Regularity Preserving but not Reflecting Encodings

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LICS 2015

Kyoto, July 9, 2015

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- *the image function $f[\_]$ of f preserves language regularity:*

$$\forall L \subseteq \Sigma^* (\, L \text{ is regular} \implies f[L] \text{ is regular} \,)\,,$$

- *but the image function $f^{-1}[\cdot]$ for the inverse function f^{-1} does not:*

$$\exists L' \subseteq \Gamma^* (\, L' \text{ is regular} \;\wedge\; f^{-1}[L] \text{ is not regular} \,) \qquad ?$$

Outline

- the problem
  - more motivation: number encodings and c-automaticity
- the solution
  - main theorem
  - encoding, and extension lemmas
  - proof sketch/idea
- consequences for comparing models of computation
- summary

c-automatic functions

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$$
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\downarrow & & \downarrow \\
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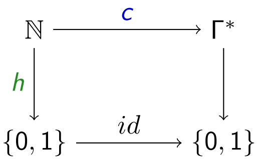

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For a number encoding $c : \mathbb{N} \to \Gamma^*$, a function $h : \mathbb{N} \to \{0,1\}$ is *c-automatic* if there is a DFA-computable $h^c : \Gamma^* \to \{0,1\}$ such that:

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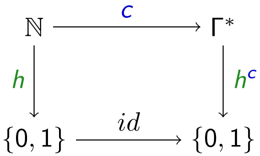

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Gives rise to a *subsumption pre-order* $\leq$:

$c \leq d$ if all c-automatic functions are d-automatic.

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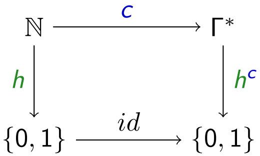

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A: For *bijective* encodings: Yes, if answer to the initial problem is yes!

Solution

Main theorem

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Let Σ, Γ *be finite alphabets, with* $|\Gamma| \geq 2$.

For every countable class $\mathcal{C}$ *of languages over* Σ,
there exists a bijective encoding $g : \Sigma^* \to \Gamma^*$ *such that:*

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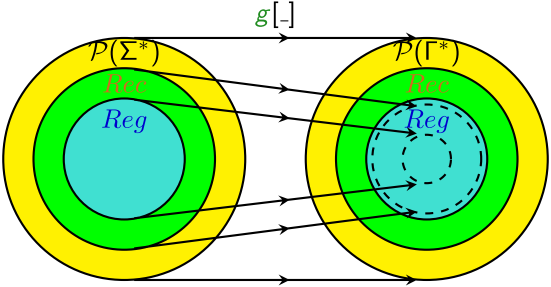

Encoding and extension lemmas

Encoding Lemma (weakening main theorem to injective encodings)

Let Σ, Γ alphabets with $|\Gamma| \geq 2$.
Then for every countable class $\mathcal{C}$ of languages over Σ,
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We say that L *is relatively regular in* M
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Extension Lemma (from injective to bijective encodings)

For every injection $f : \Sigma^* \to \Gamma^*$ there is a bijection $g : \Sigma^* \to \Gamma^*$ s.th.:

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Encoding lemma: proof

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Proof.

Let $L_1, L_2, L_3, L_4, \ldots$ be an enumeration of $\mathcal{C}$, and $w_1, w_2, w_3, \ldots$ of Σ^*.

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Define $f : \Sigma^* \to \Gamma^*$ by:

$$\begin{aligned}
f(w_1) &= L_1(w_1) \\
f(w_2) &= L_1(w_2)\, L_2(w_2) \\
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For almost all $u \in f[\Sigma^*]$: $u \in f[L_n] \iff u \in \Gamma^{n-1} 1 \Gamma^*$.

