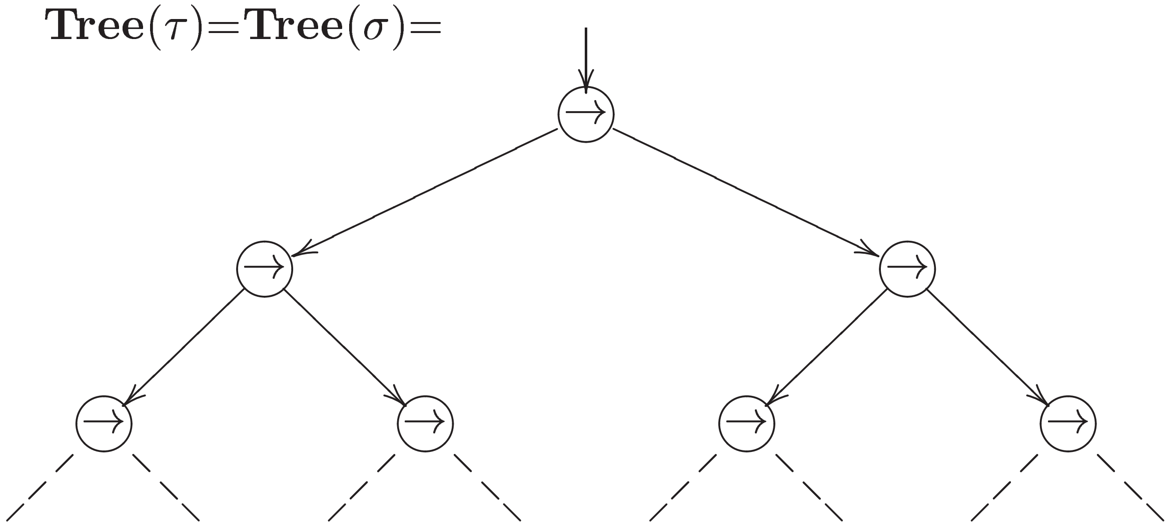


and



but possess the same tree-unfolding

Tree(τ)=**Tree**(σ)=



and hence $\tau =_\mu \sigma$ holds, i.e. recursive type equality holds between τ and σ.

■

Definition 4. (The axiom-system $\mathbf{AC}^{=}$ for recursive type equality $=_\mu$)

Axioms:

(REFL) $\overline{\tau = \tau}$ (VAR) $\overline{\tau = \tau'}$ (if $\tau \equiv_{\mathrm{v}} \tau'$)

$(\mu - \bot)$ $\overline{\mu\alpha.\,\alpha = \bot}$ (FOLD/UNFOLD) $\overline{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau_1 = \sigma_1 \qquad \tau_2 = \sigma_2}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \text{ARROW}$$

$$\frac{\tau = \sigma}{\mu\alpha.\,\tau = \mu\alpha.\,\sigma}\ \mu\text{-COMPAT} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS}$$

$$\frac{\tau_1 = \tau[\tau_1/\alpha] \qquad \tau_2 = \tau[\tau_2/\alpha]}{\tau_1 = \tau_2}\ \text{UFP} \quad (\text{if } \alpha \downarrow \tau)$$

(where $\alpha \downarrow \tau$ means that τ is *contractive* w.r.t. α). ☒

Theorem 1 (R. Amadio, L. Cardelli, '93). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau =_\mu \sigma \quad \Longleftrightarrow \quad \vdash_{\mathbf{AC}^{=}} \tau = \sigma\ .$$

Definition 5. (The axiom-system $\mathbf{HB}^{=}$ for recursive type equality $=_\mu$)

Axioms/Marked assumptions:

$$(\text{REFL})\ \overline{\tau = \tau} \qquad (\mu-\bot)'\ \overline{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \bot}$$

$$(\text{VAR})\ \underset{(\text{if } \tau \equiv_{\text{v}} \tau')}{\overline{\tau = \tau'}} \qquad (\text{FOLD/UNFOLD})\ \overline{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$$

$$(\text{HYP})\quad (\tau = \sigma)^x$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\,\text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\,\text{TRANS}$$

$$\frac{\begin{array}{c}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_1 \\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_2 \\ \tau_2 = \sigma_2\end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\,(\text{ARROW/FIX})_x$$

⊠

Theorem 2 (M. Brandt (and F. Henglein), ’98).

For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau =_\mu \sigma \iff \vdash_{\mathbf{HB}^{=}} \tau = \sigma\,.$$

Definition 6. (The syntactic-matching inference-system SM$^=$ for consistency-checks concerning recursive type equality $=_\mu$)

Axioms:

$$\text{(REFL)}\ \overline{\tau = \tau} \qquad (\mu - \bot)'\ \overline{\mu\alpha\alpha_1\dots\alpha_n.\,\alpha = \bot}$$

$$\text{(VAR)}\ \overline{\tau = \tau'}\ (\text{if } \tau \equiv_{\mathrm{v}} \tau') \qquad \text{(FOLD/UNFOLD)}\ \overline{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS}$$

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \text{DECOMP}\quad (i = 1, 2)$$

☒

Definition 7. For all recursive types $\tau, \sigma \in \mu Tp$:

$\tau = \sigma$ is **SM**$^=$-*consistent* $:\iff$

$\iff$ For all $\chi_1, \chi_2 \in \mu Tp$:
$\tau = \sigma \vdash_{\mathbf{SM}^=} \chi_1 = \chi_2 \implies \mathcal{L}'(\chi_1) \equiv \mathcal{L}'(\chi_2)$.

