On the Star Height of Regular Expressions Under Bisimulation

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August 26, 2006 / EXPRESS'06

Overview

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The Process Interpretation P (Milner)

$\mathbf{0} \stackrel{P}{\longmapsto}$ deadlock δ

$1 \stackrel{P}{\longmapsto}$ empty process ϵ

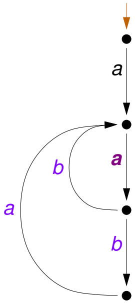
$a \stackrel{P}{\longmapsto}$ atomic action a

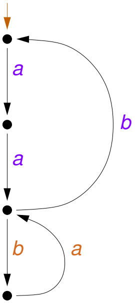
$e + f \stackrel{P}{\longmapsto}$ alternative composition between $P(e)$ and $P(f)$

$e \cdot f \stackrel{P}{\longmapsto}$ sequential composition of $P(e)$ and $P(f)$

$e^{*} \stackrel{P}{\longmapsto}$ unbounded iteration of $P(e)$

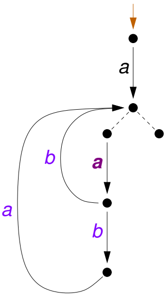
The Process Interpretation P

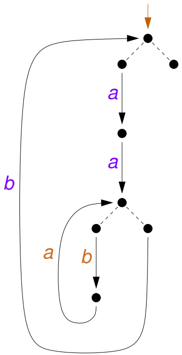




$$P\left(a \cdot \left(a \cdot (b + b \cdot a)\right)^* \cdot 0\right) \qquad P\left(\left(a \cdot a \cdot (b \cdot a)^* \cdot b\right)^* \cdot 0\right)$$

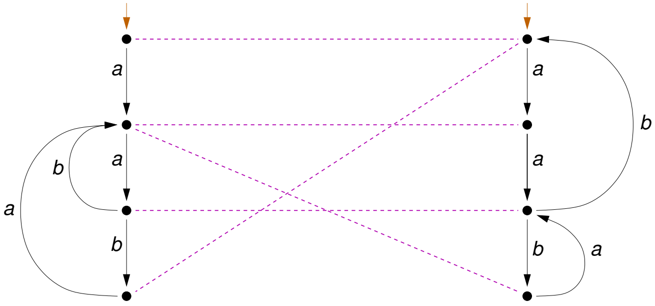
The Process Interpretation *P*





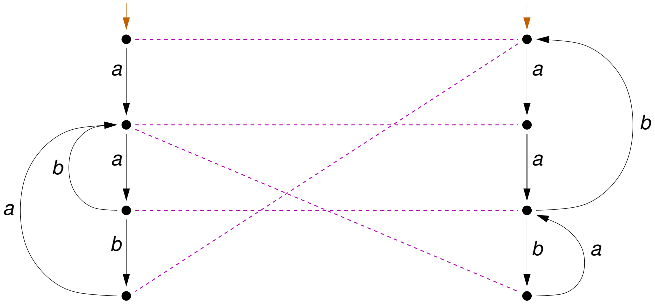
$$P\big(a\cdot(a\cdot(b+b\cdot a))^{*}\cdot 0\big) \qquad P\big((a\cdot a\cdot(b\cdot a)^{*}\cdot b)^{*}\cdot 0\big)$$

Regular Expressions under Bisimulation



$$\boldsymbol{P}\big(\, a(a(b+ba))^* \cdot 0 \,\big) \quad \Leftrightarrow \quad \boldsymbol{P}\big(\, (aa(ba)^* a)^* \cdot 0 \,\big)$$

Regular Expressions under Bisimulation



$$\left(a(a(b+ba))^{*}\cdot 0 \quad \underline{\leftrightarrow}_{P} \quad (aa(ba)^{*}a)^{*}\cdot 0\right.$$

The Process Interpretation P (Transition System)

$$\frac{}{P(a) \xrightarrow{a} 1} \qquad \frac{}{1\downarrow}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e+f) \xrightarrow{a} P(e')} \qquad \frac{P(e)\downarrow}{P(e+f)\downarrow}$$

$$\frac{P(f) \xrightarrow{a} P(f')}{P(e+f) \xrightarrow{a} P(f')} \qquad \frac{P(f)\downarrow}{P(e+f)\downarrow} \qquad \frac{P(e)\downarrow \quad P(f)\downarrow}{P(e \cdot f)\downarrow}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e \cdot f) \xrightarrow{a} P(e' \cdot f)} \qquad \frac{P(e)\downarrow \quad P(f) \xrightarrow{a} P(f')}{P(e \cdot f) \xrightarrow{a} P(f')}$$

$$\frac{P(e) \xrightarrow{a} P(e')}{P(e^*) \xrightarrow{a} P(e' \cdot e^*)} \qquad \frac{}{P(e^*)\downarrow}$$

