Confluent unfolding in the λ-calculus with letrec

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Abstract

We show that a rewriting system for unfolding terms in the λ-calculus with letrec is confluent. This system is from previous work, where we formulate letrec-unfolding as a Combinatory Reduction System (CRS). We prove confluence by applying the decreasing diagrams method to a partitioning of the parallel rewriting relation into relations that are induced by parallel steps in which a given rule contracts redexes at a given letrec-depth.

In [1] (see also [2])[1] we study infinite λ-terms and present two characterisations of those λ-terms that are expressible in the λ-calculus with letrec ($\lambda_{\mathsf{letrec}}$), in the sense that they can be obtained as the infinite unfoldings of a $\lambda_{\mathsf{letrec}}$-term. One characterisation is by a structural analysis of the term: a term is $\lambda_{\mathsf{letrec}}$-expressible if and only if there it has no infinite 'binding–capturing chains'. The other characterisation is via the concept of 'strong regularity': a term is $\lambda_{\mathsf{letrec}}$-expressible if and only if it is strongly regular.

We define a Combinatory Reduction System (CRS) for unfolding $\lambda_{\mathsf{letrec}}$-terms. In the paper confluence of the CRS is important since it guarantees that the unfolding of a term is unique. We think however, that the proof itself is of independent interest. For simplicity we use in this extended abstract an informal formulation of $\lambda_{\mathsf{letrec}}$-terms and the unfolding rewriting system instead. The set of $\lambda_{\mathsf{letrec}}$-terms $Ter(\boldsymbol{\lambda}_{\mathsf{letrec}})$ is inductively defined by the following grammar:

$$
\begin{array}{llcll}
(\textit{term}) & L & ::= & \lambda x.L & (\textit{abstraction}) \\
 & & \mid & L\,L & (\textit{application}) \\
 & & \mid & x & (\textit{variable}) \\
 & & \mid & \mathsf{letrec}\ B\ \mathsf{in}\ L & (\textit{letrec}) \\
(\textit{binding group}) & B & ::= & f_1 = L\ \ldots\ f_n = L & (\textit{equations})
\end{array}
$$

On this set we describe letrec-unfolding in the rewriting system $\boldsymbol{R}_{\triangledown}$ as follows. The names of the first four rules are chosen to reflect the kind of term that resides directly inside of the in-part of the letrec-term, which helps to see that the rules are complete in the sense that every term of the form letrec B in L is a redex.

$$
\begin{array}{rrcl}
(\varrho_{\triangledown}^{@}): & \mathsf{letrec}\ B\ \mathsf{in}\ L_0\,L_1 & \to & (\mathsf{letrec}\ B\ \mathsf{in}\ L_0)\,(\mathsf{letrec}\ B\ \mathsf{in}\ L_1) \\
(\varrho_{\triangledown}^{\lambda}): & \mathsf{letrec}\ B\ \mathsf{in}\ \lambda x.L_0 & \to & \lambda x.\mathsf{letrec}\ B\ \mathsf{in}\ L_0 \\
(\varrho_{\triangledown}^{\mathrm{letrec}}): & \mathsf{letrec}\ B_0\ \mathsf{in}\ \mathsf{letrec}\ B_1\ \mathsf{in}\ L & \to & \mathsf{letrec}\ B_0, B_1\ \mathsf{in}\ L \\
(\varrho_{\triangledown}^{\mathrm{rec}}): & \mathsf{letrec}\ B\ \mathsf{in}\ f_i & \to & \mathsf{letrec}\ B\ \mathsf{in}\ L_i \quad (\text{if } B \text{ is } f_1 = L_1 \ldots f_n = L_n) \\
(\varrho_{\triangledown}^{\mathrm{nil}}): & \mathsf{letrec}\ \ \mathsf{in}\ L & \to & L \\
(\varrho_{\triangledown}^{\mathrm{red}}): & \mathsf{letrec}\ f_1 = L_1 \ldots f_n = L_n\ \mathsf{in}\ L & \to & \mathsf{letrec}\ f_{j_1} = L_{j_1} \ldots f_{j_{n'}} = L_{j_{n'}}\ \mathsf{in}\ L \\
 & & & (\text{if } f_{j_1}, \ldots, f_{j_{n'}} \text{ are the recursion variables reachable from } L)
\end{array}
$$

'Reachable' from L in the last rule refers to recursion variables that either occur in L or on the right hand side of any equation that is reachable from L. Thus, the condition on the rule ensures that only superfluous equations are removed from the binding group.

