Lecture 2: Graph width notions, dynamical programming

An Introduction to Parameterized Complexity

Clemens Grabmayer

Ph.D. Program, Advanced Period
Gran Sasso Science Institute
L'Aquila, Italy

Tuesday, June 11, 2024

Course overview

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

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- comparing parameterizations

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- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
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- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width

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- comparing width-notions

Fixed-Parameter tractable

A *parameterized problem* is a triple $\langle Q, \Sigma, \kappa \rangle$ (short: $\langle Q, \kappa \rangle$) where:

- $Q \subseteq \Sigma^*$ is the set of *(classical) problem instances*,
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Parameterized problem $\langle Q, \Sigma, \kappa \rangle$
 Instance: $x \in \Sigma^*$.
 Parameter: $\kappa(x)$.
 Problem: Is $x \in Q$?

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Definition

A parameterized problem $\langle Q, \Sigma, \kappa \rangle$ is *fixed-parameter tractable* (is in FPT) if:

$\exists f : \mathbb{N} \to \mathbb{N}$ computable $\exists p \in \mathbb{N}[X]$ polynomial

$\exists \mathbb{A}$ algorithm, takes inputs in Σ^*

$\forall x \in \Sigma^* [\, \mathbb{A}$ decides whether $x \in Q$ holds in time $\leq f(\kappa(x)) \cdot p(|x|) \,]$

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†) Assumptions for a robust fpt-theory

$\kappa(x)$ is polynomially computable, or itself fpt-computable: for all $x \in \Sigma^*$ in time $\leq g(\kappa(x)) \cdot q(|x|)$ for g computable, $q \in \mathbb{N}[X]$.

Comparing parameterizations

Definition (computably bounded below)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
- $\kappa_1 \approx \kappa_2 \;:\Longleftrightarrow\; \kappa_1 \succeq \kappa_2 \;\wedge\; \kappa_2 \succeq \kappa_1$.
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For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

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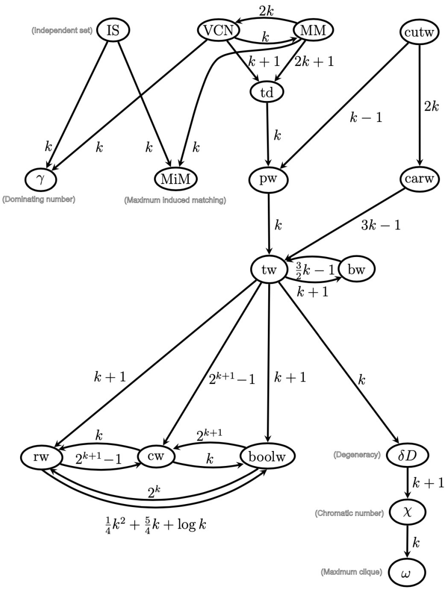
$$\langle Q, \kappa_1 \rangle \in \text{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \text{FPT}$$
$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 \ :\Leftrightarrow\ wd_1 \xrightarrow{g} wd_2$$



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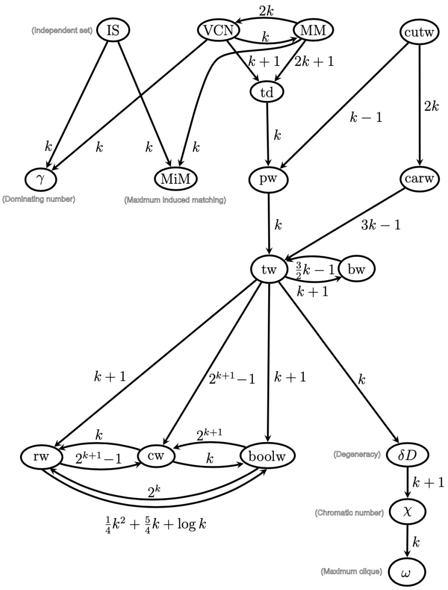

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- FPT-results
  transfer upwards
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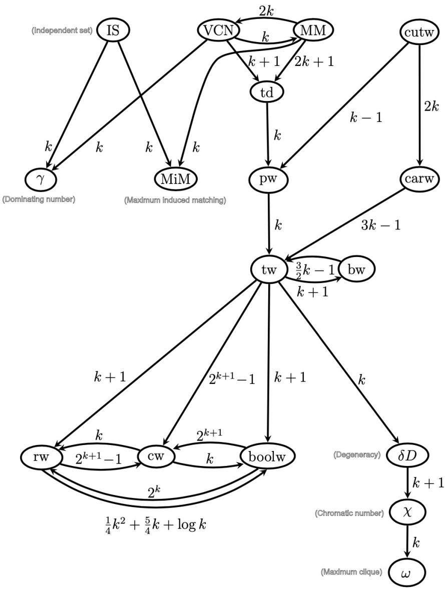
- ($\notin$ FPT)-results
  transfer downwards
  (and along $\overset{g}{\to}$)



You Always Walk Alone (with your children)

Attività motoria con i figli:

'la possibilità di uscire con i figli minori è consentita a un solo genitore per camminare purché questo avvenga in prossimità della propria abitazione'

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PHYSICAL-DISTANCE-WALKING

Instance: Graph $\mathcal{G} = \langle V, E \rangle$ with V people who want to go for a walk in the next hour in a radius of 200m of their home, and edges in E between them if they live closer than 400m of each other. A number $\ell \in \mathbb{N}$.