Extension lemma: proof idea

Extension Lemma (from injective to bijective encodings)

Let Σ^* be a countably infinite set.
For every injection $f : \Sigma^* \to \Gamma^*$ there is a bijection $g : \Sigma^* \to \Gamma^*$ s.th.:

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- Let $v_1, v_2, v_3, \ldots$ be an enumeration of all words Σ^*.
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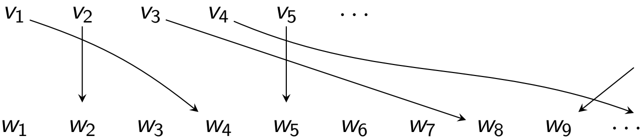
We start with an injective function f:



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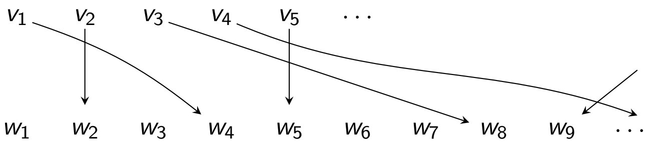

Idea: Change arrows such that every element is in the image, but so that:

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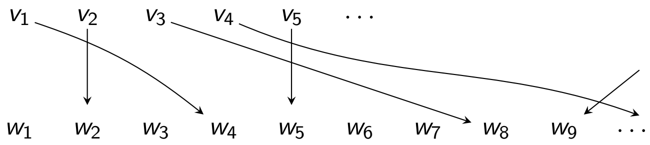

Idea: Change arrows such that every element is in the image, but so that:

- the language whose image is accepted by A_n is changed only at finitely many words (preserving relative regularity in the limit)

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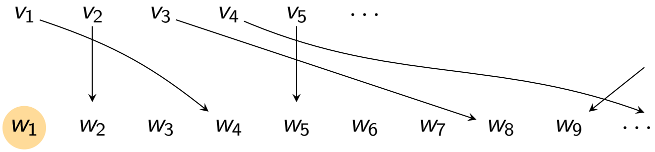

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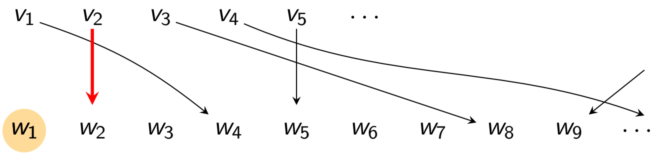

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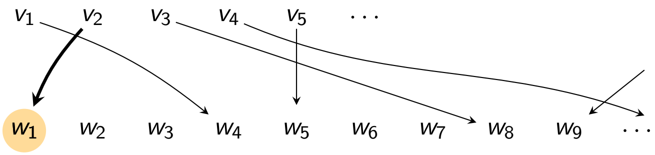

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- we pick the arrow $v_2 \to w_2$
- we redirect this arrow to target w_1

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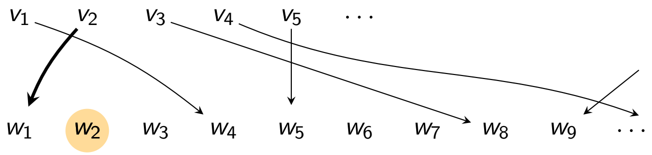

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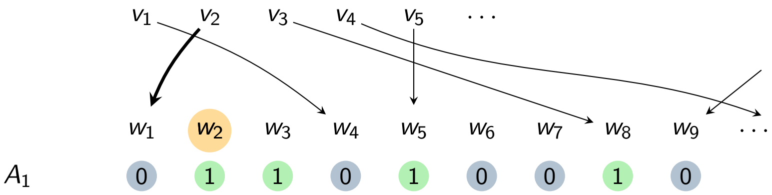

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- we take into account acceptance of the automaton A_1

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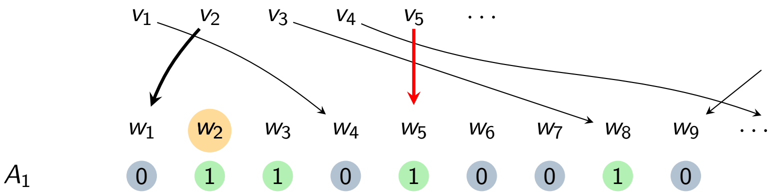

