Confluent Unfolding in the λ-calculus with letrec

Jan Rochel and Clemens Grabmayer

Dept. of Information & Computing Sciences
Dept. of Philosophy
Utrecht University

28 June 2013

$\boldsymbol{\lambda}_{\text{letrec}}$ and unfolding semantics

$$
\begin{array}{llcll}
(\textit{term}) & L & ::= & \lambda x. L & (\textit{abstraction}) \\
 & & \mid & L\, L & (\textit{application}) \\
 & & \mid & x & (\textit{variable}) \\
 & & \mid & \mathsf{letrec}\ B\ \mathsf{in}\ L & (\textit{letrec}) \\
(\textit{binding group}) & B & ::= & f_1 = L\ \ \ldots\ \ f_n = L & (\textit{equations})
\end{array}
$$

Example

$[\![\mathsf{letrec}\ f = \lambda x. f\, x\ \mathsf{in}\ f]\!] \;=\; \lambda x. (\lambda x. (\ldots)\, x)\, x$

$\boldsymbol{\lambda}_{\text{letrec}}$ and unfolding semantics

$$
\begin{array}{llcll}
(\textit{term}) & L & ::= & \lambda x.L & (\textit{abstraction}) \\
 & & \mid & L\,L & (\textit{application}) \\
 & & \mid & x & (\textit{variable}) \\
 & & \mid & \text{letrec } B \text{ in } L & (\textit{letrec}) \\
(\textit{binding group}) & B & ::= & f_1 = L \ \ \ldots \ \ f_n = L & (\textit{equations})
\end{array}
$$

Example

letrec $f = \lambda x. f\, x$ in $f \twoheadrightarrow_{\nabla} \lambda x.(\lambda x.(\ldots)\, x)\, x$

A CRS for unfolding $\boldsymbol{\lambda}_{\mathsf{letrec}}$-terms

Example

$$
\begin{aligned}
& \mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ f \\
\to_{\mathsf{rec}}\ & \mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ \lambda x. f\ x \\
\to_{\lambda}\ & \lambda x. \mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ f\ x \\
\to_{@}\ & \lambda x. (\mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ f)\, (\mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ x) \\
\to_{\mathsf{red}}\ & \lambda x. (\mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ f)\, (\mathsf{letrec}\ \ \mathsf{in}\ x) \\
\to_{\mathsf{nil}}\ & \lambda x. (\mathsf{letrec}\ f = \lambda x. f\ x\ \mathsf{in}\ f)\, x \\
\to_{\mathsf{rec}}\ & \lambda x.\ (\ldots)\ \ x
\end{aligned}
$$

A CRS for unfolding $\boldsymbol{\lambda}_{\mathsf{letrec}}$-terms

$$
\begin{array}{rrcl}
(\varrho^{@}_{\nabla}): & \mathsf{letrec}\ B\ \mathsf{in}\ L_0\, L_1 & \to & (\mathsf{letrec}\ B\ \mathsf{in}\ L_0)\,(\mathsf{letrec}\ B\ \mathsf{in}\ L_1) \\
(\varrho^{\lambda}_{\nabla}): & \mathsf{letrec}\ B\ \mathsf{in}\ \lambda x. L_0 & \to & \lambda x.\mathsf{letrec}\ B\ \mathsf{in}\ L_0 \\
(\varrho^{\mathsf{letrec}}_{\nabla}): & \mathsf{letrec}\ B_0\ \mathsf{in}\ \mathsf{letrec}\ B_1\ \mathsf{in}\ L & \to & \mathsf{letrec}\ B_0, B_1\ \mathsf{in}\ L \\
(\varrho^{\mathsf{rec}}_{\nabla}): & \mathsf{letrec}\ B\ \mathsf{in}\ f_i & \to & \mathsf{letrec}\ B\ \mathsf{in}\ L_i \quad (\text{if } B \text{ is } f_1 = L_1 \ldots f_n = L_n) \\
(\varrho^{\mathsf{nil}}_{\nabla}): & \mathsf{letrec}\ \ \mathsf{in}\ L & \to & L \\
(\varrho^{\mathsf{red}}_{\nabla}): & \mathsf{letrec}\ f_1 = L_1 \ldots f_n = L_n\ \mathsf{in}\ L & \to & \mathsf{letrec}\ f_{j_1} = L_{j_1} \ldots f_{j_{n'}} = L_{j_{n'}}\ \mathsf{in}\ L \\
 & & & (\text{if } f_{j_1}, \ldots, f_{j_{n'}} \text{ are the recursion variables reachable from } L)
\end{array}
$$