Problem:

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corresponds to: INDEPENDENT-SET

Weighted Independent Set, and Vertex Cover

Let $\mathcal{G} = \langle V, E \rangle$ a graph. For all $S \subseteq V$:

S is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \in E \,(\, \neg(u \in S \wedge v \in S)\,)$

$\Longleftrightarrow \forall e = \{u, v\} \in E \,(\, u \notin S \vee v \notin S)\,)$

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

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$S \subseteq V$ is *minimal* vertex cover $\Longleftrightarrow$ $V \setminus S$ is *maximal* independent set
Hence: solution of WEIGHTED-INDEPENDENT-SET

$\Longrightarrow$ solution of VERTEX-COVER.

Weighted Ind. Set / Vertex Cover, width-parameterized

p^*-WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $w : V \to \mathbb{R}_0^+$.

Parameter: path-width / tree-width k.

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Dynamical programming on trees (example)

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Solution: value of $A[r]$, can be computed bottom-up in linear time.

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On trees with n nodes,

WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(O(n))$.

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Corollary

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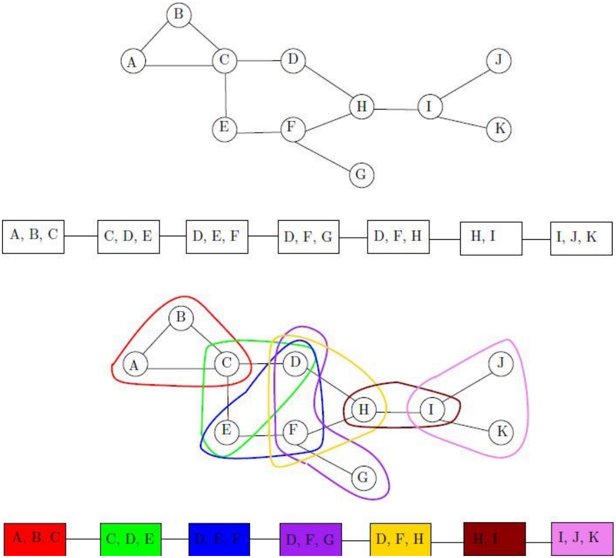
Path-decomposition (example)



Path decompositions, and path-width

Definition (Robertson–Seymour, 1983)

A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

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(P3) $(\forall v \in V)\, (\exists i, k \in \{1, \ldots, r\},\, i \le k) \big[\{j \mid v \in B_j\} = [i, k] \big]$
(the list of bags that contains a vertex of $\mathcal{G}$ is $\langle B_i, \ldots, B_k \rangle$ for some interval $[i, k]$)

Path decompositions, and path-width

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A *path decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a sequence $\langle B_1, B_2, \ldots, B_r \rangle$ of bags $B_i \subseteq V$ such that:

(P1) $V = \bigcup_{i=1}^{r} B_i$ (every vertex of $\mathcal{G}$ is in some bag).

(P2) $(\forall \{u, v\} \in E)\, (\exists i \in \{1, 2, \ldots, r\}) \big[\, \{u, v\} \subseteq B_i \,\big]$
(every edge of $\mathcal{G}$ is realized in some bag).

(P3) $(\forall v \in V)\, (\exists i, k \in \{1, \ldots, r\}\, ,\, i \le k) \big[\, \{j \mid v \in B_j\} = [i, k] \,\big]$
(the list of bags that contains a vertex of $\mathcal{G}$ is $\langle B_i, \ldots, B_k \rangle$ for some interval $[i, k]$)

The *width* of path decomp. $\langle B_1, B_2, \ldots, B_r \rangle$ is $\max \{ |B_t| - 1 \mid 1 \le t \le r \}$.

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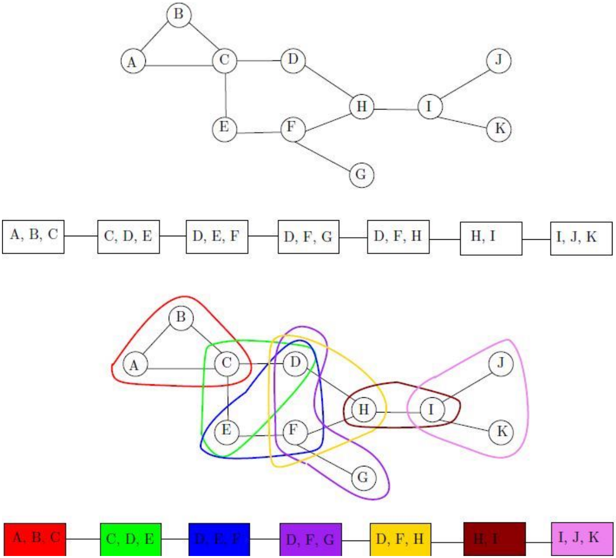
The *path-width* $pw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E\rangle$ is defined by:

$pw(\mathcal{G})$:= minimal width of a path decomposition of $\mathcal{G}$.