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- we pick the first arrow whose target has equal acceptance to w_2 (accepted by A_1)

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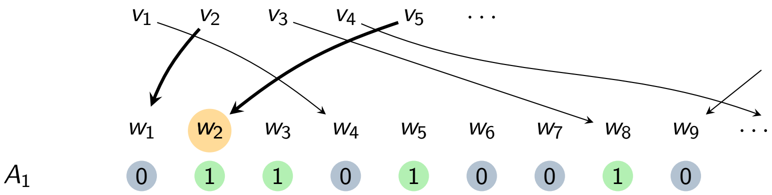

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- we redirect the arrow to w_2

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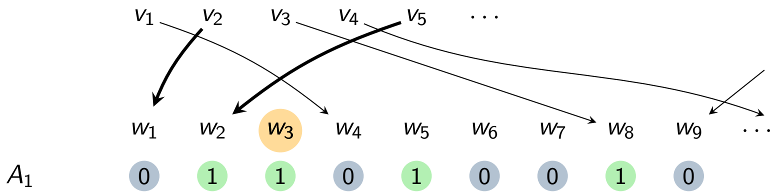

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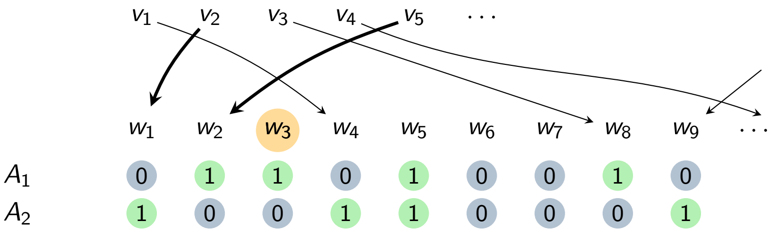

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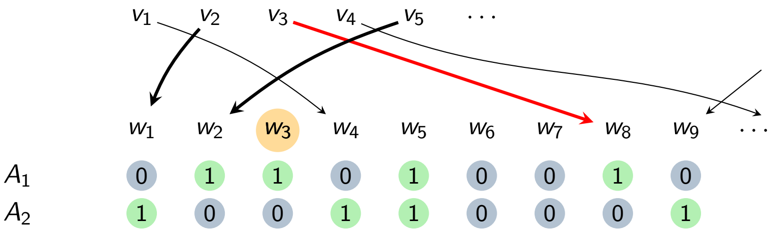

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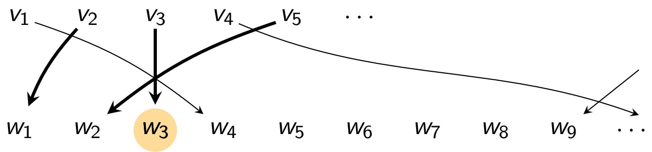

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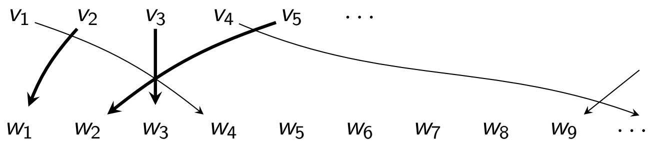

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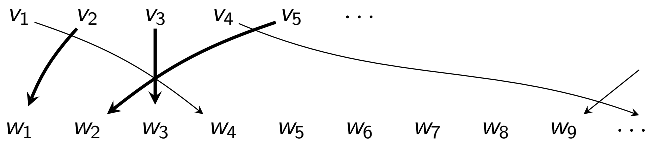

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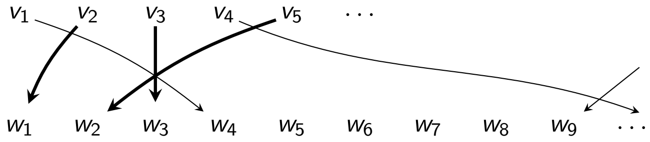

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Thus the language whose image is accepted by A_n is only disturbed with respect to finitely many words.