[1] In [2] (to appear) we consider the simpler case of expressibility in the λ-calculus with μ.

Theorem. $\boldsymbol{R}_{\triangledown}$ is confluent.

Proof sketch. First all, we cannot use Newman's Lemma to prove the theorem, since $\boldsymbol{R}_{\triangledown}$ is not terminating. To show confluence of $\boldsymbol{R}_{\triangledown}$ we use the method of 'decreasing diagrams' [4, Sec. 2.3] [3, Sec. 14.2]. We use it however not to prove confluence of $\boldsymbol{R}_{\triangledown}$ directly, but of the abstract reduction system $\mathcal{A} = (Ter(\boldsymbol{\lambda}_{\mathsf{letrec}}), \{\mathrel{-\!\!\Vert\!\!\rightarrow}_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times R\})$ with R as the set of rules of $\boldsymbol{R}_{\triangledown}$ where $\mathrel{-\!\!\Vert\!\!\rightarrow}_{\rho_d}$ denotes the parallel rewriting relation on $Ter(\boldsymbol{\lambda}_{\mathsf{letrec}})$ induced by rule ρ at letrec-depth d. As a precedence order we take the order induced by the letrec-depth:

$$\rho_d \geq \sigma_{d'} \iff d \geq d'$$

The letrec-depth of a redex in $\lambda_{\mathsf{letrec}}$-term denotes the number of letrec-nodes passed on the path from the root of the term tree to the corresponding position. We write $\rightarrow_{\rho_d}$ to denote the relation induced by applying rule ρ contracting a redex at letrec-depth d.

Let us denote the rewriting relation induced by $\mathcal{A}$ by $\rightarrow_{\mathcal{A}}$:

$$\rightarrow_{\mathcal{A}} = \bigcup \{\mathrel{-\!\!\Vert\!\!\rightarrow}_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times R\}$$

If $\rightarrow_{\mathcal{A}}$ is confluent then the rewriting relation $\rightarrow_{\triangledown}$ induced by $\boldsymbol{R}_{\triangledown}$ is confluent because it holds: $\rightarrow_{\triangledown} \subseteq \rightarrow_{\mathcal{A}} \subseteq \twoheadrightarrow_{\triangledown}$ or equivalently $\twoheadrightarrow_{\mathcal{A}} = \twoheadrightarrow_{\triangledown}$ (see also [4, Lemma 2.2.5]).

We use the approach with the parallel steps because the preceding attempt to prove confluence of $\boldsymbol{R}_{\triangledown}$-steps by decreasing diagrams more directly was unsuccessful. As a precedent order we considered an ordering on the rules and lexicographic extensions of such orderings with the letrec-depth of the contracted redex. We came to the conclusion that no such order could ensure decreasingness of the elementary diagrams of both the critical pairs as well as the strictly nested redexes. This was due to redex duplication induced by the diverging steps, so that joining the diagram required a multi-step that disrupted decreasingness. In order to resolve this problem we considered parallel steps as above such that the problematic multi-step would become a single parallel step. This led to more intricate diagrams but turned out to be a viable solution.

We will prove confluence of $\rightarrow_{\mathcal{A}}$ by showing that two diverging parallel steps in $\boldsymbol{R}_{\triangledown}$ can be joined in an elementary diagram of the following form with $d \leq e$.

$$\begin{tikzcd}
{} \arrow[rr, "\rho_d"] \arrow[dd, "\sigma_e"'] & & {} \arrow[d, dotted, "\sigma_e"] \\
& & {} \arrow[d, dotted, "\sigma_{e-1}"] \\
{} \arrow[r, dotted, "\sigma_{e-1}"'] & {} \arrow[r, dotted, "\rho_d"'] & {}
\end{tikzcd} \tag{1}$$

With the precedence as above the diagram is decreasing. Note that in all the diagrams we implicitly assume the reflexive closure for all arrows. The rest of the proof is structured as follows. To justify the diagram we distinguish the cases $d = e$ and $d < e$, for which we construct diagrams that are instances of (1).