Path-decomposition (example)



Path decomposition defines separations

Lemma

Let $\langle B_1, B_2, \ldots, B_r \rangle$ *be a path decomposition of a graph* $\mathcal{G} = \langle V, E \rangle$. *Then for all* $i \in \{1, \ldots, r-1\}$ *it holds:*

- $\langle \bigcup_{j=1}^{i} B_j, \bigcup_{j=i+1}^{r} B_j \rangle$ *is a separation of* $\mathcal{G}$ *with separator* $B_i \cap B_{i+1}$.

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- A pair $\langle A, B \rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
  - there is no edge between $A \setminus B$ and $B \setminus A$.

  $A \cap B$ is called the *separator* of a separation $\langle A, B \rangle$, and $|A \cap B|$ is called its *order*.

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- $\partial(\bigcup_{j=1}^{i} B_j) \subseteq B_i \cap B_{i+1}$.

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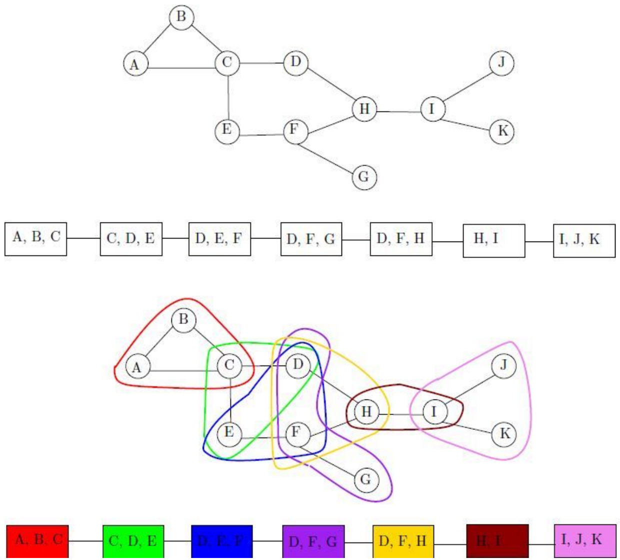
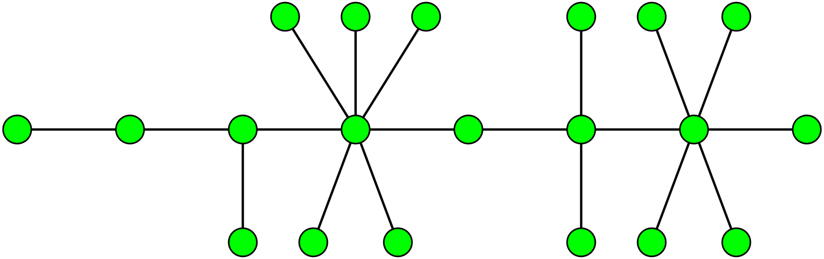
  and $|A \cap B|$ is called its *order*.
- The *border (set of border vertices)* $\partial(A)$ *of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \setminus A$. Note that:
  - $\partial(A) = \partial(V \setminus A)$.
  - $\langle A, (V \setminus A) \cup \partial(A) \rangle$ is a separation of $\mathcal{G}$, for all $A \subseteq V$.

Path-decomposition (example)



Caterpillar

Path-width?



Nice path decomposition

Definition

A *path decomposition* $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ is *nice* if:

- $B_1 = B_r = \varnothing$
- Every index $i > 1$ is either of:
  - introduce index: there is $v \in V$ such that $B_{i+1} = B_i \cup \{v\}$ and $v \notin B_i$,
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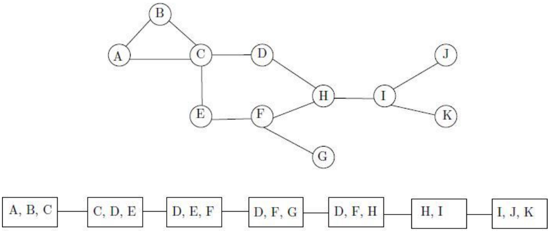

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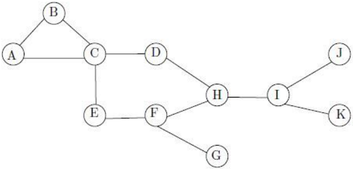

A, B, C — C, D, E — D, E, F — D, F, G — D, F, H — H, I — I, J, K

Nice path decomposition:

$\varnothing$ — A — A,B — A,B,C — B,C — C — C,D — C,D,E — D,E — D,E,F —

— D,F — D,F,G — D,F — D,F,H — F,H — H — H,I — I — I,J — I,J,K — J,K — K — $\varnothing$

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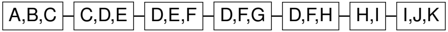
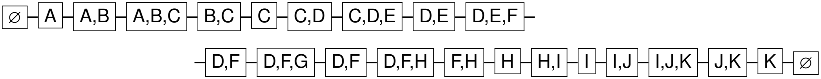
Lemma

From every path decomposition $\langle B_1, B_2, \ldots, B_r \rangle$ of a graph $\mathcal{G} = \langle V, E \rangle$ of width k a nice path decomposition $\langle B'_1, B'_2, \ldots, B'_{r'} \rangle$ of width k can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.





Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

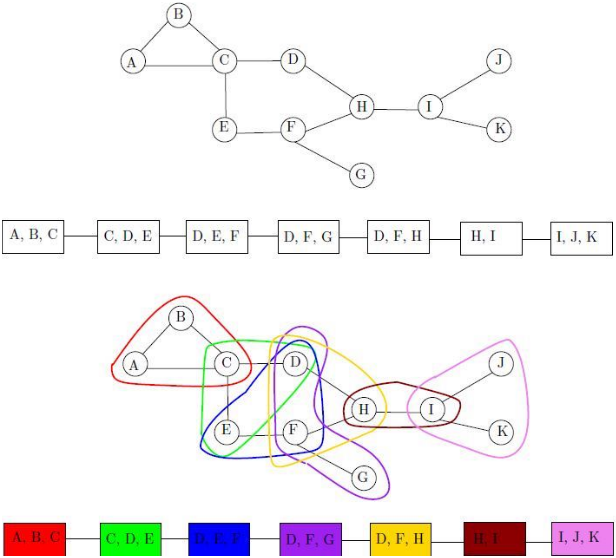
WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: path-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Path-decomposition (example)



Dyn. programming using path-width (Weigh. Ind. Set)

Let $\langle B_1, \ldots, B_r \rangle$ be a nice path decomposition of $\mathcal{G} = \langle V, E \rangle$.
Then for every $i \in \{1, \ldots, r\}$, and every $S \subseteq B_i$, we define:

$$c[i,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_i = \bigcup_{j=1}^{i} B_j \wedge \hat{S} \cap B_i = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

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Recursive equations for computing $c[i,S]$ for independent S:

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  - $i + 1$ forgets v: $B_{i+1} = B_i \smallsetminus \{v\}$ and $v \in B_i$,
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- time for comp. all values at i, using values at $i-1$: $2^{k+1} \cdot O(k^2)$

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- $\Rightarrow$ number of values $c[i, S]$ at index i: $2^{|B_i|} = 2^{k+1}$,
- $\Rightarrow$ adjacency/independence check for $S \subseteq B_t$ possible in: $O(k^2)$ using a datastructure computable in time $O(k^{O(1)} \cdot n)$,
- time for comp. a value at i, using map of values at $i - 1$: $\sim O(k)$
- time for comp. all values at i, using values at $i - 1$: $2^{k+1} \cdot O(k^2)$

$\Rightarrow$ the time for computing all values at r:

$$(2^{k+1} \cdot O(k^2)) \cdot r + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n), \text{ since } r = 2n.$$

Dynamical programming with path width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and path-width* pw$(\mathcal{G}) = k$,

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Corollary

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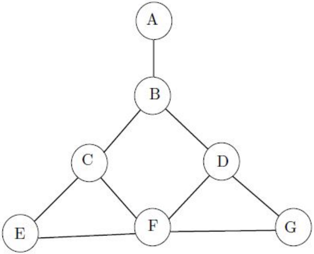

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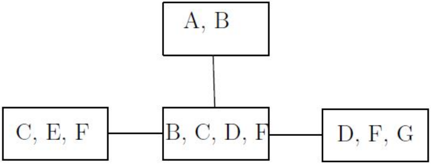

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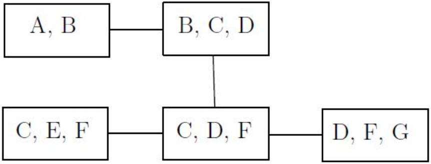

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Tree decompositions, and tree-width

Definition (Bertelé–Brioschi, 1972, Halin, 1976, Robertson–Seymour, 1984)

A *tree decomposition* of a graph $\mathcal{G} = \langle V, E \rangle$ is a pair $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ where $\mathcal{T} = \langle T, F \rangle$ a (undirected, unrooted) tree, and $B_t \subseteq V$ such that:

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The *tree-width* $tw(\mathcal{G})$ of a graph $\mathcal{G} = \langle V, E \rangle$ is defined by:

$$tw(\mathcal{G}) := \text{minimal width of a tree decomposition of } \mathcal{G}.$$

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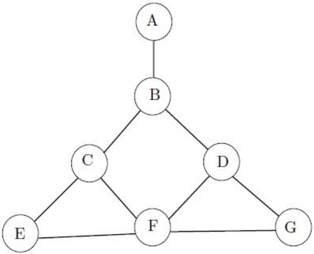

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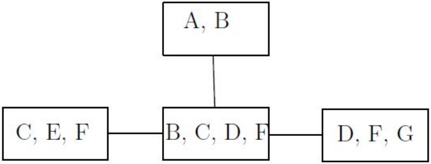

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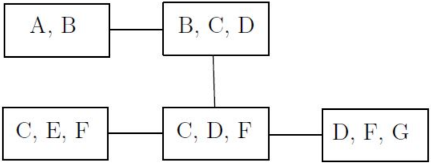

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Tree decomposition defines separations

Lemma

*Let $\langle \mathcal{T}, \{B_t\}_{t \in T} \rangle$ be a tree decomposition of a graph $\mathcal{G} = \langle V, E \rangle$.
Let $e = \langle a, b \rangle$ be an edge of $\mathcal{T}$. The $\mathcal{T} \smallsetminus e$ is the union of a tree $\mathcal{T}_a$ containing a, and a tree $\mathcal{T}_b$ containing b.
Then for $A := \bigcup_{t \in V(\mathcal{T}_a)} B_t$ and $B := \bigcup_{t \in V(\mathcal{T}_b)} B_t$ it holds:*

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Recall, for a graph $\mathcal{G} = \langle V, E\rangle$:

- A pair $\langle A, B\rangle$ of subsets $A, B \subseteq V$ is a *separation* of $\mathcal{G}$ if:
  - $V = A \cup B$
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  $A \cap B$ is called the *separator* of a separation $\langle A, B\rangle$, and $|A \cap B|$ is called its *order*.