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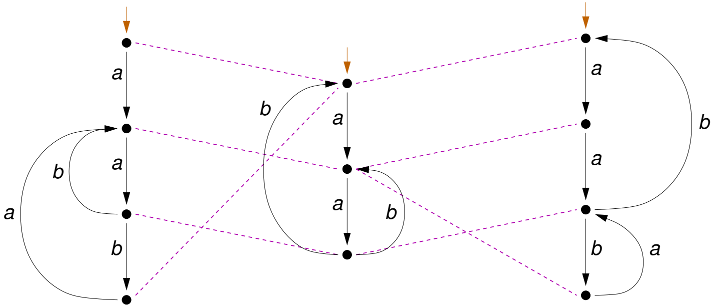
Regular Expressions under Bisimulation

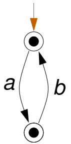


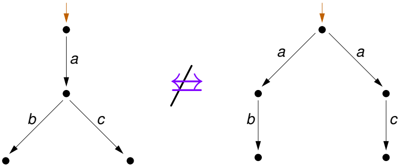
"*Two-exit iteration*"

$\notin im(\boldsymbol{P})$

Properties of the Process Interpretation P

- There are finite transition graphs that are *not isomorpic* to any process graph $P(e)$ in the image of P.
- *What is more:* there are finite transition graphs that are *not bisimilar* to any process graph $P(e)$ in the image of P.
- Identities $e \leftrightarrow_P f$ under P also hold as identities $e =_L f$ under the language intepretation L. The converse is false:





$a \cdot (b + c) \quad \not\leftrightarrow_P \quad a \cdot b + a \cdot c$

Milner's Questions (1984)

1. *Is a variant of Salomaa's axiomatisation for language equality complete for* $\leftrightarrow_{P}$ *?*

   – To my knowledge: Yet unsolved. (Partial & related results by Sewell; Fokkink; Corradini/De Nicola/Labella; C.G.)

2. *What structural property characterises the finite-state proc's that are bisimilar to proc's in the image of* P *?*

   – Definiability by "well-behaved" specifications ([BC05]); this is decidable ([BCG05]).

3. *Does "minimal star height" over single-letter alphabets define a hierarchy modulo* $\leftrightarrow_{P}$*?*

   – Yes! (Hirshfeld and Moller, 1999).

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Star Height, and Star Height of Regular Languages

The *star height* $sh(e)$ of a regular expression e is the maximum number of nested stars in e.

For example: $sh(\,(a+b)c\,) = 0\,,\;\; sh(\,(a(ba)^*a)^*dc^*\,) = 2\,.$

Definition

The *(restricted) star height* $sh(L)$ of a regular language L is the least natural number n such that $sh(e) = n$ for some regular expression e that represents L.

Generalised Star Height: concerning
generalised regular expressions in which
complementation and *intersection* may occur.

Classical Results on (Restricted) Star Height

1. *Every regular language over a single-letter alphabet has star height 1 at most.*
2. *There are regular languages with any preassigned star height* (Eggan, 1963);
   ... even over a two-letter alphabet (McNaughton, 1965, Dejean/Schützenberger, 1966);
3. *There exists an algorithm for computing the star height of the regular language given by a regular expression* (Hashiguchi, 1983).
   (The (Restricted) Star Height Problem is solvable).

Minimal Star Height under *P*

Definition

The *minimal star height* $msh(e)$ *(under P)* of a regular expression e is the least natural number n such that there exists a regular expression e_{min} with $sh(e_{min}) = n$ and $e_{min} \leftrightarrows_{P} e$.

Remark. For all $e \in RegExps$ it holds: $sh(L(e)) \leq msh(e)$.

Results for Minimal Star Height under *P*?

1. *For every $n \in \mathbb{N}$, there exists a regular expression f_n over the single-letter alphabet such that the minimal star height of f_n is n* (Hirshfeld/Moller, 2000).
2. Consequently: *For the set regular expressions over a non-empty alphabet, "minimal star height under **P**" defines a proper hierarchy.*
3. Is the Star-Height Problem under ***P*** solvable?

   *The Star Height Problem under **P***
   Instance: $e \in RegExps(\Sigma)$
   Question: What is the minimal star height of e under ***P***?