Case 1. For $d = e$ we need to consider parallel diverging steps contracting redexes at the same letrec-depth d. We construct the diagram below which is an instance of the diagram above where the diverging parallel steps are in sequentialised form. We write terms as fillings of a multihole context C with all its holes at letrec-depth d such that the contracted ρ_d- and σ_d-redexes are filled into these holes. In this way we can make explicit at which position a step takes place, i.e.

at the root of the context hole fillings. The topmost row and the leftmost column are respective sequentialisations of the parallel diverging ρ_d- and σ_d-steps into single steps.

$$\begin{tikzcd}
C[L_0,\dots,L_n] \arrow[r, "\rho_d"] \arrow[d, "\sigma_d"'] & C[L_0^\bullet, L_1\dots,L_n] \arrow[r, "\rho_d"] \arrow[d, dotted, "\sigma_d^{=}"'] & C[L_0^\bullet, L_1^\bullet, L_2\dots,L_n] \arrow[r, "\rho_d"] \arrow[d, dotted, "\sigma_d^{=}"'] & \cdots \arrow[r, "\rho_d"] & C[L_0^\bullet,\dots,L_n^\bullet] \arrow[d, dotted, "\sigma_d^{=}"'] \\
C[L_0^\circ, L_1,\dots,L_n] \arrow[r, dotted, "\rho_d", "\parallel" description] \arrow[d, "\sigma_d"'] & C[L_0^\odot, L_1,\dots,L_n] \arrow[r, dotted, "\rho_d"] \arrow[d, dotted, "\sigma_d"'] & C[L_0^\odot, L_1^\bullet, L_2,\dots,L_n] \arrow[r, dotted, "\rho_d"] \arrow[d, dotted, "\sigma_d^{=}"'] & \cdots \arrow[r, dotted, "\rho_d"] & C[L_0^\odot, L_1^\bullet,\dots,L_n^\bullet] \arrow[d, dotted, "\sigma_d^{=}"'] \\
C[L_0^\circ, L_1^\circ, L_2,\dots,L_n] \arrow[r, dotted, "\rho_d", "\parallel" description] \arrow[d, "\sigma_d"'] & C[L_0^\odot, L_1^\circ, L_2,\dots,L_n] \arrow[r, dotted, "\rho_d", "\parallel" description] \arrow[d, dotted, "\sigma_d"'] & C[L_0^\odot, L_1^\odot, L_2,\dots,L_n] \arrow[r, dotted, "\rho_d"] \arrow[d, dotted, "\sigma_d"'] & \cdots \arrow[r, dotted, "\rho_d"] & C[L_0^\odot, L_1^\odot, L_2^\bullet,\dots,L_n^\bullet] \arrow[d, dotted, "\sigma_d^{=}"'] \\
\vdots \arrow[d, "\sigma_d"'] & \vdots \arrow[d, dotted, "\sigma_d"'] & \vdots \arrow[d, dotted, "\sigma_d"'] & \ddots & \vdots \arrow[d, dotted, "\sigma_d^{=}"'] \\
C[L_0^\circ,\dots,L_n^\circ] \arrow[r, dotted, "\rho_d", "\parallel" description] & C[L_0^\odot, L_1^\circ,\dots,L_n^\circ] \arrow[r, dotted, "\rho_d", "\parallel" description] & C[L_0^\odot, L_1^\odot, L_2^\circ,\dots,L_n^\circ] \arrow[r, dotted, "\rho_d", "\parallel" description] & \cdots \arrow[r, dotted, "\rho_d", "\parallel" description] & C[L_0^\odot,\dots,L_n^\odot]
\end{tikzcd}$$

Only the tiles on the diagonal require closer attention because for all other tiles the vertical and horizontal steps take place in different holes of the context, therefore they are disjoint and consequently commute. In the tiles on the diagonal the diverging steps may be either due to a critical pair or to identical steps. In the latter case the diagram is easily joined. In case of a critical pair, since all steps take place at the same letrec-depth any such critical pair must arise from a root overlap. An exhaustive scrutiny of all these critical pairs reveals that they can be joined in a way that conforms to the tiles on the diagonal. Below two exemplary cases are shown. Note that the letrec-depths of the steps have to be increased by d according to the lifting into a context with its hole at letrec-depth d.