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- The *border (vertices) $\partial(A)$ of* a set $A \subseteq V$ of vertices consists of all vertices that have a neighbor in $V \smallsetminus A$.

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Instance: A graph $\mathcal{G}$ and $k \in \mathbb{N}$.

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p-TREE-WIDTH *is fixed-parameter tractable,*
in time $2^{p(k)} \cdot n$ *where* $n := |V|$.

Nice tree decomposition

Definition

A *tree decomposition* $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ of graph $\mathcal{G} = \langle V, E\rangle$ is *nice* if it is based on the choice of a leaf as *root* r and the parent–children relation away from r such that:

- $B_r = \varnothing$, and $B_\ell = \varnothing$ for every leaf $\ell \in T$.
- Every non-leaf node $t \in T$ is of one of three types:
  - introduce node: t has exactly one child t' such that $B_t = B_{t'} \cup \{v\}$; we say v is introduced at t.
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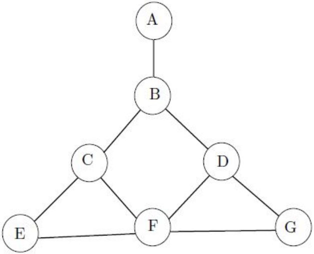
Lemma

From every tree decomposition $\langle \mathcal{T}, \{B_t\}_{t\in T}\rangle$ of a graph $\mathcal{G} = \langle V, E\rangle$ of width k a nice tree decomposition $\langle \mathcal{T}', \{B'_t\}_{t\in T'}\rangle$ of width k and with $r := |V(\mathcal{T})| \in O(kn)$ vertices can be constructed in time $O(k^2 \cdot \max\{r, n\})$ where $n := |V|$.

Tree decomposition (example)



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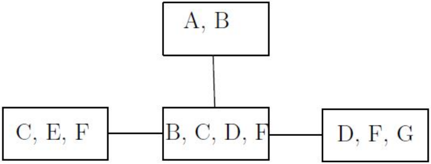

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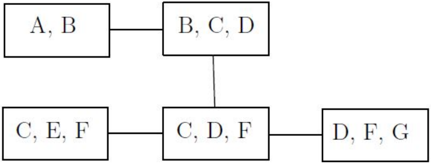

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Weighted Independent Set

Let $\mathcal{G} = \langle V, E \rangle$ a graph.

$S \subseteq V$ is independent set in $\mathcal{G}$ $:\Longleftrightarrow \forall e = \{u, v\} \big(\neg(u \in S \wedge v \in S) \big)$.

WEIGHTED-INDEPENDENT-SET

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, and a weight function $\boldsymbol{w} : V \to \mathbb{R}_0^+$.

Parameter: tree-width k.

Problem: What is the max. weight of an independent set of $\mathcal{G}$?

Dynamical programming using tree-width (example)

For every node t of a nice tree decomposition, and every $S \subseteq B_t$, we define:

$$c[t,S] := \begin{cases} -\infty & \text{if } S \text{ is not independent,} \\ \text{maximum possible weight of a set } \hat{S} \text{ such that} \\ \hat{S} \text{ is independent} \wedge S \subseteq \hat{S} \subseteq V_t \wedge \hat{S} \cap B_t = S \\ & \text{if } S \text{ is independent.} \end{cases}$$

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$\Rightarrow$ the time for computing all values at the root r :

$(2^{k+1} \cdot O(k^2)) \cdot |T| + O(k^{O(1)} \cdot n) \in O(2^k \cdot k^{O(1)} \cdot n)$, since $|T| \in O(k \cdot n)$.

Dynamical programming with tree width (example)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tree-width tw$(\mathcal{G}) = k$,

p^*-WEIGHTED-INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$.

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Corollary

For every graph $\mathcal{G} = \langle V, E\rangle$ with $|V| = n$ and tree-width tw($\mathcal{G}$) $= k$,
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Dyn. programming with tree-width: general strategy

We consider problem P for graphs $\mathcal{G} = \langle V, E \rangle$ of size n and nice tree decompositions $\langle \mathcal{T} = \langle T, F, r \rangle, \{B_t\}_{t \in T} \rangle$ of tree width k.

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- Obtain an estimate of the time needed to compute the properties in a node t depending on n and k.
- Sum up the time needed to compute the solution(s) at root r of $\mathcal{T}$.
- Add time needed to obtain the solution of P from properties at r.