Hereby for all $\tilde{\tau} \in \mu Tp$ the notation $\mathcal{L}'(\tilde{\tau})$ designates a slight variant of the *leading symbol* $\mathcal{L}(\tilde{\tau})$ of $\tilde{\tau}$. ☒

Theorem 3 (($\sim$) Z.M. Ariola, J.W. Klop, '95). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau =_\mu \sigma \iff \tau = \sigma \text{ is } \mathbf{SM}^=\text{-consistent}.$$

Definition 8. **(The axiom-system $\mathbf{HB}_0^{=}$ for recursive type equality $=_\mu$)**

Axs./M. Assms.: (REFL) $\overline{\tau = \tau}$ (HYP) $(\tau = \sigma)^x$

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR} \qquad \frac{\bot = \sigma}{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}\ (\mu{-}\bot)_l^{\bot\mathrm{der}} + (\mu{-}\bot)_r^{\bot\mathrm{der}}$$

$$\frac{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}{\mu\alpha.\,\tau_0 = \sigma}\ \mathrm{FOLD}_l \qquad \frac{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}{\tau = \mu\beta.\,\sigma_0}\ \mathrm{FOLD}_r$$

$$\frac{\begin{array}{c}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_1 \\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_2 \\ \tau_2 = \sigma_2\end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ (\mathrm{ARROW}/\mathrm{FIX})_x$$

$\boxtimes$

Definition 9. **(The axiom-system $\mathbf{SM}_0^{=}$ for recursive type equality $=_\mu$)**

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR} \qquad \frac{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}{\bot = \sigma}\ (\mu{-}\bot)_l^{\mathrm{der}\bot} + (\mu{-}\bot)_r^{\mathrm{der}\bot}$$

$$\frac{\mu\alpha.\,\tau_0 = \sigma}{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}\ \mathrm{UNFOLD}_l \qquad \frac{\tau = \mu\beta.\,\sigma_0}{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}\ \mathrm{UNFOLD}_r$$

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \mathrm{DECOMP}\quad (i = 1, 2)$$

$\boxtimes$

Definition 10 (Leading Symbol of a recursive type).

The partial function

$$\begin{aligned} \mathcal{L} \,:\, \mu Tp &\longrightarrow \{\bot, \top, \to\} \cup \{\alpha : \alpha \in TVar\} \;, \\ \tau &\longmapsto \mathcal{L}(\tau) \end{aligned}$$

which to every recursive type τ assigns its *leading symbol* $\mathcal{L}(\tau)$ (if it exists), is defined by the following clauses:

$$\mathcal{L}(\bot) :\equiv \bot \,, \; \mathcal{L}(\top) :\equiv \top \,,$$

$$\mathcal{L}(\alpha) :\equiv \alpha \quad (\text{for } \alpha \in TVar) \,,$$

$$\mathcal{L}(\tau_1 \to \tau_2) :\equiv \to \,,$$

$$\mathcal{L}(\mu\alpha_1 \ldots \alpha_n.\, \alpha) :\equiv \alpha \quad (\text{if } \alpha \not\equiv \alpha_i \text{ for } 1 \le i \le n) \,,$$

$$\mathcal{L}(\mu\alpha_1 \ldots \alpha_n.\, (\rho_1 \to \rho_2)) :\equiv \to \;.$$

Only in the (single remaining) situation, that

$$\tau \equiv \mu\alpha_1 \ldots \alpha_n.\, \alpha_i \quad (\text{for } \; n \in \mathbb{N} \;\text{ and }\; 1 \le i \le n\,) \,,$$

where $n \in \mathbb{N}$, $\mathcal{L}(\tau)$ is undefined. — The *total* function

$$\begin{aligned} \mathcal{L}' \,:\, \mu Tp &\longrightarrow \{\bot, \top, \to\} \cup \{\alpha : \alpha \in TVar\} \;, \\ \tau &\longmapsto \mathcal{L}'(\tau) \end{aligned}$$

which is a slight variant of $\mathcal{L}$ and is defined by the clauses in the definition of $\mathcal{L}$ and by the additional clause

$$\mathcal{L}'(\mu\alpha_1 \ldots \alpha_n.\, \alpha_i) :\equiv \bot \quad (\text{for } 1 \le i \le n) \,.$$

$\boxtimes$

Remark. For all $\tau \in \mu Tp$ it holds: $\mathcal{L}'(\tau) = \mathbf{Tree}(\tau)(\epsilon)$.

Example 3 (“Consistency-Unfolding” in $\mathbf{SM}_{\mathbf{0}}^{=}$).

$$
\begin{array}{c}
\left(\overbrace{\mu\alpha.\,((\alpha\to\alpha)\to\alpha)}^{\equiv:\,\tau} = \overbrace{\mu\alpha.\,(\alpha\to(\alpha\to\alpha))}^{\equiv:\,\sigma}\right)^{y} \\
\hline\hline
(\tau\to\tau)\to\tau = \sigma\to(\sigma\to\sigma) \\
\hdashline
\begin{array}{cc}
\begin{array}{c}
(\tau\to\tau=\sigma)^{x} \\
\hline
\tau\to\tau=\sigma\to(\sigma\to\sigma) \\
\hdashline
\begin{array}{cc}
(\tau=\sigma)^{y} &
\begin{array}{c}
\tau=\sigma\to\sigma \\
\hline
(\tau\to\tau)\to\tau=\sigma\to\sigma \\
\hdashline
(\tau\to\tau=\sigma)^{x} \qquad (\tau=\sigma)^{y}
\end{array}
\end{array}
\end{array}
&
\begin{array}{c}
(\tau=\sigma\to\sigma)^{z} \\
\hline
(\tau\to\tau)\to\tau=\sigma\to\sigma \\
\hdashline
\begin{array}{cc}
\begin{array}{c}
\tau\to\tau=\sigma \\
\hline
\tau\to\tau=\sigma\to(\sigma\to\sigma) \\
\hdashline
(\tau=\sigma)^{y} \qquad (\tau=\sigma\to\sigma)^{z}
\end{array}
& (\tau=\sigma)^{y}
\end{array}
\end{array}
\end{array}
\end{array}
$$

$\blacksquare$

Reflection in upwards-direction of this "Consisteny-Unfolding":

$$
\begin{array}{c}
\dfrac{
\dfrac{
\dfrac{(\tau = \sigma)^{y} \qquad \dfrac{\dfrac{(\tau \to \tau = \sigma)^{x} \qquad (\tau = \sigma)^{y}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}}{\tau = \sigma \to \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}
}{(\tau \to \tau = \sigma)^{x}}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{(\tau = \sigma)^{y} \qquad (\tau = \sigma \to \sigma)^{z}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma} \qquad (\tau = \sigma)^{y}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}
}{(\tau = \sigma \to \sigma)^{z}}
}{(\tau \to \tau) \to \tau = \sigma \to (\sigma \to \sigma)} \\
\hline\hline
\Big(\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\ \tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\ \sigma}\Big)^{y}
\end{array}
$$

Example 4 (Proof in $\mathbf{HB}_{\mathbf{0}}^{=}$).

$$
\begin{prooftree}
\AxiomC{$(\dots)^{y}$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\AxiomC{$(\dots)^{x}$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\dots)^{y}$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to \sigma$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\RightLabel{x}
\BinaryInfC{$\tau \to \tau = \sigma \to (\sigma \to \sigma)$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\dots)^{y}$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\AxiomC{$(\dots)^{z}$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\BinaryInfC{$\tau \to \tau = \sigma \to (\sigma \to \sigma)$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\dots)^{y}$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\RightLabel{z}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to \sigma$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\RightLabel{y}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to (\sigma \to \sigma)$}
\doubleLine
\UnaryInfC{$\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\ \tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\ \sigma}$}
\end{prooftree}
$$

∎

Definition 11. (The rule FIX).