Confluence by Decreasing Diagrams

Usually:

$$\to_\nabla \;=\; \bigcup\{\to_i \mid i \in I\}$$

Here:

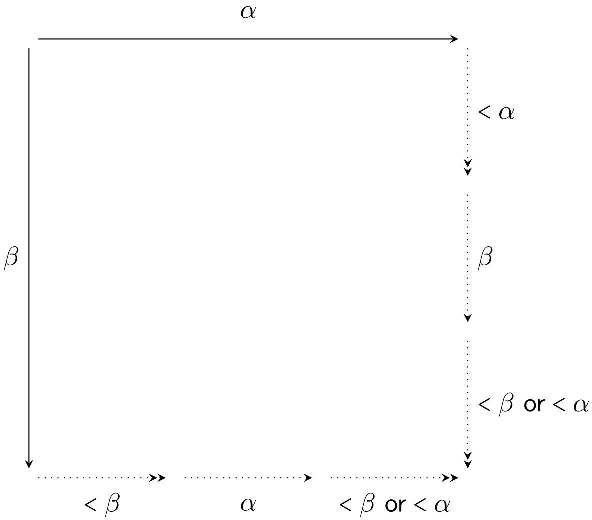
$$\to_{\mathcal{A}} = \bigcup\{\mathrel{+\!\!\!|\!|\!\!\!\to}_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$$

where $\mathrel{+\!\!\!|\!|\!\!\!\to}_{\rho_d}$ is a parallel ρ-step at letrec-depth d.

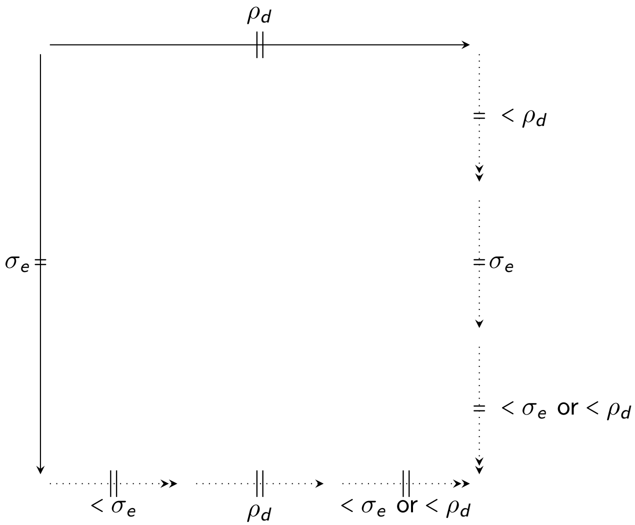
It holds:

$$\to_\nabla \subseteq \to_{\mathcal{A}} \subseteq \twoheadrightarrow_\nabla \quad \text{or equivalently} \quad \twoheadrightarrow_{\mathcal{A}} = \twoheadrightarrow_\nabla$$

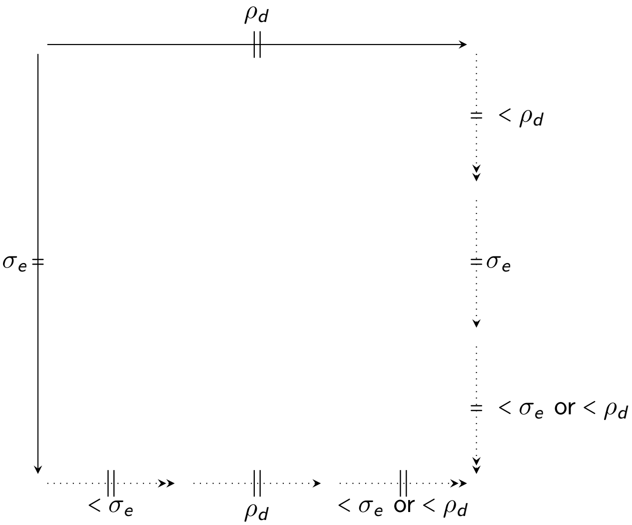
Confluence by Decreasing Diagrams



Elementary Diagram for $\bigcup\{\twoheadrightarrow\!\!\!\!\!\!\|\ _{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$



