A Coinductive Axiomatisation of Regular Expressions Under Bisimulation

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Abstract

Milner (1984) introduced a (non-standard) interpretation of regular expressions as finite-state processes. He asked whether a certain slight modification of Salomaa's axiomatisation for language equivalence of regular expressions is complete for bisimilarity of regular expressions under the process interpretation. We present two results addressing this apparently still open problem: Based on the observation that Antimirov's notion of "partial derivative" for regular expressions is closely linked to the process interpretation, we give a proof system with a coinductive motivation that is complete for bisimilarity under the process interpretation. And as a corollary, we show that the system obtained from Milner's adaptation of Salomaa's system by the following modification is in fact complete: replacing the single-equation solving rule by a rule that expresses that systems of guarded linear equations are uniquely solvable.

In his paper [2] on a complete axiomatisation of bisimilarity between finite-state processes, Milner introduced an interpretation (here denoted by) P of regular expressions e as finite-state processes $P(e)$. Different from the standard language interpretation L, the interpretation P assigns the following meaning to the regular operators: $+$ and . respectively denote *alternative composition* (also called *choice*) between processes, and *sequential composition* of processes; * denotes unbounded iteration of processes with the possibility of termination before each iteration; letters a stand for processes that are executions of *atomic actions*; and the constants 0 and 1 are interpreted as *deadlock*, and as the *empty process* (also called *skip*), respectively. The interpretation P gives rise to the equivalence relation "bisimilarity of process interpretations" on regular expressions, which we denote by $\leftrightarrows_P$ here: for all regular expressions e and f, $e \leftrightarrows_P f$ is defined to hold if and only if $P(e) \leftrightarrows P(f)$ (where $\leftrightarrows$ stands for bisimilarity between processes).

As an axiomatisation of $\leftrightarrows_P$, Milner in [2] also suggested an adaptation of Salomaa's complete axiom system for language equivalence $=_L$ on regular expressions. For this purpose, in particular the left-distributivity law $x.(y+z) = x.y + x.z$ as well as the axiom $x.0 = 0$, which are not valid in process theory, have to be removed from Salomaa's system. Milner recognised that Salomaa's method of completeness proof cannot be applied, at least not directly, to yield also a completeness proof for the modified system with respect to $\leftrightarrows_P$. He pinned down the reason to the fact that there exist finite-state processes that are not bisimilar to processes described by regular expressions. As a consequence, even linear systems of guarded recursive equations are not always solvable in process interpretations of regular expressions. Milner's question (Q) in [2], of whether the modified version of Salomaa's system (or some extension of it by a finite number of other axioms) is complete for $\leftrightarrows_P$, has frequently been studied and a number of related results have been obtained (we mention work by Sewell (1997), Fokkink (1997), Corradini, De Nicola and Labella (1998)). Yet, apparently (Q) is still unsolved. Due to the connection with Basic Process Algebra BPA, we denote Milner's system for $\Leftrightarrow_P$ by $\mathbf{BPA^*_{0,1}}$+1-$\mathrm{RSP}^*_{0,1}$, where the rule 1-$\mathrm{RSP}^*_{0,1}$ is a variant of the *recursive specification principle* for solving single linear guarded equations: from $e = f.e + g$ the equation $e = f^*.e + g$ may be deduced under the assumption that the empty word ϵ is not contained in $L(f)$.