Let $\mathcal{S}$ be one of the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ or $\mathbf{HB}^{=}$.

Applications of the additional rule FIX are of the form

$$\begin{array}{c} [\tau = \sigma]^{x} \\ \mathcal{D}_0 \\ \dfrac{\tau = \sigma}{\tau = \sigma} \; (\mathrm{FIX})_x \end{array} \qquad \begin{array}{l} (\text{if } \mathcal{D}_0 \text{ is } \textit{contractive} \text{ w.r.t.} \\ \quad \text{open marked assms. } (\tau = \sigma)^{x}\,) \end{array} \tag{1}$$

in derivations in $\mathcal{S}$+FIX. Hereby the displayed side-condition has to be understood as:

For every undischarged marked assumption of the form $(\tau = \sigma)^{x}$ in $\mathcal{D}_0$ the thread down to the conclusion $\tau = \sigma$ of $\mathcal{D}_0$ *crosses* an application of ARROW or ARROW/FIX *at least once*.

$\boxtimes$

In the case of $\mathcal{S} := \mathbf{HB}_{\mathbf{0}}^{=}$ the side-condition on applications of the rule FIX in derivations $\mathcal{D}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$+FIX of the form (1) is actually equivalent to the weaker formulation:

> There are *either no* undischarged marked assumptions in $\mathcal{D}_0$ of the form $(\tau = \sigma)^x$ *or* there is *at least one* occurrence of an application of ARROW or ARROW/FIX in $\mathcal{D}_0$.

Lemma 4. *The rule FIX is an admissible rule in $\mathbf{HB}_{\mathbf{0}}^{=}$, which can be eliminated effectively from arbitrary derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$+FIX .*

More precisely, every derivation $\mathcal{D}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$+FIX can effectively be transformed into a derivation $\mathcal{D}'$ in $\mathbf{HB}_{\mathbf{0}}^{=}$ with the same conclusion and the same (if any) open assumption-classes.

Definition 12. (The extension-system e-$\mathbf{HB}_{\mathbf{0}}^{=}$ of the system $\mathbf{HB}_{\mathbf{0}}^{=}$).

The extension **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$ of the system $\mathbf{HB}_{\mathbf{0}}^{=}$ has the same formulas and axioms as $\mathbf{HB}_{\mathbf{0}}^{=}$, allows the same marked hypotheses and contains all derivation-rules of $\mathbf{HB}_{\mathbf{0}}^{=}$. Additionally, **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$ also contains the following 3 rules:

$$\frac{\begin{array}{c}[\tau' = \sigma']^{x} \\ \mathcal{D}_0 \\ \tau = \sigma\end{array}}{\tau' = \sigma'}\;(\mathrm{VAR/FIX})_x \qquad \begin{array}{l}(\text{where } \tau' \equiv_{\mathrm{v}} \tau \text{ and } \sigma' \equiv_{\mathrm{v}} \sigma \\ \quad \text{and if side-cond.s. } \mathbf{C} \text{ and } \mathbf{I})\end{array}$$

$$\frac{\begin{array}{c}\left[\mu\alpha.\,\tau_0 = \sigma_0\right]^{x} \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{array}}{\mu\alpha.\,\tau_0 = \sigma_0}\;(\mathrm{FOLD}_l/\mathrm{FIX})_x \qquad (\text{if side-cond.s. } \mathbf{C} \text{ and } \mathbf{I})$$

and $\mathrm{FOLD}_r/\mathrm{FIX}$, which is of an analogous form as $\mathrm{FOLD}_l/\mathrm{FIX}$. Hereby:

- *Side-condition* **C**: $\mathcal{D}_0$ is *contractive* w.r.t. the assumptions $(\ldots)^x$, which get discharged in the respective rule-application.
- *Side-condition* **I**: The respective class of assumptions $[\ldots]^x$ in $\mathcal{D}_0$, which is discharged, is in fact *inhabitated*.

$\boxtimes$

Lemma 5. (Equivalence of the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and e-$\mathbf{HB}_{\mathbf{0}}^{=}$).
The rules VAR/FIX, FOLD$_l$/FIX and FOLD$_r$/FIX are admissible rules in $\mathbf{HB}_{\mathbf{0}}^{=}$*, which can effectively be eliminated from derivations in* **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$*. – Thus the systems* $\mathbf{HB}_{\mathbf{0}}^{=}$ *and* **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$ *are equivalent (i.e. they possess the same theorems).*
Furthermore every derivation $\mathcal{D}$ in **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$ *can effectively be transformed into a derivation $\mathcal{D}'$ in* $\mathbf{HB}_{\mathbf{0}}^{=}$ *with the same conclusion and the same (if any) open assumption-classes.*

Proof. As an example we consider the case, in which we want to eliminate the application of a rule FOLD$_l$/FIX from the bottom of a derivation $\mathcal{D}$ in **e-**$\mathbf{HB}_{\mathbf{0}}^{=}$ with immediate subderivation $\mathcal{D}_0$ in $\mathbf{HB}_{\mathbf{0}}^{=}$. This can be done as follows:

(1) Replacing the application of $(\mathrm{FOLD}_l/\mathrm{FIX})_x$ by an application of FOLD$_l$, which is succeeded by an application of $(\mathrm{FIX})_x$:

$$
\dfrac{\begin{matrix}[\,\mu\alpha.\,\tau_0 = \sigma_0\,]^x \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{matrix}}{\mu\alpha.\,\tau_0 = \sigma_0}\,(\mathrm{FOLD}_l/\mathrm{FIX})_x
\quad\Longrightarrow\quad
\dfrac{\dfrac{\begin{matrix}[\,\mu\alpha.\,\tau_0 = \sigma_0\,]^x \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{matrix}}{\mu\alpha.\,\tau_0 = \sigma_0}\,\mathrm{FOLD}_l}{\mu\alpha.\,\tau_0 = \sigma_0}\,(\mathrm{FIX})_x
$$

(2) Eliminating $(\mathrm{FIX})_x$ in an effective way (which is possible due to Lemma 4). $\square$

Definition 13 (Finite Trees of Consequences in $\mathbf{SM}_{\mathbf{0}}^{=}$).