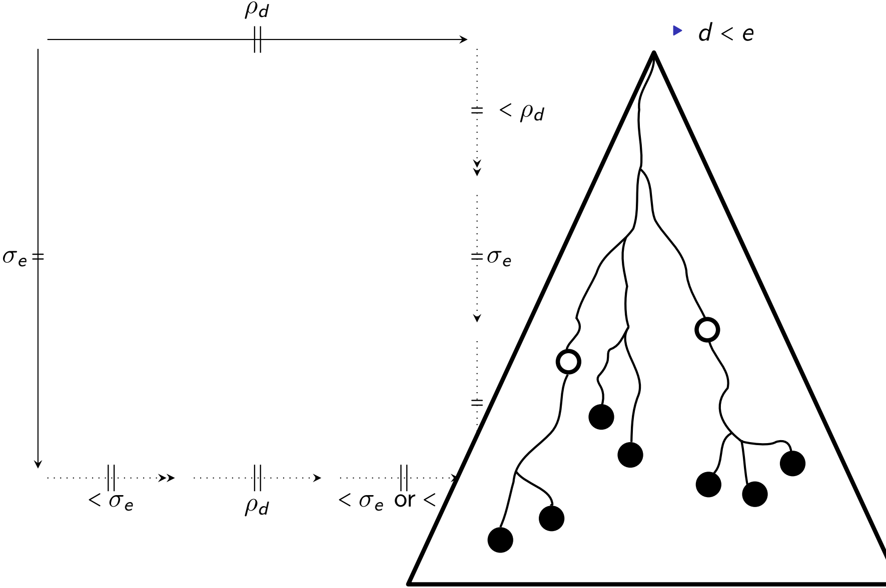
Elementary Diagram for $\bigcup\{\xrightarrow{\|}_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$



Two cases:

- $d = e$
- $d < e$

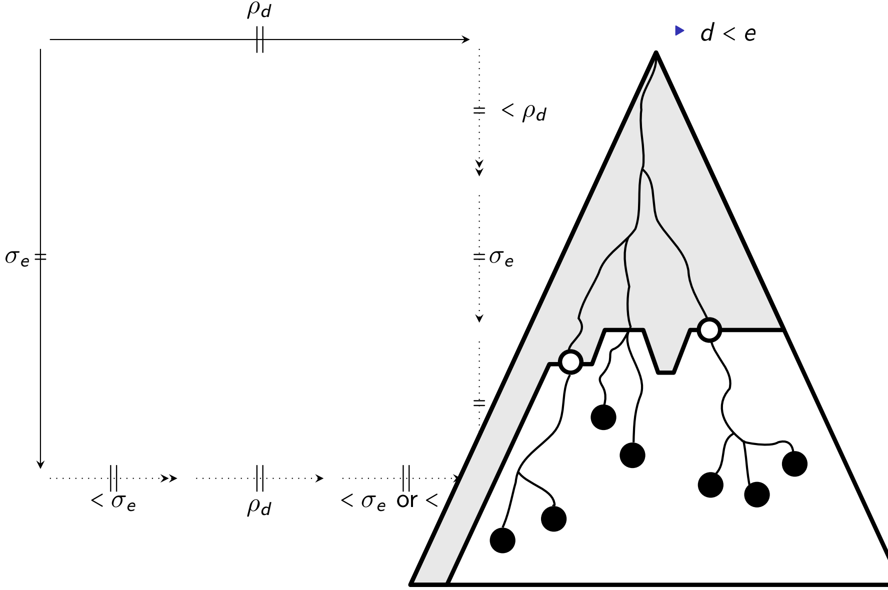
Elementary Diagram for $\bigcup\{\mathrel{+\!\!\!|\!\!\!\rightarrow}_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$