Well-Behaved Specifications (Motivation): A Correspondence Theorem

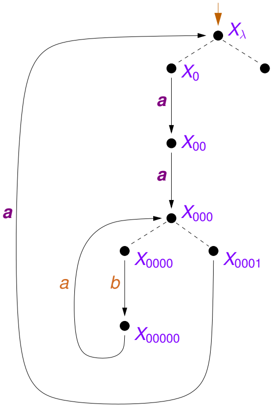
Theorem ([BC05])

Expressibility as a regular expression under ***P***
is equivalent to
definability by a well-behaved specification:

For all processes p ,

$$(\exists\, e \in RegExps)\,\big[\, p \leftrightarrow \boldsymbol{P}(e) \,\big] \iff (\exists\, \mathcal{E} \in WBSpecs)\,\big[\, p \text{ is a solution of } \mathcal{E} \,\big]$$

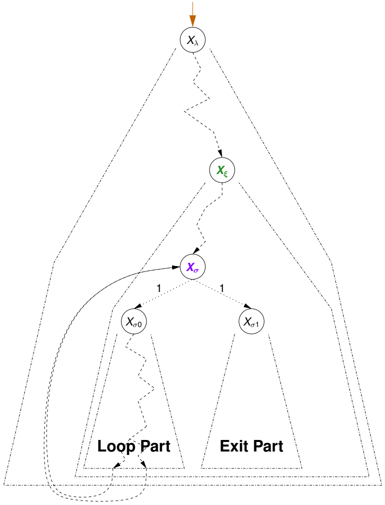
Well-Behaved Specifications (Example)



$$
\begin{aligned}
X_\lambda &= 1 \cdot X_0 + 1 \cdot X_1 \\
X_0 &= a \cdot X_{00} \\
X_{00} &= a \cdot X_{000} \\
X_{000} &= 1 \cdot X_{0000} + 1 \cdot X_{0001} \\
X_{0000} &= b \cdot X_{00000} \\
X_{00000} &= a \cdot X_{000} \\
X_{0001} &= a \cdot X_\lambda \\
X_1 &= 0
\end{aligned}
$$

$$P((aa(ba)^* a)^*.0)$$

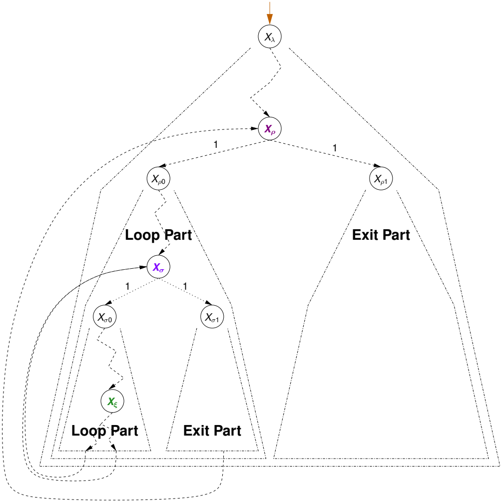
Well-Behaved Specifications (Some Intuition, I)



X_ξ, X_λ … *well-behaved variables*
(X_ξ "does not return" to a recursion variable above itself)

X_σ is a *cycling variable*
(Some recursion variable below X_σ "returns to" X_σ)

Well-Behaved Specifications (Some Intuition, II)



X_σ, X_ρ ... cycling variables

X_ξ *cycles back to* X_σ

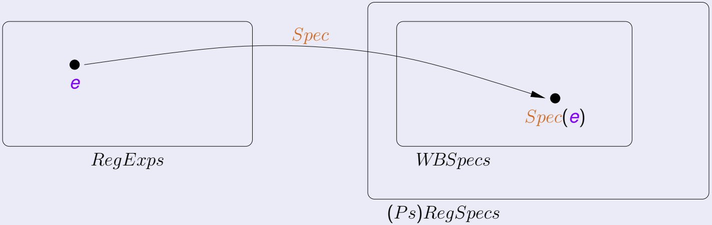
(The nearest return of X_ξ to a rec.var. above is to X_σ)

X_σ *cycles back to* X_ρ

Definability Lemma

Lemma (Definability by well-behaved spec's [BC05])

The processes represented by regular expressions under ***P*** *are definable by well-behaved specifications.*
Moreover: there is an effectively computable mapping
$Spec : RegExps(\Sigma) \rightarrow WBSpecs(\Sigma)$ *such that*

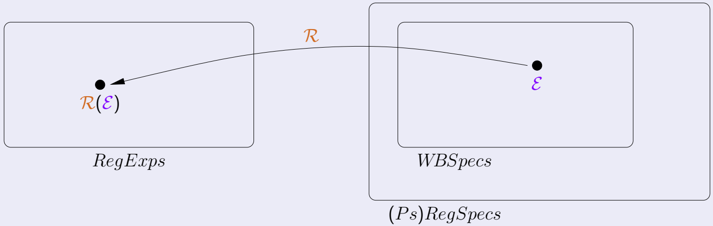


for all $e \in RegExps(\Sigma)$ *,* ***P***(e) *is a solution of* $Spec(e)$.