A *finite tree of consequences* (a *f.t.o.c.*) $\mathcal{C}$ of (or from) an equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is an object, which can be formed by a finite number of applications of the following three generation rules. The depth $|\mathcal{C}|$ of $\mathcal{C}$ is defined in parallel:

(i) $\mathcal{C}$ is the recursive type equality $\boxed{\tau = \sigma}$. – Here $|\mathcal{C}| := 0$.

(ii) $\mathcal{C}$ is of the form $\boxed{\begin{array}{c} \dfrac{\tau = \sigma}{\tau_0 = \sigma_0}\, R \\ \mathcal{C}_0 \end{array}}$, given that $\mathcal{C}_0$ is a f.t.o.c. of $\tau_0 = \sigma_0$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and that R is a rule-application of one of the rules $\text{UNFOLD}_{l/r}$, $(\mu{-}\bot)_{l/r}^{\text{der}\bot}$ or VAR of $\mathbf{SM}_{\mathbf{0}}^{=}$. – Here $|\mathcal{C}| := |\mathcal{C}_0| + 1$.

(iii) $\mathcal{C}$ is of the form $\boxed{\begin{array}{cc} \multicolumn{2}{c}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}} \\ \hdashline \tau_{01} = \sigma_{01} & \tau_{02} = \sigma_{02} \\ \mathcal{C}_{01} & \mathcal{C}_{02} \end{array}\ \text{DECOMP}}$, given that $\mathcal{C}_{01}$ is a f.t.o.c. of $\tau_{01} = \sigma_{01}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and that $\mathcal{C}_{02}$ is a f.t.o.c. of $\tau_{02} = \sigma_{02}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$.

Hereby the part $\begin{array}{cc} \multicolumn{2}{c}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}} \\ \hdashline \tau_{01} = \sigma_{01} & \tau_{02} = \sigma_{02} \end{array}\ \text{DECOMP}$ is called a *branching*.

– In this case $|\mathcal{C}| := 1 + \max\{|\mathcal{C}_{01}|, |\mathcal{C}_{02}|\}$.

⊠

Definition 14. (Finite trees of consequences with marked formulas; back-bound leaf-occurrences of marked formulas).

Let τ and σ be recursive types.

(i) A finite tree of consequences *with marked formulas* (a f.t.o.c. with m.f.) $\mathcal{C}$ of the equality $\tau = \sigma$ in $\mathbf{SM}^{=}_{\mathbf{0}}$ results from a f.t.o.c. $\mathcal{C}^{(u)}$ of $\tau = \sigma$ in $\mathbf{SM}^{=}_{\mathbf{0}}$ by marking some formula-occurrences in $\mathcal{C}^{(u)}$ through attaching subscript-markers to them.

(ii) If a f.t.o.c. with m.f. $\mathcal{C}$ with a single displayed leaf at its bottom of the form

$$\begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ (\rho = \chi)^{x} \\ \mathcal{C}_1 \\ (\rho = \chi)^{x} \end{array} \quad . \tag{2}$$

with respect to two f.t.o.c.'s with m.f. $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\mathbf{SM}^{=}_{\mathbf{0}}$ from $\tau = \sigma$ and from $\rho = \chi$ is given, such that $|\mathcal{C}_1| \geq 1$ and there is no occurrence of $(\rho = \chi)^x$ in the thread from the root of $\mathcal{C}_1$ to the displayed leaf-occurrence of $(\rho = \chi)^x$, then we say that the occurrence of $(\rho = \chi)^x$ at the considered leaf is *bound back* in $\mathcal{C}$ to the displayed occurrence of $(\rho = \chi)^x$ at the position *inside* $\mathcal{C}$ corresponding to the root of $\mathcal{C}_1$ in the symbolic denotation (2).

$\boxtimes$

Definition 15. (Finite trees of consequences with back-bound leaf-occurrences of marked formulas).

A *finite tree of consequences with back-bound leaf-occurrences of marked formulas* (a f.t.o.c. with b.l.o.m.f.) $\mathcal{C}$ from the equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is a f.t.o.c. with m.f., which satisfies the following two conditions:

(i) Each leaf at the bottom of $\mathcal{C}$ carries either an occurrence of an unmarked formula $\rho = \rho$ (which corresponds to an axiom (REFL) of e.g. the system $\mathbf{SM}^{=}$) or an occurrence of a marked formula $(\rho = \chi)^x$ (which may or may not be bound back in $\mathcal{C}$).

(ii) For each occurrence of a marked formula $(\rho = \chi)^x$ *inside* $\mathcal{C}$, i.e. such that $\mathcal{C}$ with this occurrence displayed can be written as

$$\begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ (\rho = \chi)^x \\ \mathcal{C}_1 \end{array}$$

for some f.t.o.c. with m.f. $\mathcal{C}_0$ from $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and a f.t.o.c. $\mathcal{C}_1$ from $\rho = \chi$ with depth $|\mathcal{C}_1| \geq 1$, there exists another leaf-occurrence of $(\rho = \chi)^x$ in $\mathcal{C}$ (in a position in $\mathcal{C}$ corresponding to a leaf-occurrence of this marked formula in $\mathcal{C}_1$), which is bound back to the considered occurrence of $(\rho = \chi)^x$ inside $\mathcal{C}$.

$\boxtimes$

Definition 16. ((Partial) Consistency-Unfoldings).

Let τ and σ be recursive types.