Dynamical programming: similar results (I)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ *with* $|V| = n$ *and tw*$(\mathcal{G}) = k$,

- p^*-VERTEX-COVER, INDEPENDENT-SET $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-DOMINATING-SET $\in$ DTIME$(4^k \cdot k^{O(1)} \cdot n)$,
- p^*-ODD CYCLE TRAVERSAL $\in$ DTIME$(3^k \cdot k^{O(1)} \cdot n)$,
- p^*-MAXCUT $\in$ DTIME$(2^k \cdot k^{O(1)} \cdot n)$,
- p^*-q-COLORABILITY $\in$ DTIME$(q^k \cdot k^{O(1)} \cdot n)$.

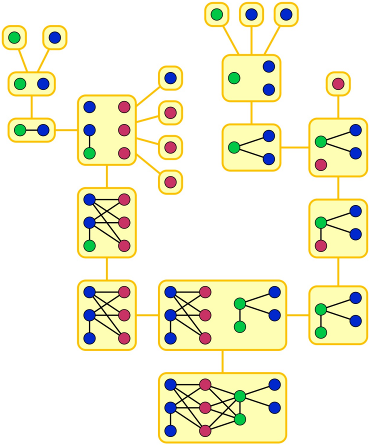
Dynamical programming: similar results (II)

Theorem

For every graph $\mathcal{G} = \langle V, E \rangle$ with $|V| = n$ and tw($\mathcal{G}$) $= k$, the following problems are in DTIME$(k^{O(k)} \cdot n)$ *:*

- p^*-STEINER-TREE,
- p^*-FEEDBACK-VERTEX-SET,
- p^*-HAMILTONIAN-PATH *and* p^*-LONGEST-PATH,
- p^*-HAMILTONIAN-CYCLE *and* p^*-LONGEST-CYCLE,
- p^*-CHROMATIC-NUMBER,
- p^*-CYCLE-PACKING,
- p^*-CONNECTED-VERTEX-COVER,
- p^*-CONNECTED-FEEDBACK-VERTEX-SET.

Clique width (example)



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For $k \in \mathbb{N}$, the *k-expressions* are defined by:

$$\varphi, \varphi_1, \varphi_2 \ ::= \ \boldsymbol{i} \mid \text{edge}_{i-j}(\varphi) \mid \text{recolor}_{i \to j}(\varphi) \mid (\varphi_1 \oplus \varphi_2)$$

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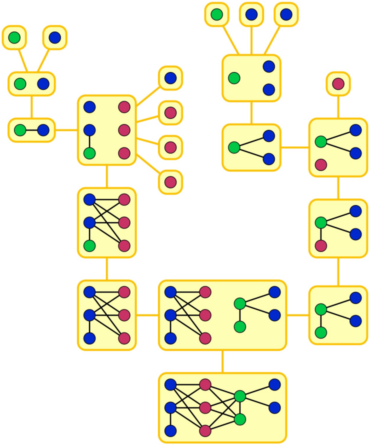
Definition (Courcelle, Engelfriet, Rozenberg, 1993, [2])

The *clique-width clw($\mathcal{G}$)* of $\mathcal{G} = \langle V, E \rangle$ is defined by:

clw($\mathcal{G}$) := the least $k \in \mathbb{N}$ such that, for some k-expression φ, $\mathcal{G} = \mathcal{G}(\varphi)$ (when removing colors)

Clique width (example)

Building a graph $\mathcal{G}$ of clique-width *clw*$(\mathcal{G}) = 3$:



Clique-Width (examples, properties, computability)

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- Every graph property expressible in MSO (monadic second-order logic) can be decided in linear time w.r.t. the graph's clique-width.

f-Width (of sets)

By a *cut function* or a *connectivity function* we mean a function $f : 2^U \to \mathbb{R}_0^+$ such that:

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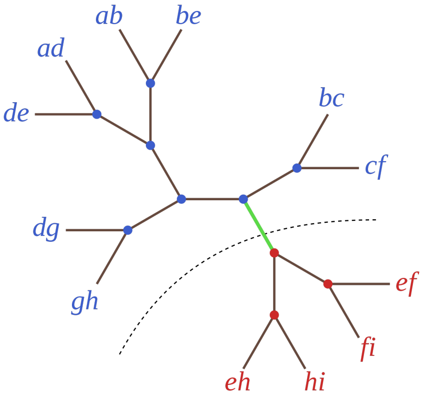

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Let U be a set, $f : 2^U \to \mathbb{R}_0^+$ a cut function.
A *branch decomposition* of U is a pair $\langle \mathcal{T}, \eta \rangle$ where:

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The *width* of an edge $e \in T$ (with respect to f) is $f(X_e) = f(Y_e)$. The *width of* $\langle \mathcal{T}, \eta \rangle$ *w.r.t.* f is the maximum width over the edges of $\mathcal{T}$.

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- $\eta : U \to \textit{Leafs}(\mathcal{T})$ a bijective function.

Every edge $e \in T$ splits the tree into two connected parts, and, via η, splits U into a partition $\langle X_e, Y_e \rangle$.
The *width* of an edge $e \in T$ (with respect to f) is $f(X_e) = f(Y_e)$. The *width of* $\langle \mathcal{T}, \eta \rangle$ *w.r.t.* f is the maximum width over the edges of $\mathcal{T}$.