Solvability Lemma

Lemma (Solvability of well-behaved spec's [BC05])

Every well-behaved specification is solved by a process represented by a regular expression.
Moreover: there is an effectively computable mapping $\mathcal{R} : \mathit{WBSpecs}(\Sigma) \to \mathit{RegExps}(\Sigma)$ *such that*



$\boldsymbol{P}(\mathcal{R}(\mathcal{E}))$ *is a solution of* $\mathcal{E}$, *for all* $\mathcal{E} \in \mathit{WBSpecs}(\Sigma))$.

The Correspondence Theorem

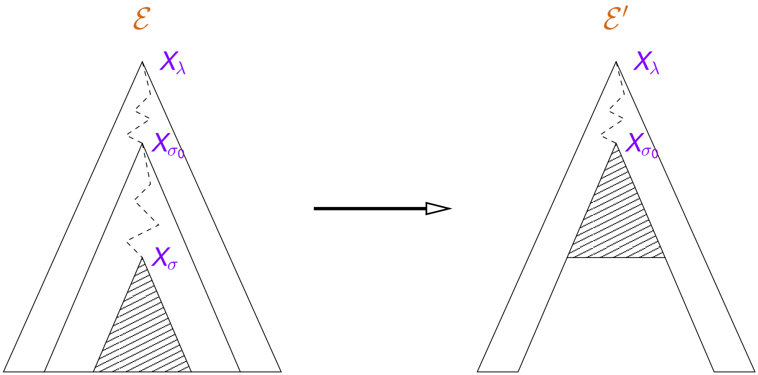
Theorem ([BC05])

*Expressibility as a regular expression under **P***
is equivalent to
definability by a well-behaved specification:

For all processes p ,

$$(\exists\, e \in RegExps)\,\big[\, p \leftrightarrows \mathbf{P}(e) \,\big]$$
$$\Leftrightarrow \quad (\exists\, \mathcal{E} \in WBSpecs)\,\big[\, p \text{ is a solution of } \mathcal{E} \,\big]$$

Reducible Well-Behaved Specifications (Example)



$$\langle X_\sigma \,|\, \mathcal{E} \rangle \leftrightarrows \langle X_{\sigma_0} \,|\, \mathcal{E} \rangle$$

X_σ, X_{σ_0} are well-behaved

Reducibility Lemma, Decidability Theorem

Lemma (Reducibility of well-behaved spec's [BCG05])

Let $\mathcal{E}$ be a well-behaved specification that has a finite-state process p with n states and maximal branching degree k as a solution.

Then $\mathcal{E}$ is equivalent to a well-behaved specification $\mathcal{E}_{red}$ with

- *depth less or equal to $(n+1)^3 \cdot 2^{3k}$, and*
- *less or equal to k summands in each defining equation.*

Theorem ([BCG05])

Expressibility by a regular expression under the process interpretation is decidable.

Overview

Solution of the Star Height Problem

Four Steps:

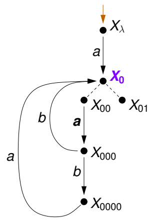
1. Introduction of the notion "star height" for well-behaved specifications.
2. Refined versions of the Definability, Solvability, and Reducibility Lemmas.
3. The algorithm CSH for computing the minimal star height under ***P*** of a regular expression.
4. Correctness Proof for the algorithm CSH.

Star Height of Well-Behaved Specifications

Definition

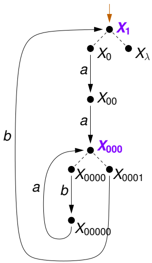
The *star height* $sh(\mathcal{E})$ of a well-behaved specification $\mathcal{E}$ is the maximum number of nested cycling variables in $\mathcal{E}$.