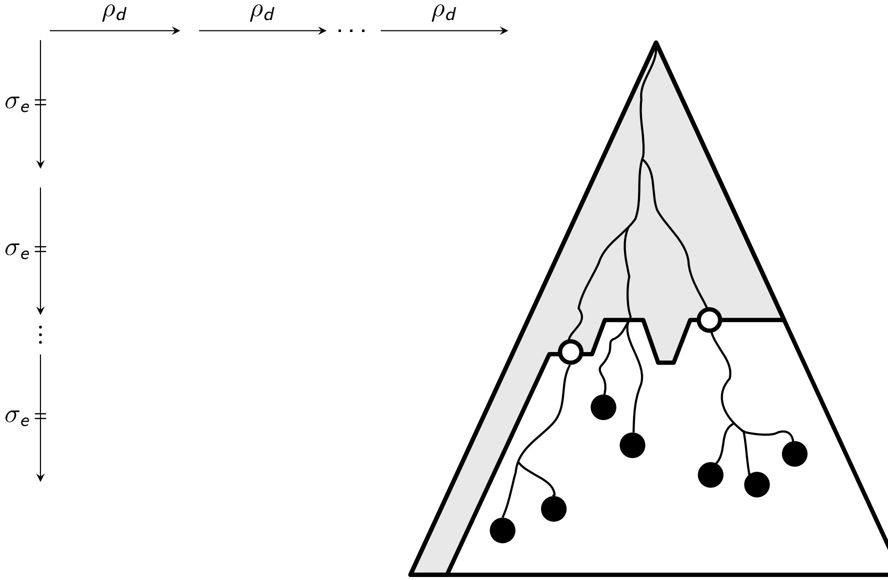
- $d < e$

Elementary Diagram for $\bigcup\{\twoheadrightarrow\!\!\!\!\!\|\;_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$