(1) A *partial consistency-unfolding* $\mathcal{C}$ of the equation $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is a f.t.o.c. $\mathcal{C}$ with b.l.o.m.f., which satisfies the following property:

For all leaf-occurrences $(\rho = \chi)^x$ of marked formulas, which are bound back in $\mathcal{C}$, there is at least one occurrence of a branching DECOMP in the thread in $\mathcal{C}$ from the considered leaf-occurrence of $(\rho = \chi)^x$ up to that occurrence of $(\rho = \chi)^x$ inside $\mathcal{C}$, to which the leaf-occurrence is bound back.

(2) A *consistency-unfolding* $\mathcal{C}$ of the equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is a partial consistency-unfolding $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$,

in which each leaf-occurrence of a marked formula $(\rho = \chi)^x$ at the bottom of $\mathcal{C}$ is bound back in $\mathcal{C}$.

$\boxtimes$

Theorem 6. *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau = \sigma \text{ is } \mathbf{SM}_{\mathbf{0}}^{=}\text{-consistent} \iff \text{There exists a consistency-unfolding } \mathcal{C} \text{ of } \tau = \sigma \text{ in } \mathbf{SM}_{\mathbf{0}}^{=} .$$

Sketch of Proof.

(a) "$\Rightarrow$": If $\tau = \sigma$ is $\mathbf{SM}_{\mathbf{0}}^{=}$-consistent, then it is always possible to *build up inductively* bigger and bigger partial consistency-unfoldings of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$, which ultimately lead to a consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$.

(b) "$\Leftarrow$": Let $\mathcal{C}$ be a consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$.

(i) For all equalities $\rho = \chi$ in $\mathcal{C}$ it holds: $\mathcal{L}'(\rho) \equiv \mathcal{L}'(\chi)$.

(ii) By induction on $|\mathcal{C}|$ it can be shown:

For all $\mathbf{SM}_{\mathbf{0}}^{=}$-derivations $\begin{array}{c}(\tau = \sigma)\\ \mathcal{D}\\ \rho = \chi\end{array}$ either (1) there exists an equality $\rho_0 = \chi_0$ in $\mathcal{C}$ s.th. $\mathcal{L}'(\rho) \equiv \mathcal{L}'(\rho_0)$ and $\mathcal{L}'(\chi) \equiv \mathcal{L}'(\chi_0)$, or (2) $\rho =_\mu \chi$ holds directly.

(i) and (ii) imply $\mathbf{SM}_{\mathbf{0}}^{=}$-consistency of $\tau = \sigma$.

$\square$

Notation (Dual Rules in $\mathbf{HB}_{\mathbf{0}}^{=}$ and in $\mathbf{SM}_{\mathbf{0}}^{=}$).

In the two forthcoming definitions we will use the following notation for related rules in the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and in $\mathbf{SM}_{\mathbf{0}}^{=}$:

Rule $R^{(d)}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$	FOLD_l	FOLD_r	VAR	$(\mu{-}\bot)_l^{\bot\mathrm{der}}$	$(\mu{-}\bot)_r^{\bot\mathrm{der}}$
Rule $R^{(cu)}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$	UNFOLD_l	UNFOLD_r	VAR	$(\mu{-}\bot)_l^{\mathrm{der}\bot}$	$(\mu{-}\bot)_r^{\mathrm{der}\bot}$

$\boxtimes$

Remark. For all one-premise-rules $R^{(d)}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$ the following is true:

$$\left.\begin{array}{l} \text{For all } \tau,\sigma,\tau_0,\sigma_0 \in \mu Tp \text{ it holds:} \\ \qquad \dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\,R^{(d)} \quad \text{is a rule-application in } \mathbf{HB}_{\mathbf{0}}^{=} \quad \Longleftrightarrow \\ \qquad\qquad \Longleftrightarrow \quad \dfrac{\tau = \sigma}{\tau_0 = \sigma_0}\,R^{(cu)} \quad \text{is a rule application in } \mathbf{SM}_{\mathbf{0}}^{=}. \end{array}\right\} \tag{3}$$

And respectively (3) holds also for all rules $R^{(cu)}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ *except* the rule DECOMP.

Definition 17. (The Reflection-Operation $\mathcal{D}(\cdot)$ for f.t.o.c.'s with b.l.o.m.f.).

The *reflection* $\mathcal{D}(\mathcal{C})$ of an arbitrary f.t.o.c. with b.l.o.m.f. $\mathcal{C}$ in $\mathbf{SM}^{=}_{\mathbf{0}}$ via the *reflection-operation* $\mathcal{D}(\cdot)$ is defined inductively according to the following 5 clauses:

$$\boxed{(\tau = \sigma)^{\boldsymbol{m}}} \quad \stackrel{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{(\tau = \sigma)^{\boldsymbol{m}}}$$

$$\boxed{\begin{array}{c} \dfrac{\tau = \sigma}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}} \, R^{(cu)} \\ \mathcal{C}_0 \end{array}} \quad \stackrel{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} \mathcal{D}(\mathcal{C}_0) \\ \dfrac{\tau_0 = \sigma_0}{\tau = \sigma} \, R^{(d)} \end{array}}$$

(for $R^{(cu)} \in \{\text{UNFOLD}_{l/r},\ \text{VAR},\ (\mu{-}\perp)^{\text{der}\perp}_{l/r}\}$)

$$\boxed{\begin{array}{c} \dfrac{(\tau = \sigma)^{\color{red}{x}}}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}} \, R^{(cu)} \\ \mathcal{C}_0 \\ [\tau = \sigma]^{\color{red}{x}} \end{array}} \quad \stackrel{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} [\tau = \sigma]^{\color{red}{x}} \\ \mathcal{D}(\mathcal{C}_0) \\ \dfrac{\tau_0 = \sigma_0}{\tau = \sigma} \, (R^{(d)}/\text{FIX})_{\color{red}{x}} \end{array}}$$