Here we present two results related to Milner's question (Q) that provide partial answers. Firstly, we formulate a coinductively motivated, natural-deduction style proof system $\mathbf{c\text{-}BPA^*_{0,1}}$ and show that it is sound and complete for $\Leftrightarrow_P$. And secondly, we use the completeness of $\mathbf{c\text{-}BPA^*_{0,1}}$ to demonstrate the completeness with respect to $\Leftrightarrow_P$ of $\mathbf{BPA^*_{0,1}}$+$\mathrm{USP}^*_{0,1}$, the system obtained from Milner's system by replacing the rule 1-$\mathrm{RSP}^*_{0,1}$ with the *unique solvability principle* $\mathrm{USP}^*_{0,1}$: this rule allows to infer $e_1 = e_2$ for regular expressions e_1 and e_2 if both of e_1 and e_2 are provable to be the solution of the same finite system $\mathcal{E}$ of guarded linear equations. The second result is closely connected to (Q) because, as is easy to show, (Q) is equivalent to the following question: Is the system $\mathbf{BPA^*_{0,1}}$+1-$\mathrm{USP}^*_{0,1}$ complete for $\Leftrightarrow_P$? Herein 1-$\mathrm{USP}^*_{0,1}$ is the unique solvability principle for systems $\mathcal{E} = \{x = f.x + g\}$ of *single* guarded (which means $\epsilon \notin L(f)$) linear equations: an application of 1-$\mathrm{USP}^*_{0,1}$ amounts to deducing $e_1 = e_2$ from $e_1 = f.e_1 + g$ and $e_2 = f.e_2 + g$.

Here are some details concerning the coalgebraic motivation for the system $\mathbf{c\text{-}BPA^*_{0,1}}$. It turns out that there is a close link between the process interpretation P and the notion of "partial derivatives" of regular expressions due to Antimirov in [1]. There, "partial derivatives" are introduced as mathematically motivated refinements of Brzozowski's "word derivatives" (1964). For all $w \in A^*$ and $e \in \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the set of regular expressions over alphabet A, the set $\partial_w(e)$ of partial derivatives of e with respect to w is an inductively defined, finite set of regular expressions, in symbols: $\partial_w(e) \in \mathcal{P}_{\mathrm{f}}(\mathcal{R}(A))$. Antimirov established a close connection between partial derivatives and word derivatives, and he proved that every regular expression has only finitely many partial derivatives (a statement analogous to a result for word derivatives by Brzozowski).

We observe that "partial derivatives" adequately describe the behaviour of regular expressions under the process interpretation: for all $w \in A^*$ and $e, f \in \mathcal{R}(A)$, $P(e) \xrightarrow{w} P(f)$ holds (process $P(e)$ is able to evolve, via the sequence w of actions, to process $P(f)$) if and only if $f \in \partial_w(e)$. Using this, we show that $\Leftrightarrow_P$ coincides with the notion of bisimilarity $\sim$ on the F-coalgebra $(\mathcal{R}(A), \langle o, t\rangle)$, where F is the functor $F(X) = 2 \times \mathcal{P}_{\mathrm{f}}(A \times X)$, and the functions $t : \mathcal{R}(A) \to \mathcal{P}_{\mathrm{f}}(A \times \mathcal{R}(A))$ and $o : \mathcal{R}(A) \to 2$ are defined, for all $e, f \in \mathcal{R}(A)$, by $t(e) =_{\mathrm{def}} \{\langle a, e'\rangle \mid a \in A, e' \in \partial_a(e)\}$, and respectively, by the clause: $o(e) = 1$ if and only if $\epsilon \in L(e)$. Using that every regular expression has only finitely many partial derivatives, we establish a *finitary coinduction principle* for $\Leftrightarrow_P$: for showing $e \Leftrightarrow_P f$, it suffices to prove $e \sim_{\mathrm{fin}} f$, the existence of a finite bisimulation R in $(\mathcal{R}(A), \langle o, t\rangle)$ with $\langle e, f\rangle \in R$. The natural-deduction system $\mathbf{c\text{-}BPA^*_{0,1}}$ is constructed in such a way that derivations without open assumptions correspond to finite bisimulations in $(\mathcal{R}(A), \langle o, t\rangle)$. From this we prove that $\mathbf{c\text{-}BPA^*_{0,1}}$ is sound and (utilising the finitary coinduction principle) also complete for $\Leftrightarrow_P$.

References

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[2] R. Milner. A complete inference system for a class of regular behaviours. *Journal of Computer and System Sciences*, 28:439–466, 1984.