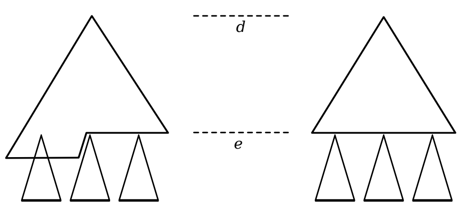
$$\begin{tikzcd}
\mathsf{letrec}\ \mathsf{in}\ L\,P \arrow[r, "@_0"] \arrow[d, "\mathrm{nil}_0"'] & (\mathsf{letrec}\ \mathsf{in}\ L)\,(\mathsf{letrec}\ \mathsf{in}\ P) \arrow[d, dotted, "{=}\mathrm{nil}_0"] \\
L\,P \arrow[r, equal] & L\,P
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{letrec}\ B\ \mathsf{in}\ L\,P \arrow[r, "@_0"] \arrow[d, "\mathrm{red}_0"'] & (\mathsf{letrec}\ B\ \mathsf{in}\ L)\,(\mathsf{letrec}\ B\ \mathsf{in}\ P) \arrow[d, dotted, "{=}\mathrm{red}_0"] \\
\mathsf{letrec}\ B'\ \mathsf{in}\ L\,P \arrow[r, dotted, "@_0"'] & (\mathsf{letrec}\ B'\ \mathsf{in}\ L)\,(\mathsf{letrec}\ B'\ \mathsf{in}\ P)
\end{tikzcd}$$

Case 2. For $d < e$ we use the same approach as for $d = e$, the diagram is however more involved. Again, we use a context C with context holes at letrec-depth d. But since $e > d$, more than one σ_e-contraction may take place in one such hole. Therefore a per-hole partitioning of the vertical steps requires a sequence of parallel steps.

The diagram below fits the scheme of the elementary diagram (1) when interleaving the σ_e-steps with the σ_{e-1}-steps in the rightmost column such that steps at depth e preceed those at depth $e-1$. Similarly for the bottommost row where the ρ_{e-1}-steps have to preceed the σ_d-steps. These reorderings are possible since the segments represent contractions within different holes of C. As in the previous diagram the tiles which do not lie on the diagonal are unproblematic, which leaves us to complete the proof by constructing the tiles on the diagonal.

$$
\begin{tikzcd}
C[L_0,\dots,L_n] \arrow[rr,"\rho_d"] \arrow[dd,"\sigma_e^{=}"'] & & C[L_0^\bullet,L_1\dots,L_n] \arrow[rr,"\rho_d"] \arrow[d,dotted,"\sigma_e^{=}"] & & C[L_0^\bullet,L_1^\bullet,L_2\dots,L_n] \arrow[r,"\rho_d"] \arrow[d,dotted,"\sigma_e^{=}"] & \cdots \arrow[r,"\rho_d"] & C[L_0^\bullet,\dots,L_n^\bullet] \arrow[d,dotted,"\sigma_e^{=}"] \\
 & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] \\
C[L_0^\circ,L_1,\dots,L_n] \arrow[r,dotted,"\sigma_{e-1}"] \arrow[dd,"\sigma_e^{=}"'] & {} \arrow[r,dotted,"\rho_d","\|" marking] & C[L_0^\odot,L_1,\dots,L_n] \arrow[rr,dotted,"\rho_d"] \arrow[dd,dotted,"\sigma_e^{=}"'] & & C[L_0^\odot,L_1^\bullet,L_2,\dots,L_n] \arrow[r,dotted,"\rho_d"] \arrow[d,dotted,"\sigma_e^{=}"] & \cdots \arrow[r,dotted,"\rho_d"] & C[L_0^\odot,L_1^\bullet,\dots,L_n^\bullet] \arrow[d,dotted,"\sigma_e^{=}"] \\
 & & & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] \\
C[L_0^\circ,L_1^\circ,L_2,\dots,L_n] \arrow[r,dotted,"\sigma_{e-1}"] \arrow[dd,"\sigma_e^{=}"'] & {} \arrow[r,dotted,"\rho_d","\|" marking] & C[L_0^\odot,L_1^\circ,L_2,\dots,L_n] \arrow[r,dotted,"\sigma_{e-1}"] \arrow[dd,dotted,"\sigma_e^{=}"'] & {} \arrow[r,dotted,"\rho_d","\|" marking] & C[L_0^\odot,L_1^\odot,L_2,\dots,L_n] \arrow[r,dotted,"\rho_d"] \arrow[dd,dotted,"\sigma_e^{=}"'] & \cdots \arrow[r,dotted,"\rho_d"] & C[L_0^\odot,L_1^\odot,L_2^\bullet,\dots,L_n^\bullet] \arrow[d,dotted,"\sigma_e^{=}"] \\
 & & & & & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] \\
\vdots \arrow[dd,"\sigma_e^{=}"'] & & \vdots \arrow[dd,dotted,"\sigma_e^{=}"'] & & \vdots \arrow[dd,dotted,"\sigma_e^{=}"'] & \ddots & \vdots \arrow[d,dotted,"\sigma_e^{=}"] \\
 & & & & & & {} \arrow[d,dotted,"\sigma_{e-1}^{=}"] \\
C[L_0^\circ,\dots,L_n^\circ] \arrow[r,dotted,"\sigma_{e-1}"] & {} \arrow[r,dotted,"\rho_d","\|" marking] & C[L_0^\odot,L_1^\circ,\dots,L_n^\circ] \arrow[r,dotted,"\sigma_{e-1}"] & {} \arrow[r,dotted,"\rho_d","\|" marking] & C[L_0^\odot,L_1^\odot,L_2^\circ,\dots,L_n^\circ] & \cdots & C[L_0^\odot,\dots,L_n^\odot]
\end{tikzcd}
$$