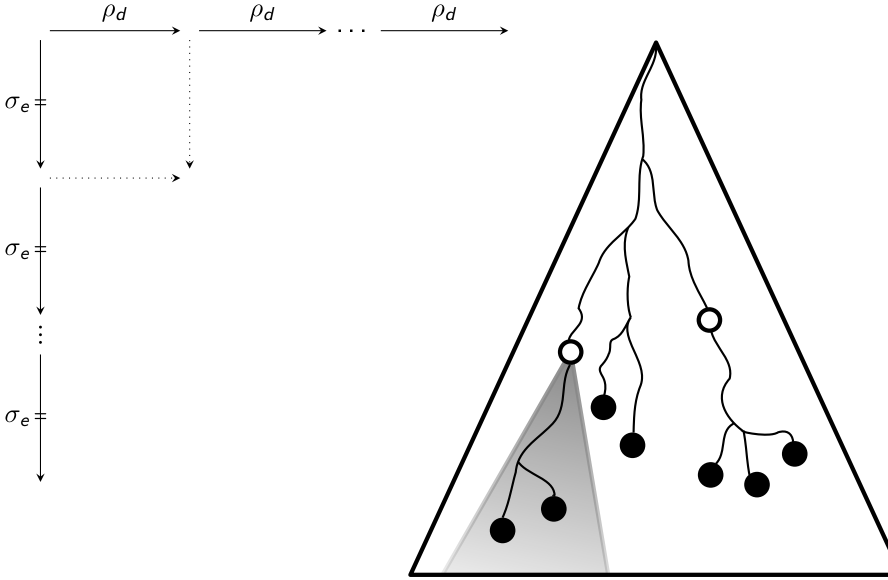
- $d < e$



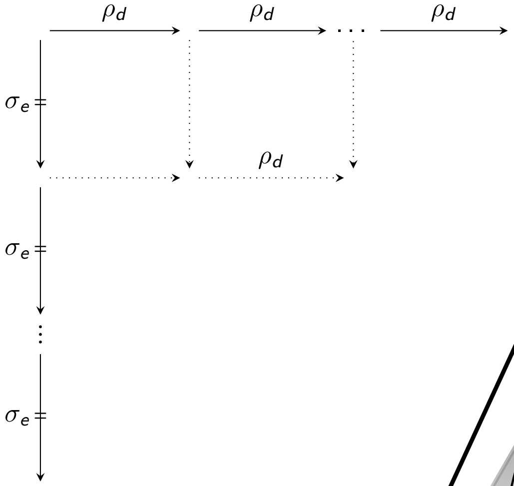
Elementary Diagram for $\bigcup\{\twoheadrightarrow_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$

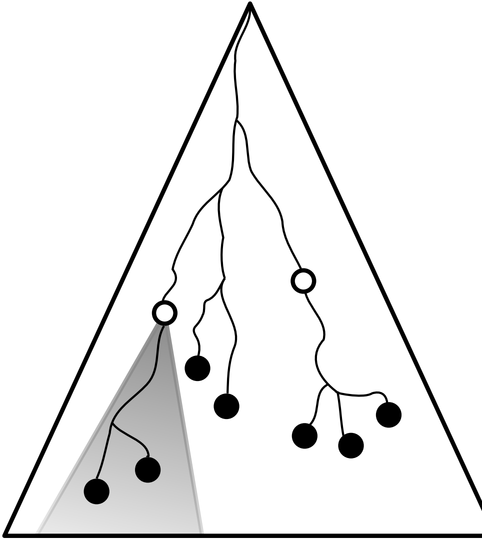


Elementary Diagram for $\bigcup\{\twoheadrightarrow_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$

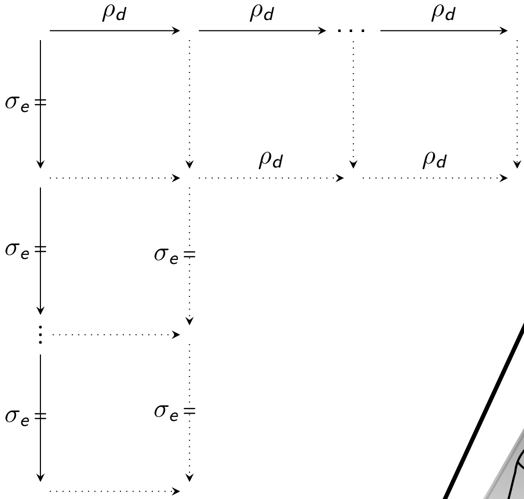


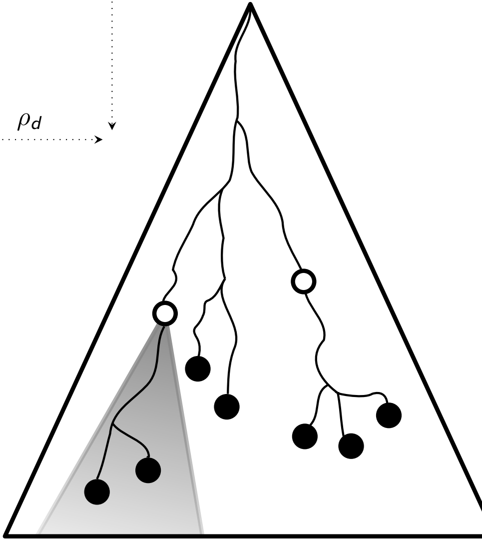
Elementary Diagram for $\bigcup\{ -\!\!\!\|\!\!\!\rightarrow_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules} \}$