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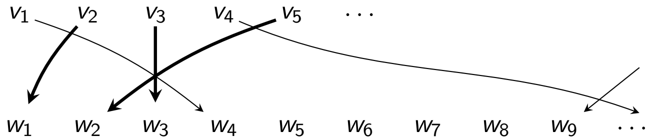

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(Relative) regularity is preserved in steps and in the limit!

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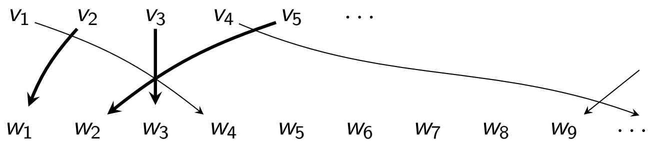

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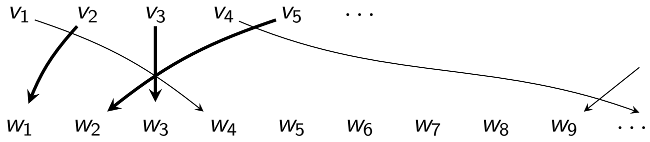

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- We show that, if not, A_n can be changed (once) to A'_n with almost the same acceptance behavior such that the choice is possible.

Consequences for comparing models of computation

Comparing computational power via encodings

- Simulation of models of computation $\mathcal{M}_1 = \langle D_1, \mathcal{F}_1 \rangle$, $\mathcal{M}_2 = \langle D_2, \mathcal{F}_2 \rangle$:

Comparing computational power via encodings

- Simulation of functions:

  function f_2 *simulates* function f_1 via *encoding* ρ if:

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\mathcal{M}_1 \qquad
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Weak requirements on encodings (Boker/Dershowitz)

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- *informally computable*/*effective*/*mechanizable in principle*
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Definition (power subsumption pre-order [Boker/Dershowitz 2006])

(i) $\mathcal{M}_1 \lesssim \mathcal{M}_2$ if: there is an injective ρ such that $\mathcal{M}_1 \lesssim_\rho \mathcal{M}_2$

(ii) $\mathcal{M}_1 \lesssim_{\text{bijective}} \mathcal{M}_2$ if: there is a bijective ρ such that $\mathcal{M}_1 \lesssim_\rho \mathcal{M}_2$

Anomalies for decision models

Our main result implies anomalies of these definitions.

$\mathcal{M} = \langle D, \mathcal{F} \rangle$ is a *decision model* if $\{0,1\} \subseteq D$, $\forall f \in \mathcal{F}\,(f[D] \subseteq \{0,1\})$.

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$\mathsf{TMD}(\Sigma)$: class of Turing machine deciders with input alphabet Σ

Anomaly (example)

$$\mathsf{TMD}(\Sigma) \lesssim_{\text{bijective}} \mathsf{DFA}(\Gamma)$$

Extensions and moral of these anomalies

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- depends on uncomputable encodings,
- highlights that uncomputable encodings must be excluded.
  - Sometimes the structure of the models $\mathcal{M}_1$ and $\mathcal{M}_2$ excludes uncomputable, bijective encodings ρ such that $\mathcal{M}_1 \lesssim_\rho \mathcal{M}_2$.
  - We give a sufficient condition for this, extending work by Shapiro (1982).

Summary

we solved a problem in language theory:

- there exist bijective word encodings
  that are regularity preserving, but not reflecting.

by showing:

- for all countable sets $\mathcal{C}$ of languages,
  there is a bijective encoding g
  such that $g[L]$ is regular for all $L \in \mathcal{C}$.

some consequences:

- for comparing models of computation via encodings:
  - use of *unrestricted* bijective encodings leads to anomalies
- in the paper: for c-automatic sequences