$\mathcal{E}_1$



$sh(\mathcal{E}_1) = 1$

$\mathcal{E}_2$

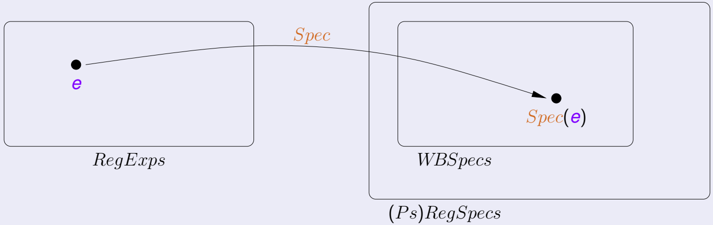


$sh(\mathcal{E}_2) = 2$

Refined Definability Lemma

Lemma (Definability by well-behaved spec's)

There is an effectively computable mapping
$Spec : RegExps \rightarrow WBSpecs$ *such that*



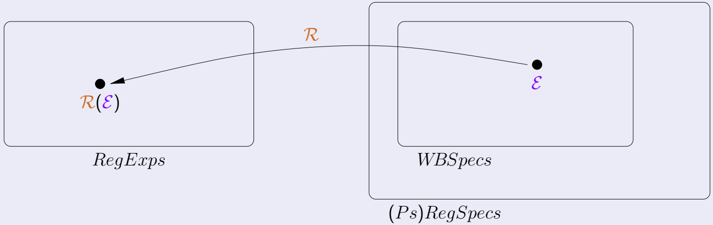
for all $\mathcal{E} \in WBSpecs$,

P(*e*) *is a solution of* $Spec$(*e*),
and $sh(Spec(e)) = sh(e)$,

Refined Solvability Lemma

Lemma (Solvability of well-behaved spec's)

There is an effectively computable mapping $\mathcal{R} : \mathit{WBSpecs} \rightarrow \mathit{RegExps}$ *such that*



P$(\mathcal{R}(\mathcal{E}))$ *is a solution of* $\mathcal{E}$,
and $sh(\mathcal{R}(\mathcal{E})) = sh(\mathcal{E})$,

for all $\mathcal{E} \in \mathit{WBSpecs}$.

Refined Reducibility Lemma

Lemma (Reducibility of well-behaved spec's)

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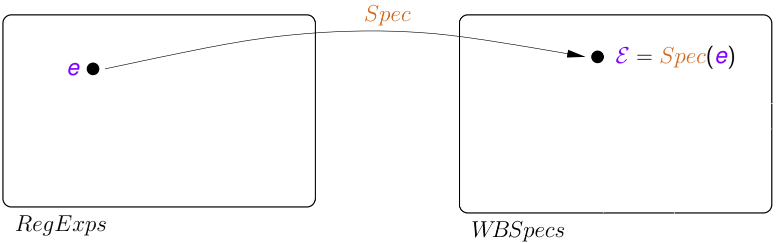
Then $\mathcal{E}$ is equivalent to a well-behaved specification $\mathcal{E}_{\boldsymbol{red}}$ with

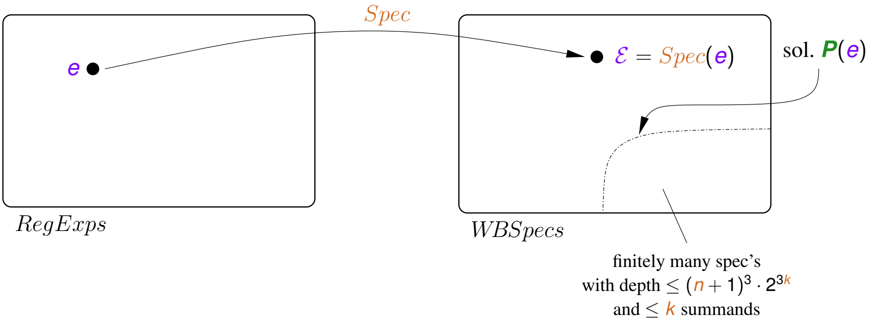
- *depth less or equal to $(n+1)^3 \cdot 2^{3k}$,*
- *less or equal to k summands in each defining equation,*
- *and $sh(\mathcal{E}_{\boldsymbol{red}}) \leq sh(\mathcal{E})$.*

The Algorithm CSH (Step CSH1)



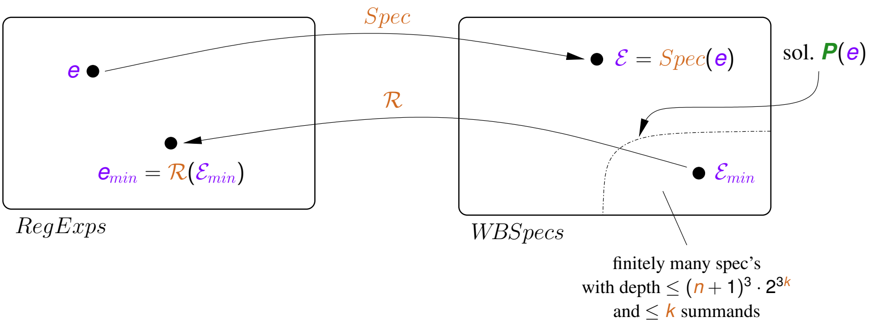
$P(e)$ solves $\mathcal{E}$, $sh(\mathcal{E}) = sh(e)$
(by the Ref.Def.Lemma)

The Algorithm CSH (Step CSH2)



$P(e)$ solves $\mathcal{E}$, $sh(\mathcal{E}) = sh(e)$
(by the Ref.Def.Lemma)
$P(e)$: n states, $\leq k$ branch.degr.