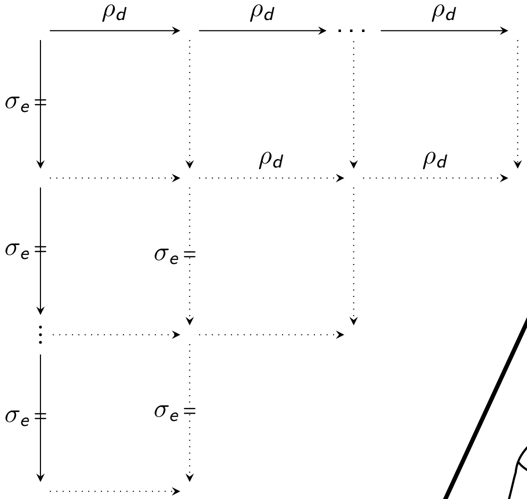


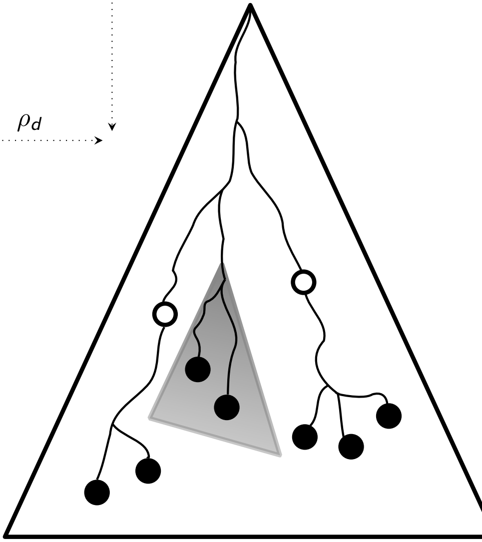
Elementary Diagram for $\bigcup\{\mathrel{-\!\!\!\mid\!\mid\!\!\!\rightarrow}_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$





Elementary Diagram for $\bigcup\{\twoheadrightarrow\!\!\!\!\!\!\mid\;\;_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$





Elementary Diagram for $\bigcup\{\twoheadrightarrow_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$