Every hole on the diagonal is filled with at most one ρ_d-redex (at the root of the context hole fillings) but because of $d < e$ with possibly many σ_e-redexes (properly inside of the fillings). There may or may not be an overlap between the ρ_d-step and a σ_e-step, but there can be at most one, which is due to the rules of $\boldsymbol{R}_\triangledown$.



Therefore σ_e contracts either an overlap and a number of nested redexes, or only nested redexes without an overlap. These constellations are depicted on the figure on the left. There is one ρ_d-redex and three σ_e-redexes. On the left, one of the σ_e-redexes overlaps with the ρ_d-redex while on the right all σ_e-redexes are strictly nested inside the ρ_d-redex.

For the critical pairs due to a non-root overlap, and for all situations with nested redexes, we construct diagrams of the following shape, respectively:

$$
\begin{tikzcd}
{} \arrow[rr,"\rho_0"] \arrow[d,"\sigma_1"'] & & {} \arrow[d,dotted,"\sigma_0"] \\
{} \arrow[r,dotted,"\sigma_0"'] & {} \arrow[r,dotted,"\rho_0"',"\|" marking] & {}
\end{tikzcd}
\qquad (2)
$$

$$
\begin{tikzcd}
{} \arrow[r,"\rho_0"] \arrow[d,"\sigma_e"'] & {} \arrow[d,dotted,"\sigma_{e'}"] \\
{} \arrow[r,dotted,"\rho_0"'] & {}
\end{tikzcd}
\qquad e' \in \{e, e-1\} \qquad (3)
$$

When lifted into a context of letrec-depth d both of the diagrams comply to the shape necessary for the diagonal tiles, but we need to be able to handle situations as on the left of the above figure, where both nested redexes as well as the overlapping redex are contracted. Firstly, since all σ-redexes occur at the same letrec-depth, it must hold that $d = 0$ and $e = 1$, which is due to the rules of $\boldsymbol{R}_{\triangledown}$. Secondly, none of the involved redex contractions affect any of the nested redexes except for duplicating or erasing them, which means that the residuals of the σ-steps after these steps are part of a parallel $\sigma_{e'}$-step (mind that we assume the reflexive closure of all steps). Or in a diagram:

$$
\begin{tikzcd}
{} \arrow[rr, "\rho_0"] \arrow[dd, "\sigma_1 ="'] & & {} \arrow[d, dotted, "=\sigma_{e_1}"] & \\
& & \vdots \arrow[d, dotted, "=\sigma_{e_n}"] & e_i \in \{0,1\} \\
{} \arrow[rr, dotted, "\rho_0"] \arrow[d, "\sigma_1"'] & & {} \arrow[d, dotted, "\sigma_0"] & \\
{} \arrow[r, dotted, "\sigma_0"'] & {} \arrow[r, dotted, "\parallel \rho_0"'] & {} &
\end{tikzcd}
$$

The diagram is composed from the previous two diagrams. A parallel version of (3) constitutes the top part, while the bottom part is an exact replica of (2). The top part settles the portion arising from the nested redexes, the bottom part settles the portion arising from the overlapping redex.

At last in order to fit that diagram into the scheme of the diagonal tiles the steps on the right have to be reordered such that σ_{e_i}-steps with $e_i = 1$ preceed σ_{e_i}-steps with $e_i = 0$. The reordering is viable because every σ_{e_i}-step takes place in its own residual of the σ_1-step from the left.