The Algorithm CSH (Step CSH3)



$P(e_{min})$ solves $\mathcal{E}_{min}$, $sh(e_{min}) =$
$= sh(\mathcal{E}_{min})$ (by the Ref.Solv.Lemma)

$e_{min} \leftrightarrows_P e$

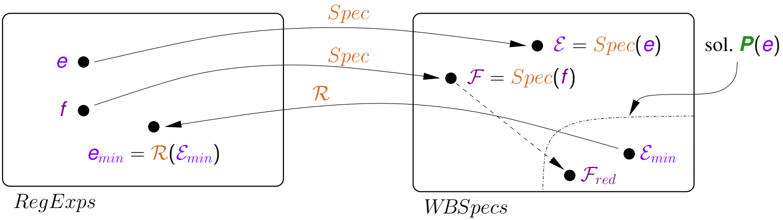
min. star height of $e = sh(e_{min}) = n_0$

$P(e)$ solves $\mathcal{E}$, $sh(\mathcal{E}) = sh(e)$
(by the Ref.Def.Lemma)

$P(e)$: n states, $\leq k$ branch.degr.

$\mathcal{E}_{min} \sim \mathcal{E}$, $sh(\mathcal{E}_{min}) =$ min!

Correctness of the Algorithm CSH



Let $f \in RegExps$ be arbitrary with $f \leftrightarrows_P e$. Then it follows for $\mathcal{F} = Spec(f)$ by the Ref.Def.Lem.: $sh(\mathcal{F}) = sh(f)$, $P(f)$ and also $P(e)$ are solutions of $\mathcal{F}$. By the Ref.Red.Lem., a reduced specification $\mathcal{F}_{red}$ exists with solution $P(e)$ and $sh(\mathcal{F}) \geq sh(\mathcal{F}_0)$. Then by the choice of $\mathcal{E}_{min}$ and e_{min} it follows:

$$sh(f) = sh(\mathcal{F}) \geq sh(\mathcal{F}_{red}) \geq sh(\mathcal{E}_{min}) = sh(e_{min}) = n_0 .$$

Hence: $n_0 = sh(e_{min}) = msh(e)$.

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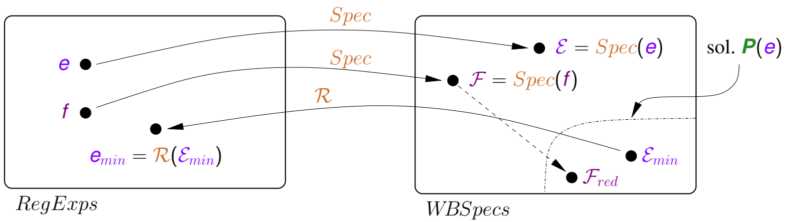

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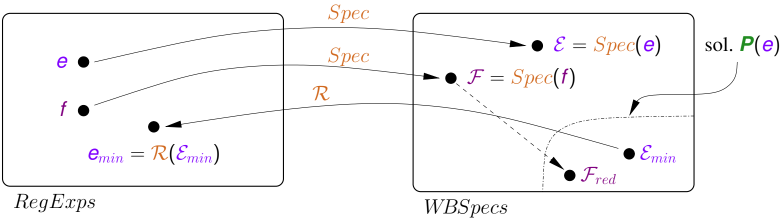

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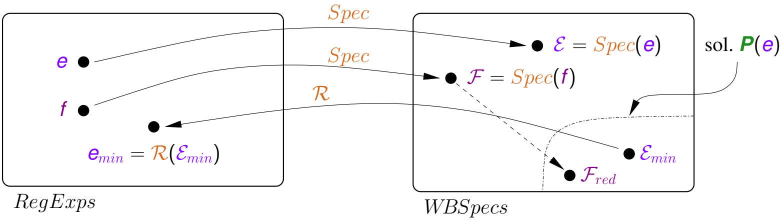

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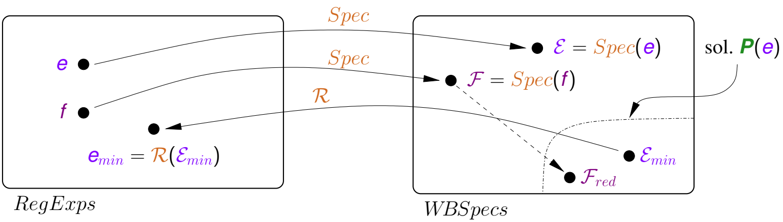
Correctness of the Algorithm CSH