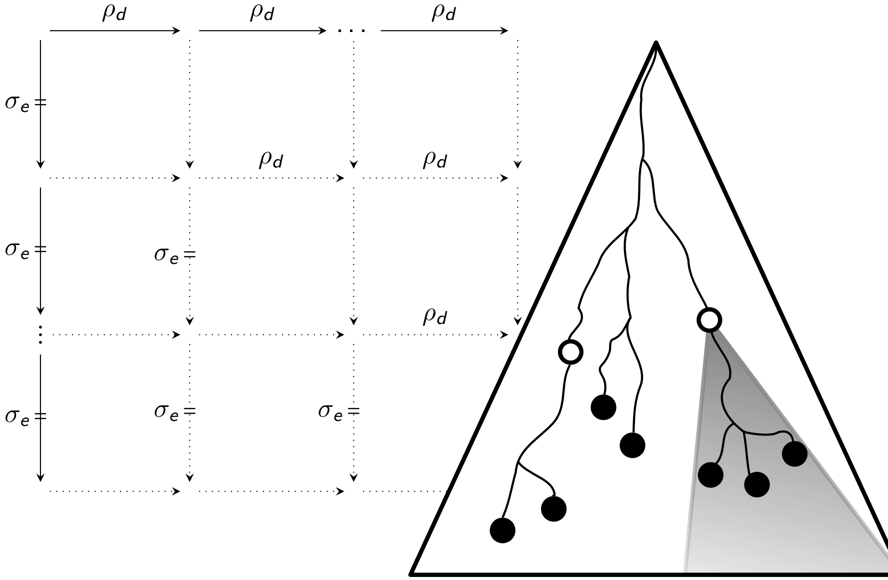


Diagram for a Diagonal Tile

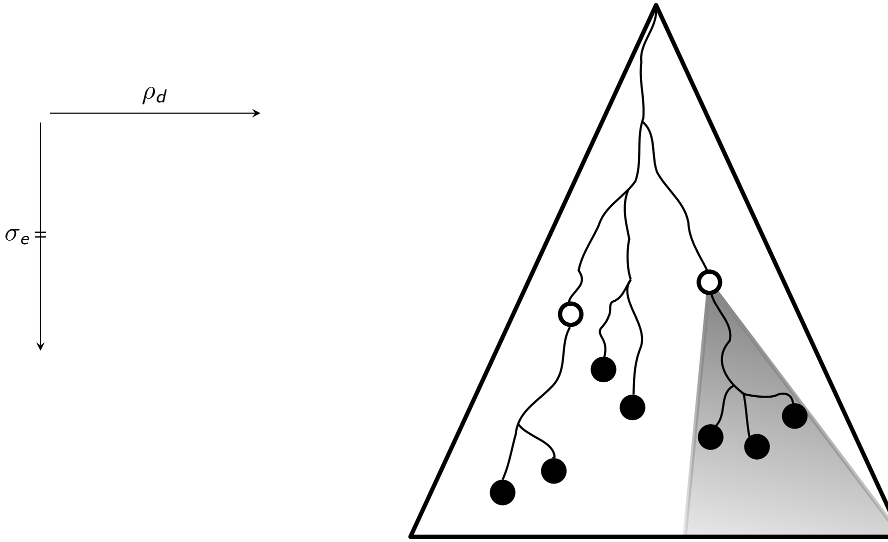
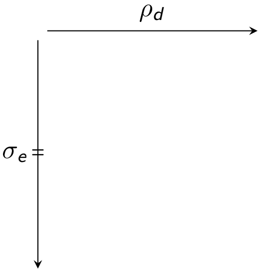


Diagram for a Diagonal Tile



Two cases:

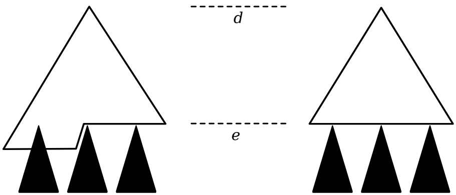
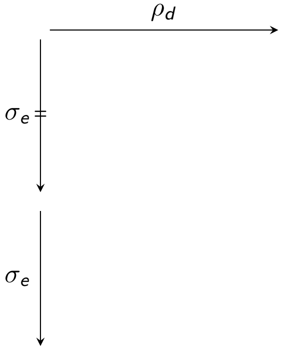


Diagram for a Diagonal Tile



letrec B_0 in letrec B_1 in L

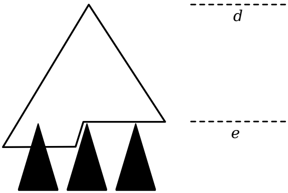
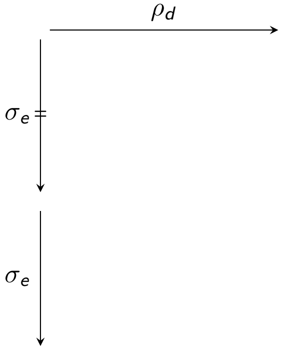


Diagram for a Diagonal Tile



$e = d + 1$

letrec B_0 in letrec B_1 in L

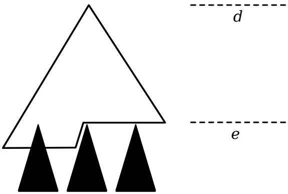
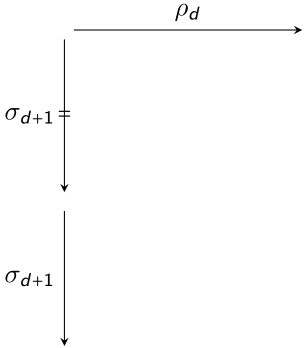


Diagram for a Diagonal Tile



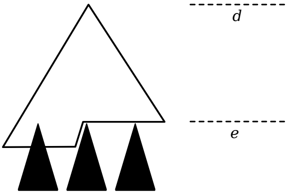
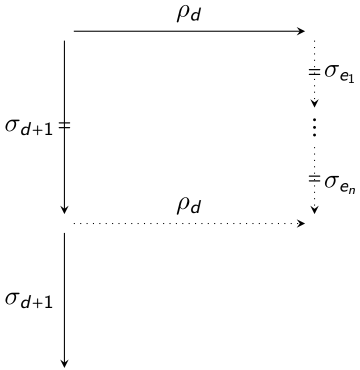