We conclude the proof by a comprehensive analysis of all critical pairs that arise from non-root overlaps in $\boldsymbol{R}_{\triangledown}$ as well as the diagrams for joining nested redexes. Below, one critical pair is shown for each case. See [1] for an exhaustive scrutinisation.

$$
\begin{tikzcd}
\mathsf{letrec}\ B\ \mathsf{in}\ \mathsf{letrec}\ C\ \mathsf{in}\ L\,P \arrow[r, "\mathrm{letrec}_0"] \arrow[d, "@_1"'] & \mathsf{letrec}\ B\ C\ \mathsf{in}\ L\,P \arrow[dd, dotted, "@_0"] \\
\mathsf{letrec}\ B\ \mathsf{in}\ (\mathsf{letrec}\ C\ \mathsf{in}\ L)(\mathsf{letrec}\ C\ \mathsf{in}\ P) \arrow[d, dotted, "@_0"'] & \\
(\mathsf{letrec}\ B\ \mathsf{in}\ \mathsf{letrec}\ C\ \mathsf{in}\ L)(\mathsf{letrec}\ B\ \mathsf{in}\ \mathsf{letrec}\ C\ \mathsf{in}\ P) \arrow[r, dotted, "\parallel"', "\mathrm{letrec}_0"'] & (\mathsf{letrec}\ B\ C\ \mathsf{in}\ L)(\mathsf{letrec}\ B\ C\ \mathsf{in}\ P)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{letrec}\ B\ \mathsf{in}\ L_0\,L_1 \arrow[r, "@_0"] \arrow[d, "\sigma_e"'] & (\mathsf{letrec}\ B\ \mathsf{in}\ L_0)(\mathsf{letrec}\ B\ \mathsf{in}\ L_1) \arrow[d, dotted, "=\sigma_e"] \\
\mathsf{letrec}\ B'\ \mathsf{in}\ L_0'\,L_1' \arrow[r, dotted, "@_0"'] & (\mathsf{letrec}\ B'\ \mathsf{in}\ L_0')(\mathsf{letrec}\ B'\ \mathsf{in}\ L_1')
\end{tikzcd}
$$

$\square$

A generalisation of the proof to obtain a theorem is still very much work in progress. Below are preliminary propositions that are to capture the essential properties of $\boldsymbol{R}_\nabla$ that made the above approach possible.

The following lemma requires the rewriting steps to be partitioned in a way such that diverging parallel steps cannot be 'intertwined', permitting the construction of a decreasing diagram using parallel steps as in the proof above.

Lemma 1. *Let $\mathcal{A}$ be an ARS $(A, \{\to_\alpha \mid \alpha \in I\})$ that is induced by a TRS/CRS/HRS, where the index set I is equipped with a well-founded partial order. The steps Φ of $\mathcal{A}$ are equipped with respective indices from I as well as with the position ($pos : \Phi \to \mathbb{N}$) of the contracted redex. Indexed single steps $\to_\alpha$ induce indexed parallel steps $\mathrel{+\!\!\!\!\!\!\;\Vert\!\!\!\to}_\alpha$ (for all $\alpha \in I$). If the following conditions are met, then $\to_I = \bigcup_{i \in I} \to_i$ is confluent.*

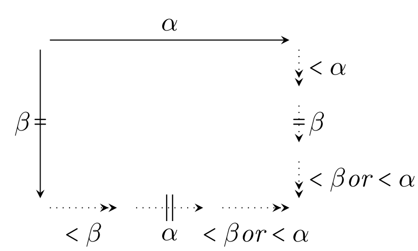
1. *There do not exist $\alpha, \beta \in I$, two α-steps ρ_1, ρ_2, and two β-steps σ_1, σ_2 such that it holds:*

$$pos(\rho_1) \parallel pos(\rho_2) \wedge pos(\sigma_1) \parallel pos(\sigma_2)$$

$$pos(\rho_1) < pos(\sigma_1) \wedge pos(\rho_2) > pos(\sigma_2)$$

   Here we use $\parallel$ for incomparable ('parallel') positions: $p \parallel q \Leftrightarrow p \nleq q \wedge q \nleq p$.

2. *A diverging α-step at position p and a parallel β-step with positions $q_1, \ldots, q_n$ below p ($\forall i \in \{1, \ldots, n\} : p \leq q_i$) can be joined by a diagram of the following form:*



   Thereby all closing steps need to take place below p, i.e. at positions $\geq p$.

A second specialised lemma is to be more concrete and easier to apply. It includes in the index a notion of depth (cf. letrec-depth), to which the order on the index is linked, such that condition 1 of Lemma 1 is met. Furthermore we stipulate properties of the rewriting relation, that allow for an order-respecting context embedding of rewriting steps. This will simplify condition 2 of Lemma 1 such that the diagram has only to be constructed for critical overlaps at the root of a term.

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