Let $f \in RegExps$ be arbitrary with $f \leftrightarrows_{\boldsymbol{P}} e$. Then it follows for $\mathcal{F} = Spec(f)$ by the Ref.Def.Lem.: $sh(\mathcal{F}) = sh(f)$, $\boldsymbol{P}(f)$ and also $\boldsymbol{P}(e)$ are solutions of $\mathcal{F}$. By the Ref.Red.Lem., a reduced specification $\mathcal{F}_{red}$ exists with solution $\boldsymbol{P}(e)$ and $sh(\mathcal{F}) \geq sh(\mathcal{F}_0)$. Then by the choice of $\mathcal{E}_{min}$ and e_{min} it follows:

$$sh(f) = sh(\mathcal{F}) \geq sh(\mathcal{F}_{red}) \geq sh(\mathcal{E}_{min}) = sh(e_{min}) = n_0 .$$

Hence: $n_0 = sh(e_{min}) = msh(e)$.

The Star Height Problem is Solvable

Theorem

The star-height problem for the process interpretation is solvable: there is an algorithm that, for all regular expressions e, on input e computes the minimal star height of e.

Remark. This is a theoretical result, which (on its own) does not allow to show a better than double-exponential time bound on a naive decision algorithm extracted from the proof.

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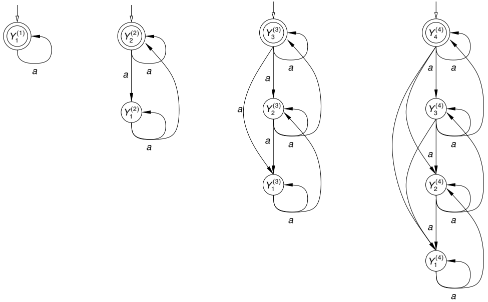
Overview

Milner's Conjecture on Star-Height

Let the sequence $\{f_n\}_n$ be defined as

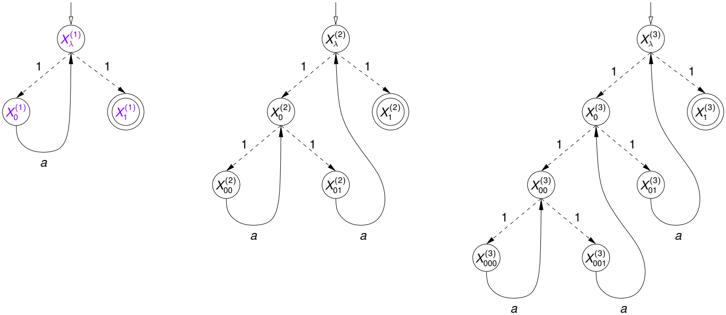
$$f_1 = a^* \qquad f_{n+1} = (f_n \cdot a)^* \ .$$

Then $\boldsymbol{P}(f_1)$, $\boldsymbol{P}(f_2)$, $\boldsymbol{P}(f_3)$, $\boldsymbol{P}(f_4)$ are of the forms:



Conjecture (Milner). *The minimal star height of f_n is n.*

Well-Behaved Spec's with solutions $P(f_1)$, $P(f_2)$, $P(f_3)$



$$X^{(1)}_\lambda = 1 \cdot X^{(1)}_0 + 1 \cdot X^{(1)}_1$$
$$X^{(1)}_0 = a \cdot X^{(\lambda)}_\lambda$$
$$X^{(1)}_1 = 1$$

… …

Alternative Proof of Milner's Conjecture (I)

Theorem (Hirshfeld and Moller, 2000)

For every n, there exists a regular expression f_n over the single-letter alphabet $\{a\}$ such that the minimal star height of f_n under the process interpretation is n. This is witnessed by the sequence $\{f_n\}_n$ in Milner's Conjecture.

Lemma (Main Lemma)

For every well-behaved specification $\mathcal{E}$ that has $\boldsymbol{P}(f_n)$ as a solution, $sh(\mathcal{E}) \geq n$ holds.

Hint at the Proof.

A careful analysis of well-behaved specifications $\mathcal{E}$ that have $\boldsymbol{P}(f_n)$ as a solution. □

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Alternative Proof of Milner's Conjecture (II)

Lemma (Main Lemma)

For every well-behaved specification $\mathcal{E}$ that has $\mathbf{P}(f_n)$ as a solution, $sh(\mathcal{E}) \geq n$ holds.

(Alternative) Proof of the Theorem.