(for $R^{(cu)} \in \{\text{UNFOLD}_{l},\ \text{UNFOLD}_{r},\ \text{VAR}\}$)

$$\frac{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}{\begin{array}{cc}(\tau_{01} = \sigma_{01})^{\boldsymbol{m}_{01}} & (\tau_{02} = \sigma_{02})^{\boldsymbol{m}_{02}} \\ \mathcal{C}_{01} & \mathcal{C}_{02}\end{array}}\ \text{DECOMP} \quad \overset{\mathcal{D}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \frac{\begin{array}{cc}\mathcal{D}(\mathcal{C}_{01}) & \mathcal{D}(\mathcal{C}_{02}) \\ \tau_{01} = \sigma_{01} & \tau_{02} = \sigma_{02}\end{array}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ \text{ARROW}$$

$$\frac{(\overbrace{\tau_{01} \to \tau_{02}}^{\equiv \tau} = \overbrace{\sigma_{01} \to \sigma_{02}}^{\equiv \sigma})^{x}}{\begin{array}{cc}(\tau_{01} = \sigma_{01})^{\boldsymbol{m}_{01}} & (\tau_{02} = \sigma_{02})^{\boldsymbol{m}_{02}} \\ \mathcal{C}_{01} & \mathcal{C}_{02} \\ \langle \tau = \sigma \rangle^{x} & \langle \tau = \sigma \rangle^{x}\end{array}}\ \text{DECOMP} \quad \overset{\mathcal{D}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \frac{\begin{array}{cc}\langle \tau = \sigma \rangle^{x} & \langle \tau = \sigma \rangle^{x} \\ \mathcal{D}(\mathcal{C}_{01}) & \mathcal{D}(\mathcal{C}_{02}) \\ \tau_{01} = \sigma_{01} & \tau_{02} = \sigma_{02}\end{array}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ (\text{ARROW}/\text{FIX})_{x}$$

$\boxtimes$

Definition 18. (The Reflection-Operation $\mathcal{C}(\cdot)$ for e-HB$_{\mathbf{0}}^{=}$-derivations).

The *reflection* $\mathcal{C}(\mathcal{D})$ of an arbitrary derivation $\mathcal{D}$ in **e-HB$_{\mathbf{0}}^{=}$** with (possibly) open assumption-classes via the *reflection-operation* $\mathcal{C}(\cdot)$ is defined inductively according to the following 5 clauses:

$$\boxed{(\tau = \sigma)^{\boldsymbol{m}}} \quad \stackrel{\mathcal{C}(\cdot)}{\longmapsto} \quad \boxed{(\tau = \sigma)^{\boldsymbol{m}}}$$

$$\boxed{\begin{array}{c} \mathcal{D}_0 \\ \dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\, R^{(d)} \end{array}} \quad \stackrel{\mathcal{C}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} \dfrac{\tau = \sigma}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)} \\ \mathcal{C}(\mathcal{D}_0) \end{array}}$$

(for $R^{(d)} \in \{\mathrm{FOLD}_{l/r},\ \mathrm{VAR},\ (\mu{-}\bot)^{\bot\mathrm{der}}_{l/r}\}$)

$$\boxed{\begin{array}{c} [\tau = \sigma]^{x} \\ \mathcal{D}_0 \\ \dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\, (R^{(d)}/\mathrm{FIX})_{x} \end{array}} \quad \stackrel{\mathcal{C}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} \dfrac{(\tau = \sigma)^{x}}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)} \\ \mathcal{C}(\mathcal{D}_0) \\ [\tau = \sigma]^{x} \end{array}}$$

(for $R^{(d)} \in \{\mathrm{FOLD}_l,\ \mathrm{FOLD}_r,\ \mathrm{VAR}\}$)

$$\dfrac{\begin{array}{cc}\mathcal{D}_{01} & \mathcal{D}_{02}\\ \tau_{01}=\sigma_{01} & \tau_{02}=\sigma_{02}\end{array}}{\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02}}\ \mathrm{ARROW} \quad \overset{\mathcal{C}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \begin{array}{c}\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02}\\ \hdashline (\tau_{01}=\sigma_{01})^{\boldsymbol{m}_{01}} \qquad (\tau_{02}=\sigma_{02})^{\boldsymbol{m}_{02}}\\ \mathcal{C}(\mathcal{D}_{01}) \qquad \mathcal{C}(\mathcal{D}_{02})\end{array}\ \mathrm{DECOMP}$$

$$\dfrac{\begin{array}{cc}\langle\tau=\sigma\rangle^{x} & \langle\tau=\sigma\rangle^{x}\\ \mathcal{D}_{01} & \mathcal{D}_{02}\\ \tau_{01}=\sigma_{01} & \tau_{02}=\sigma_{02}\end{array}}{\underbrace{\tau_{01}\to\tau_{02}}_{\equiv\tau}=\underbrace{\sigma_{01}\to\sigma_{02}}_{\equiv\sigma}}\ (\mathrm{ARROW/FIX})_{x} \quad \overset{\mathcal{C}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \begin{array}{c}(\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02})^{x}\\ \hdashline (\tau_{01}=\sigma_{01})^{\boldsymbol{m}_{01}} \qquad (\tau_{02}=\sigma_{02})^{\boldsymbol{m}_{02}}\\ \mathcal{C}(\mathcal{D}_{01}) \qquad \mathcal{C}(\mathcal{D}_{02})\\ \langle\tau=\sigma\rangle^{x} \qquad \langle\tau=\sigma\rangle^{x}\end{array}\ \mathrm{DECOMP}$$

⊠

Lemma 7. (Well-Definedness of the reflection-operations $\mathcal{D}(\cdot)$ and $\mathcal{C}(\cdot)$ in Definitions 17 and 18).

(i) *For every f.t.o.c. with b.l.o.m.f.* $\tilde{\mathcal{C}}$ *in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *its reflection* $\mathcal{D}(\tilde{\mathcal{C}})$ *is a "pseudo-derivation" in* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*.*

"Pseudo-Derivation" in $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*: Derivation in a variant-system of* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*, which differs from* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ *only insofar, as that the side-condition* $\mathbf{C}$ *on* $R^{(d)}$*/FIX-applications for* $R^{(d)} \in \{FOLD_{l/r}, VAR\}$ *may be violated.*

(ii) *For every* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*-derivation* $\tilde{\mathcal{D}}$ *with the conclusion* $\tau = \sigma$ *and with (possibly) open assumption-classes its reflection* $\mathcal{C}(\tilde{\mathcal{D}})$ *is a f.t.o.c. of* $\tau = \sigma$ *in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *with b.l.o.m.f..*

This is sketched in the picture:

$$\boxed{\begin{array}{c} (\tau = \sigma)^{m} \\ \tilde{\mathcal{C}} \\ \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \end{array}} \quad \begin{array}{c} \xmapsto{\mathcal{D}(\cdot)} \\ \xleftarrow[\mathcal{C}(\cdot)]{} \end{array} \quad \boxed{\begin{array}{c} \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \\ \tilde{\mathcal{D}} \\ \tau = \sigma \end{array}}$$

f.t.o.c. with b.l.o.m.f. of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$

"pseudo-derivation" of $\tau = \sigma$ in $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ with open assumption-classes

Theorem 8. [based on an idea by J.W. Klop] (A Duality between derivations in e-$\mathbf{HB}_{\mathbf{0}}^{=}$ and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$).