The *f-width* $w_f(U)$ of U is defined as:

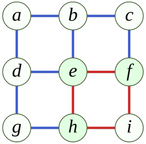
$$w_f(U) := \text{minimum width of branch decomp's of } U \text{ w.r.t. } f.$$

Branch-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph. The *border (vertices)* of a set $X \subseteq E$ of edges is defined by:

$$\partial(X) := \left\{ v \in V \mid \exists e_1 \in X \; \exists e_2 \in E \setminus X \; \left[v \text{ is incident to } e_1 \text{ and } e_2 \right] \right\}$$



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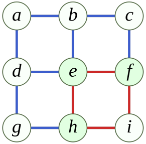

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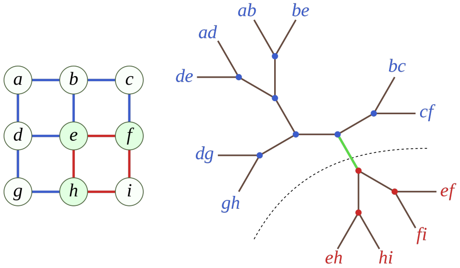

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Proposition

$\mathit{bw}(\mathcal{G}) \approx \mathit{tw}(\mathcal{G})$, for every graph; more precisely:

$$\mathit{bw}(\mathcal{G}) \leq \mathit{tw}(\mathcal{G}) + 1 \leq \tfrac{3}{2} \cdot \mathit{bw}(\mathcal{G}) \,.$$

Rank-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ we define the $GF(2)$-matrix:

$$B_{\mathcal{G}}(X) := (b_{x,y})_{x \in X, y \in V \smallsetminus X}\,, \text{ where, for all } x \in X, y \in V \smallsetminus X\text{:}$$

$$b_{x,y} = 1 \iff \{x, y\} \in E\,.$$

($B_{\mathcal{G}}(X)$ is the adjacency matrix of the bipartite graph induced by $\mathcal{G}$ between X and $V \smallsetminus X$.)

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The *rank-width rw*($\mathcal{G}$) of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{rw}(\mathcal{G}) := w_{\rho_{\mathcal{G}}}(E) \quad \text{for} \quad \rho_{\mathcal{G}} : 2^V \to \mathbb{N}_0,\ X \mapsto \text{rank of } B_{\mathcal{G}}(X)$$

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Properties

- *rw*($\mathcal{G}$) ≤ *tw*($\mathcal{G}$).
- tree-width cannot be bounded functionally by rank-width: *rw*(K_n) = 1, but *tw*(K_n) = $n - 1$.

Carving-Width

Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph.
For $X \subseteq V$ the *edge-cut* of X is:

$$\textit{cut}_{\mathcal{G}}(X) := \{e = \{u, v\} \in E \mid u \in X, v \in V \setminus X\} \ .$$

The *carving-width* $\textit{carw}(\mathcal{G})$ of a graph $\mathcal{G} = \langle G, E \rangle$ is:

$$\textit{carw}(\mathcal{G}) := w_{\textit{cut}}(E) \quad \text{for} \quad \textit{cut} : 2^V \to \mathbb{N}_0, \ X \mapsto |\textit{cut}_{\mathcal{G}}(X)| \ .$$

Carving-Width and Cut-Width

Definition

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Definition

Let $\mathcal{G} = \langle V, E \rangle$ be a graph with $n = |V|$.
For a permutation $\pi : \{1, \ldots, n\} \to V$ on V we define:

$$\textit{width}(\pi) := \max_{1 \le i \le n} \textit{cut}_{\mathcal{G}}(\{\pi(j) \mid 1 \le j \le i\}) .$$

The *cut-width* $\textit{cutw}(\mathcal{G})$ of $\mathcal{G}$ is:

$$\textit{cutw}(\mathcal{G}) := \min_{\pi \text{ perm. of } V} \textit{width}(\pi) .$$

Coverage in Multi-Interface Networks

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CMI(p) (for $p \in \mathbb{N}$)

Instance: A graph $\mathcal{G} = \langle V, E \rangle$, $W : V \rightarrow 2^{\{1,\ldots,a\}}$ available-interface allocation, $c : \{1, \ldots, a\} \rightarrow \mathbb{R}^{+}$ interface cost function.

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Problem: Obtain, if possible, a minimal solution with respect to the total cost of the interfaces that are activated, that is, $c(W_A) = \sum_{v \in V} \sum_{i \in W_A(v)} c(i)$.

Coverage in Multi-Interface Networks

Theorem

CMI$(2) \in$ NP-complete*, also for graphs with max. node degree* ≥ 4.

Coverage in Multi-Interface Networks (parameterized)

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Theorem (Aloisio, Navarra, 2020, [1])

- *For path-width pw*$(\mathcal{G}) = k$,

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Corollary

$(p^*)'$-*CMI*$(p) \in$ FPT.

Comparing parameterizations

Definition (computably bounded)

Let $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ parameterizations.