Let $n \in \mathbb{N} \setminus \{0\}$ arbitrary. It suffices to show that $sh(f_n) = n$.

Let $e \in RegExps(\{a\})$ such that $e \leftrightarrows_{\mathbf{P}} f_n$. Then by the Ref.Def.Lemma there exists a well-behaved specification $\mathcal{E}$ with $sh(\mathcal{E}) = sh(e)$ such that $\mathbf{P}(e)$, and also $\mathbf{P}(f)$ is a solution of $\mathcal{E}$. By the Lemma $sh(\mathcal{E}) \geq n$ follows, entailing $sh(e) \geq n$.

Hence the minimal star height of f_n is n. □

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Hence the minimal star height of f_n is n. □

Overview

Summary

We consider regular expressions under Milner's process interpretation.

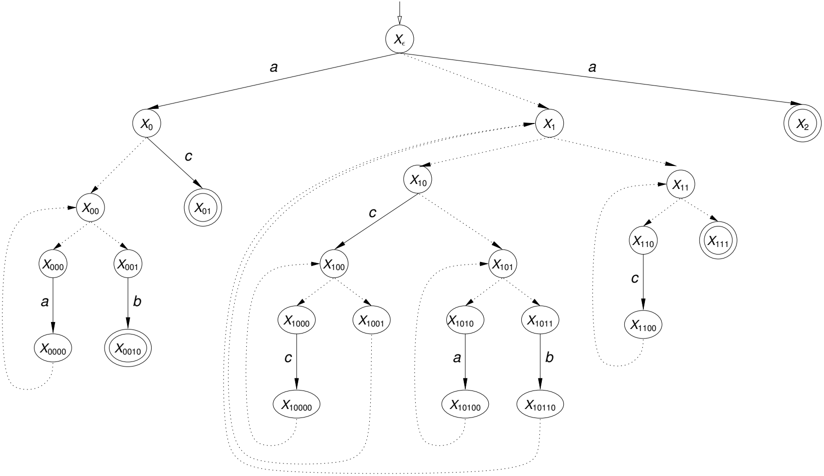
We use the correspondence with "well-behaved" specifications from [BC05] to show:

- The Star Height Problem for Regular Expressions under Milner's Process Interpretation is solvable. (Using the reducibility lemma for well-behaved spec's from [BCG05].)
- For regular expressions over a single-letter alphabet, minimal star height w.r.t. the process interpretation defines a proper hierarchy (Hirshfeld/Moller, 2000).

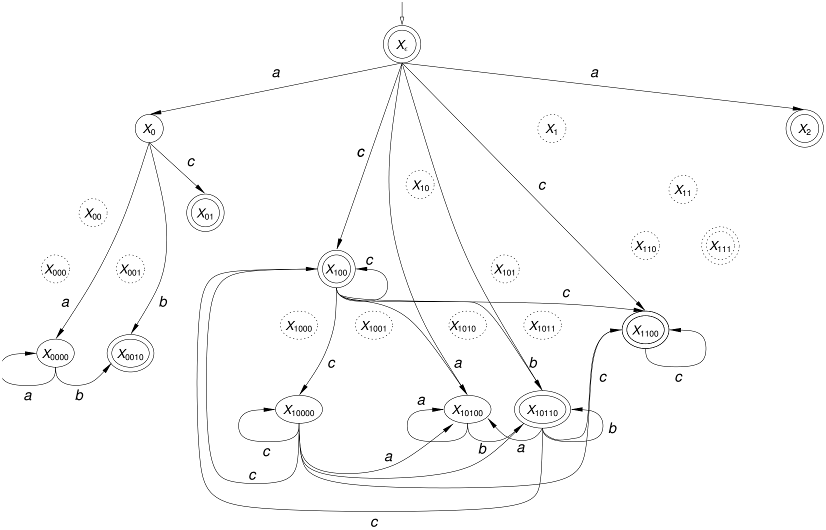
Questions for Further Research

1. How could a better algorithm for the star-height problem look like?
2. Is there a relationship with known decision algorithms for the (restricted) star-height problem for regular languages?
3. Are there appealing interpretations for (generalised) regular expressions (allowing *complementation and intersection* operators) in process theory?
4. Is it possible, to find, for all $e \in RegExps$ an $e_{min} \in RegExps$ of minimal star height such that $e_{min} \leftrightarrows_{P} e$ and $e_{min} = e$ *is provable* in Milner's adaptation for P of Salomaa's axiomatisation for L?

Example: *Spec*$(a(a^* b + c) + (c^* + a^* b)^* + a)$



Example: Canonical Solution of *Spec*$(a(a^*b + c) + \ldots)$



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