There is a bijective functional relationship between derivations in e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *without open assumption-classes and consistency-unfoldings in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *via the reflection-operations* $\mathcal{D}(\cdot)$ *and* $\mathcal{C}(\cdot)$ *defined in Definition 18 and Definition 17 in the following precise sense:*

(i) *For every derivation* $\mathcal{D}$ *in* e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *without open assumption-classes and with conclusion* $\tau = \sigma$ *its reflection* $\mathcal{C}(\mathcal{D})$ *is a consistency-unfolding of* $\tau = \sigma$ *in* $\mathbf{SM}_{\mathbf{0}}^{=}$.

(ii) *For every consistency-unfolding* $\mathcal{C}$ *of* $\tau = \sigma$ *in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *its reflection* $\mathcal{D}(\mathcal{C})$ *is a derivation in* e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *with conclusion* $\tau = \sigma$ *and without open assumption-classes.*

(iii) *The operations* $\mathcal{D}(\cdot)$ *of taking the reflection of a consistency-unfolding in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *and* $\mathcal{C}(\cdot)$ *of taking the reflection of a derivation in* e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *without open assumption-classes are each other's inverses.*

The very immediate kind of this bijective functional relationship and the possibility to visualize the reflection-operations in a geometrical way is reason to call it a Duality.

Proof. By induction on $|\tilde{\mathcal{D}}|$ of arbitrary $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivations $\tilde{\mathcal{D}}$ with n open assumption classes and respectively by induction on $|\tilde{\mathcal{C}}|$ of arbitrary partial consistency-unfoldings $\tilde{\mathcal{C}}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with n classes of unbound leaf-occurrences of marked formulas (where $n \in \mathbb{N}_0$):

$$\boxed{\begin{array}{c} \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \\ \tilde{\mathcal{D}} \\ \tau = \sigma \end{array}} \quad \begin{array}{c} \xmapsto{\mathcal{C}(\cdot)} \\ \xleftarrow[\mathcal{D}(\cdot)]{} \end{array} \quad \boxed{\begin{array}{c} (\tau = \sigma)^{\boldsymbol{m}} \\ \tilde{\mathcal{C}} \\ \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \end{array}}$$

$\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivation of $\tau = \sigma$ with n open assumption-classes

partial consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with n classes of leaf-occurrences of unbound marked formulas

$\square$

Example 5. **(Duality between a derivation in e-HB$_0^=$ and a consistency-check in SM$_0^=$).**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\tau = \sigma)^{x} \qquad \dfrac{}{\bot = \bot}}{\tau \to \bot = \sigma \to \bot}\ \text{ARROW}}{\tau = \sigma \to \bot} \qquad \dfrac{}{\bot = \bot}}{\tau \to \bot = (\sigma \to \bot) \to \bot}\ \text{ARROW}}{\tau \to \bot = \sigma}\ \text{FOLD}_r}{\underbrace{\mu\alpha.\,(\alpha \to \bot)}_{\equiv:\ \tau} = \underbrace{\mu\beta.\,((\beta \to \bot) \to \bot)}_{\equiv:\ \sigma}}\ (\text{FOLD}_l/\text{FIX})_x
$$

$$
\dfrac{\dfrac{\dfrac{\big(\mu\alpha.\,(\alpha \to \bot) = \mu\beta.\,((\beta \to \bot) \to \bot)\big)^{x}}{\tau \to \bot = \sigma}\ \text{UNFOLD}_l}{\tau \to \bot = (\sigma \to \bot) \to \bot}\ \text{UNFOLD}_r}{\dfrac{\dfrac{\tau = \sigma \to \bot}{\tau \to \bot = \sigma \to \bot}}{(\tau = \sigma)^{x} \qquad \bot = \bot}\ \text{DECOMP} \qquad \bot = \bot}\ \text{DECOMP}
$$

■

Definition 19. (The property D for finite trees of consequences with back-binding markers for leaves).

Let τ and σ be recursive types.

A finite tree of consequences $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with back-bound leaf-occurrences of marked formulas satisfies the *property* **D** iff all occurrences of marked formulas $(\rho = \chi)^x$ in leaves at the bottom of $\mathcal{C}$ (with arbitrary $\rho, \chi \in \mu Tp$ and markers x), which are bound back in $\mathcal{C}$, are exclusively bound back to upper premises of branchings DECOMP in $\mathcal{C}$.

$\boxtimes$

This means, that a f.t.o.c. with b.l.o.m.f. $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$, which satisfies the property **D**, can be written w.r.t. an arbitrary back-bound leaf-occurrence of a m.f. $(\rho = \chi)^x$ in $\mathcal{C}$ as e.g.

$$\text{E.g.}\qquad \begin{array}{c} \tau = \sigma \\ \mathcal{C} \\ (\rho = \chi)^x \end{array} \qquad \equiv \qquad \begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ \dfrac{(\rho_1 \to \rho_2 = \chi_1 \to \chi_2)^x}{\begin{array}{cc} \rho_1 = \chi_1 & \rho_2 = \chi_2 \\ \mathcal{C}_1 & \mathcal{C}_2 \\ (\rho_1 \to \rho_2 = \chi_1 \to \chi_2)^x & \end{array}}\ \text{DECOMP} \end{array}$$

with $\rho_1, \rho_2, \chi_1, \chi_2 \in \mu Tp$ s.th. $\rho \equiv \rho_1 \to \rho_2$ and $\chi \equiv \chi_1 \to \chi_2$ and with some f.t.o.c.'s with m.f. $\mathcal{C}_0$ from $\tau = \sigma$, $\mathcal{C}_1$ from $\rho_1 = \chi_1$ and $\mathcal{C}_2$ from $\rho_2 = \chi_2$ in $\mathbf{SM}_{\mathbf{0}}^{=}$.