- $\kappa_1 \succeq \kappa_2 \;:\Longleftrightarrow\; \exists g : \mathbb{N} \to \mathbb{N}$ computable $\forall x \in \Sigma^* \big[\, g(\kappa_1(x)) \geq \kappa_2(x) \,\big]$.
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Proposition

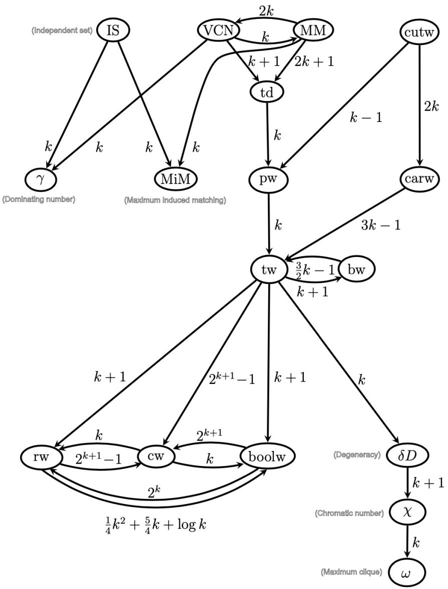
For all parameterized problems $\langle Q, \kappa_1 \rangle$ and $\langle Q, \kappa_2 \rangle$ with parameterizations $\kappa_1, \kappa_2 : \Sigma^* \to \mathbb{N}$ with $\kappa_1 \succeq \kappa_2$:

$$\langle Q, \kappa_1 \rangle \in \text{FPT} \quad \Longleftarrow \quad \langle Q, \kappa_2 \rangle \in \text{FPT}$$
$$\langle Q, \kappa_1 \rangle \notin \text{FPT} \quad \Longrightarrow \quad \langle Q, \kappa_2 \rangle \notin \text{FPT}$$

Computably boundedness between notions of width

(from Sasák, [5])

$$wd_1 \geq wd_2 \;:\Leftarrow\; wd_1 \xrightarrow{g(k)} wd_2$$



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- FPT-results
  transfer upwards
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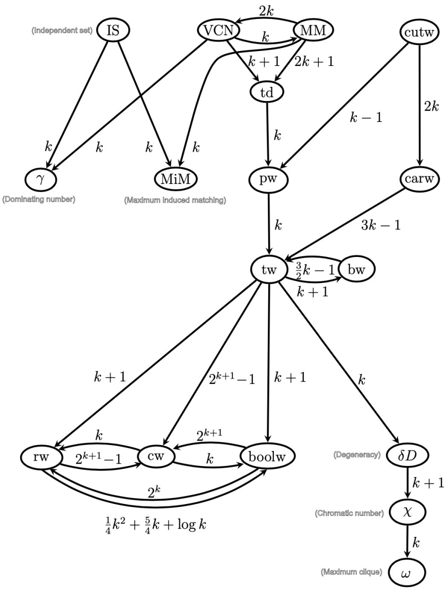

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- ($\notin$ FPT)-results
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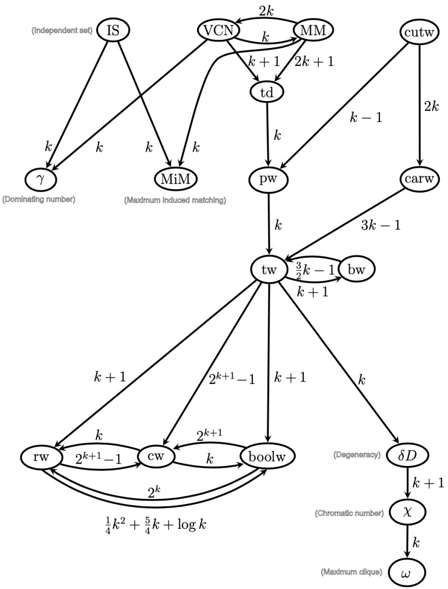

Summary

- comparing parameterizations
- dynamical programming on trees, example:
  - WEIGHTED-INDEPENDENT-SET (and VERTEX-COVER)
- path-width
  - example: fpt-algorithm for bounded path-width
- tree-width
  - example: fpt-algorithm for bounded path-width
- fpt-results for other problems, obtained similarly
- other notions of width
  - clique-width
  - using f-width to define:
    - carving-width (and cut-width)
    - branch-width
    - rank-width
- example problem: coverage in multi-interface networks
- comparing width-notions

Tomorrow

Monday, June 10 10.30 – 12.30	Tuesday, June 11	Wednesday, June 12 10.30 – 12.30	Thursday, June 13	Friday, June 14
Introduction & basic FPT results		**Algorithmic Meta-Theorems**		
motivation for FPT kernelization, Crown Lemma, Sunflower Lemma	GDA	1st-order logic, monadic 2nd-order logic, FPT-results by Courcelle's Theorems for tree and clique-width	GDA	GDA
Algorithmic Techniques		*Formal-Method* & *Algorithmic Techniques*		
	14.30 – 16.30			14.30 – 16.30
	Notions of bounded graph width			**FPT-Intractability Classes & Hierarchies**
	path-, tree-, clique width, FPT-results by dynamic programming, transferring FPT results betw. widths	GDA	GDA	motivation for FP-intractability results, FPT-reductions, class XP (slicewise polynomial), W- and A-Hierarchies, placing problems on these hierarchies

Tomorrow

- recalling notions from logic:
  - propositional, and first-order logic
  - monadic second-order logic (MSO)
- Courcelle's Theorem: obtaining FPT-results by
  - model-checking of MSO-properties
    on graphs and structures of bounded tree-/clique-width

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