Diagram for a Diagonal Tile



$e_i \in \{d, d+1\}$

$(\varrho^{\mathsf{nil}}_{\nabla})$: letrec in $L \to L$

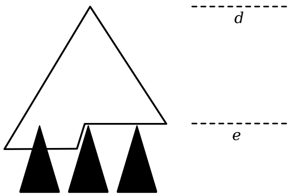
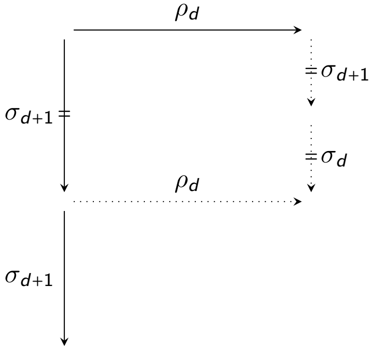


Diagram for a Diagonal Tile



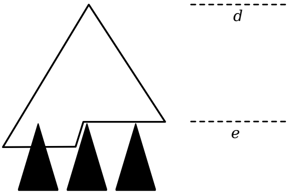
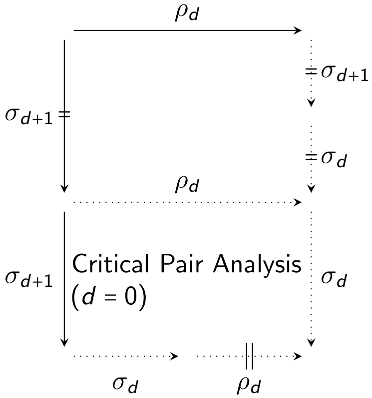


Diagram for a Diagonal Tile



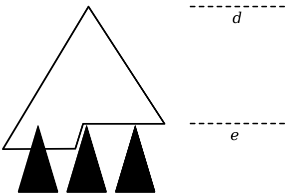


Diagram for a Diagonal Tile

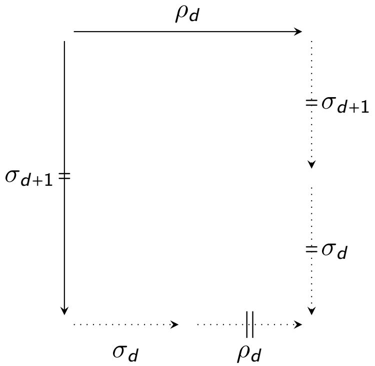
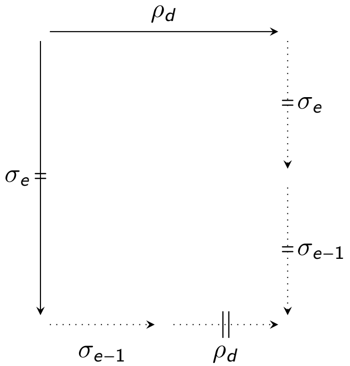
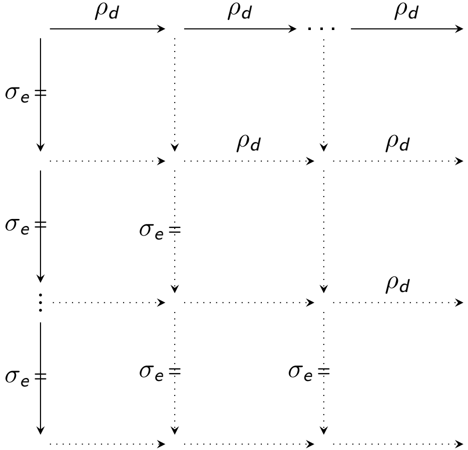


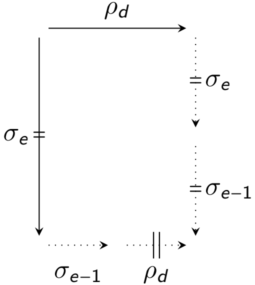
Diagram for a Diagonal Tile



$(e = d + 1)$

Elementary Diagram for $\bigcup\{\nrightarrow_{\rho_d} \mid (d,\rho) \in \mathbb{N} \times \mathit{Rules}\}$





Elementary Diagram for $\bigcup\{\twoheadrightarrow_{\rho_d} \mid (d, \rho) \in \mathbb{N} \times \mathit{Rules}\}$