Theorem 9. (Examining the "fine grain" in the picture given by Theorem 8: A Duality between derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property D).

The reflection-operations $\mathcal{D}(\cdot)$ and $\mathcal{C}(\cdot)$ defined in Definitions 18 and 17 also yield a bijective functional relationship between derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ without open assumption-classes and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$ in the following precise sense:

(i) For every derivation $\mathcal{D}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$ without open assumption-classes and with conclusion $\tau = \sigma$ its reflection $\mathcal{C}(\mathcal{D})$ is a consistency-unfolding in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$.

(ii) For every consistency-unfolding $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$ its reflection $\mathcal{D}(\mathcal{C})$ is a derivation in $\mathbf{HB}_{\mathbf{0}}^{=}$ with conclusion $\tau = \sigma$ and without open assumption-classes.

(iii) The operations $\mathcal{C}(\cdot)$ of taking the reflection of a derivation in $\mathbf{HB}_{\mathbf{0}}^{=}$ without open assumption-classes and the restriction of the reflection-operator $\mathcal{D}(\cdot)$ to the set of all consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$ are each other's inverses.

Proof. By induction on $|\tilde{\mathcal{D}}|$ of arbitrary $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations $\tilde{\mathcal{D}}$ with (possibly) open assumption-classes and respectively by induction on $|\tilde{\mathcal{C}}|$ of arbitrary partial consistency-unfoldings $\tilde{\mathcal{C}}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$:

$$\boxed{\begin{array}{c} \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \\ \tilde{\mathcal{D}} \\ \tau = \sigma \end{array}} \quad \begin{array}{c} \xmapsto{\mathcal{C}(\cdot)} \\ \xleftarrow[\mathcal{D}(\cdot)]{} \end{array} \quad \boxed{\begin{array}{c} (\tau = \sigma)^{\boldsymbol{m}} \\ \tilde{\mathcal{C}} \\ \{\, [\tau_i = \sigma_i]^{x_i} \,\}_{i=1,\dots,n} \end{array}}$$

$\mathbf{HB}_{\mathbf{0}}^{=}$-derivation of $\tau = \sigma$ with (possibly) open assumption-classes

partial consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$

$\square$

Example 6. (Duality between a derivation in $\mathbf{HB}_0^{=}$ and a consistency-check in $\mathbf{SM}_0^{=}$ with the property D).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\tau\to\bot=(\sigma\to\bot)\to\bot)^{x}}{\tau=\sigma} \qquad \dfrac{}{\bot=\bot}}{\tau\to\bot=\sigma\to\bot}\ \text{ARROW}}{\tau=\sigma\to\bot} \qquad \dfrac{}{\bot=\bot}}{\tau\to\bot=(\sigma\to\bot)\to\bot}\ (\text{ARR./FIX})_{x}}{\underbrace{\mu\alpha.\,(\alpha\to\bot)}_{\equiv:\,\tau}=\underbrace{\mu\beta.\,((\beta\to\bot)\to\bot)}_{\equiv:\,\sigma}}\ \text{FOLD}_{l/r}$$

$$\dfrac{\mu\alpha.\,(\alpha\to\bot)=\mu\beta.\,((\beta\to\bot)\to\bot)}{(\tau\to\bot=(\sigma\to\bot)\to\bot)^{x}}\ \text{UNFOLD}_{l/r}$$

$$\dfrac{(\tau\to\bot=(\sigma\to\bot)\to\bot)^{x}}{\dfrac{\dfrac{\tau=\sigma\to\bot}{\tau\to\bot=\sigma\to\bot}}{\dfrac{\tau=\sigma}{(\tau\to\bot=(\sigma\to\bot)\to\bot)^{x}} \qquad \bot=\bot}\ \text{DECOMP} \qquad \bot=\bot}\ \text{DECOMP}$$

Issues Neglected Here:

- Why is it not possible to find a similar duality by reflection-operations between the *original* systems $\mathbf{HB}^{=}$ and $\mathbf{SM}^{=}$?
- The directions "$\Leftarrow$" in the two statements, that for arbitrary $\tau, \sigma \in \mu Tp$

$$\tau = \sigma \text{ is } \mathbf{SM}^{=}\text{-consistent} \iff \tau = \sigma \text{ is } \mathbf{SM}_{\mathbf{0}}^{=}\text{-consistent} \quad \textit{and}$$
$$\tau = \sigma \text{ is derivable in } \mathbf{HB}_{\mathbf{0}}^{=} \iff \tau = \sigma \text{ is derivable in } \mathbf{HB}^{=}$$

  holds (which assertions were referred to implicitly), are not completely trivial to show.
- Showing in Theorem 6 the direction

$$\tau =_{\mu} \sigma \implies \text{there ex. a cons.-unfolding of } \tau = \sigma \text{ in } \mathbf{SM}_{\mathbf{0}}^{=}$$

  (for all $\tau, \sigma \in \mu Tp$) is—due to the duality—analogue and equally as difficult as proving completeness for $\mathbf{HB}_{\mathbf{0}}^{=}$.
  It can be done directly by an analogue construction as presented by Brandt and Henglein for the completeness of $\mathbf{HB}^{=}$. This uses the notion of a *subterm* of a recursive type as well as the lemma, that every recursive type has only finitely many subterms up to taking variants.

Summary:

- Preliminaries on recursive types.
- The variant systems $\mathbf{HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}^{=}$ and $\mathbf{SM}_{\mathbf{0}}^{=}$ of $\mathbf{SM}^{=}$.
- The extension-system $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}_{\mathbf{0}}^{=}$.
- Notions: F.t.o.c.'s, f.t.o.c.'s with m.f. and with b.l.o.m.f., (partial) consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$.
- Reflection-operations $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$ between "pseudo"-$\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivations and f.t.o.c.'s with b.l.o.m.f. in $\mathbf{SM}_{\mathbf{0}}^{=}$.
- **Main Theorem**: A *duality* induced by $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$ between $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivations and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$.
- *Finer picture of this duality*: A Duality also between $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations and *such* consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$, that fulfil the property $\mathbf{D}$.

The complete set of the LaTeX-slides presented here is available at:

`http://www.cs.vu.nl/~clemens/sl_duality.ps.gz`

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