Remarks about an Exercise by Detlef Plump

Exercise (Detlef Plump). Let $\mathcal{T} = (\Sigma, \mathcal{R})$ be a(n arbitrary) term-rewriting system with rewrite-relation $\rightarrow$, that is weakly normalizing (in symbolical notation: $WN(\mathcal{T})$, or, in the context here, just: $WN(\rightarrow)$). Let $\rightarrow_{\text{inn}}$ be the reduction-relation on $Ter(\Sigma)$, that is defined for all $s, t \in Ter(\Sigma)$ by

$$\begin{aligned} s \rightarrow_{\text{inn}} t \;\; :\Longleftrightarrow \;\; & (\exists\, C \in Ctxt(\Sigma)) \\ & \quad (\exists\, r \in Ter(\Sigma))\,(\exists\, n \in Ter(\Sigma)) \\ & \qquad \big[\, s \equiv C[r] \;\wedge\; r \text{ is an innermost redex of } s \;\wedge \\ & \qquad\quad \wedge\; r \twoheadrightarrow n \;\wedge\; n \text{ is a normal form (w.r.t. } \rightarrow) \;\wedge\; t \equiv C[n] \,\big] \;. \end{aligned} \tag{1}$$

(Thus a single step $s \rightarrow_{\text{inn}} t$ of the reduction-relation $\rightarrow_{\text{inn}}$ consists in replacing an innermost redex r (with respect to $\rightarrow$) of s by a normal form n (with respect to $\rightarrow$) of r with t as the result.)

Show that $\rightarrow_{\text{inn}}$ is terminating, i.e. strongly normalizing (in symbolical notation: $SN((Ter(\Sigma), \rightarrow_{\text{inn}}))$, or just $SN(\rightarrow_{\text{inn}})$).

Proof. The function

$$\begin{aligned} m_1 \,:\, Ter(\Sigma) \;&\longrightarrow\; \mathbb{N}_0 \\ t \;&\longmapsto\; \sum \big\{\, |p| \;\mid\; p \in \{0,1\}^*,\, p \in Pos(t),\, t|_p \text{ is innermost redex in } t \,\big\} \end{aligned} \tag{2}$$

is a measure-function between $(Ter(\Sigma), \rightarrow_{\text{inn}})$ and the linear partial order $(\mathbb{N}_0, \leq)$ (i.e. it is strictly monotonously decreasing, that is, $s \rightarrow_{\text{inn}} t \;\Rightarrow\; m_1(s) > m_1(t)$ holds for all $s, t \in Ter(\Sigma)$). – Alternatively, also the function

$$\begin{aligned} m_2 \,:\, Ter(\Sigma) \;&\longrightarrow\; \mathbb{N}_0 \\ t \;&\longmapsto\; \text{mulset}\big\{\, |p| \;\mid\; p \in \{0,1\}^*,\, p \in Pos(t),\, t|_p \text{ is a redex in } t \,\big\} \end{aligned} \tag{3}$$

is a measure-function between $(Ter(\Sigma), \rightarrow_{\text{inn}})$ and $(\mathcal{M}_{\text{fin}}(\mathbb{N}_0), \leq_{\text{mulset}})$, the latter being the well-founded partial order on the set $\mathcal{M}_{\text{fin}}(\mathbb{N}_0)$ of the finite multisets over $(\mathbb{N}_0, \leq)$ with the multiset-order $\leq_{\text{mulset}}$.

$\square$

Remarks. (1) The reduction-relation $\rightarrow_{\text{inn}}$ is no rewrite relation on $Ter(\Sigma)$ in the sense[1] of the book by T. Nipkow and F. Baader. This is the case, because $\rightarrow_{\text{inn}}$ is in general not closed under substitutions, since the notion of an "innermost redex" is not closed under substitution. – However, clearly, the reduction-relation $\rightarrow_{\text{inn}}$ has the rewrite relation $\rightarrow$ as its *refinement*.

[1] In the book *Term Rewriting and All That* by F. Baader and T. Nipkow the notion of a (term) *rewrite relation* $\rightarrow$ on a set $Ter(\tilde{\Sigma})$ of terms over the signature $\tilde{\Sigma}$ is defined as an arbitrary binary relation $\rightarrow$ on $Ter(\tilde{\Sigma})$, that is compatible with $\tilde{\Sigma}$-contexts and closed under substitutions.

(2) If the provision $\mathcal{T} \models WN$ is dropped from the assumption of the exercise, then $SN(\to_{\text{inn}})$ holds still: This provision is never used in the proof of the exercise and hence the two given measure-functions in (2) and (3) are also measure-functions in the situation that $\mathcal{T} \not\models WN$.

(3) However, if $\mathcal{T} \models WN$ is dropped as an assumption of the exercise, then it holds in the situation $\mathcal{T} \not\models WN$, that the reduction-relation $\to_{\text{inn}}$ possesses *more normal forms* in $Ter(\Sigma)$ than $\to$. Indeed, even the stronger statement is true, that $\to_{\text{inn}}$ possesses more normal forms than $\to$ precisely in the case, when $\mathcal{T} \not\models WN$ holds (cf. the below proposition).

(4) Furthermore not even the restriction in the exercise and in the assertion in item (2) above to the reduction-relation $\to_{\text{inn}}$, where $\to_{\text{inn}}$ only allows to replace *innermost redexes* r with respect to the rewrite-relation $\to$ in terms s of a TRS $\mathcal{T}$ by normal forms with respect to $\to$, is necessary. In fact the following generalization of the statement referred to above in item (2) is true:

Let $\mathcal{T} = (\Sigma, \mathcal{R})$ be a(n arbitrary) term-rewriting system with rewrite-relation $\to$. Let $\to_{\text{n}}$ be the reduction-relation on $Ter(\Sigma)$, that is defined for all $s, t \in Ter(\Sigma)$ by

$$\begin{aligned} s \to_{\text{n}} t \;\; :\Longleftrightarrow \;\; & (\exists\, C \in Ctxt(\Sigma)) \\ & \quad (\exists\, r \in Ter(\Sigma))\, (\exists\, n \in Ter(\Sigma)) \\ & \qquad \big[\, s \equiv C[r] \;\wedge\; r \text{ is a redex } \wedge\; r \twoheadrightarrow n \;\wedge \\ & \qquad\quad \wedge\; n \text{ is a normal form (w.r.t. } \to) \;\wedge\; t \equiv C[n]\,\big] \;. \end{aligned} \tag{4}$$

(Thus a single step $s \to_{\text{n}} t$ of the reduction-relation $\to_{\text{n}}$ consists in replacing an *arbitrary* redex r (with respect to $\to$) of s by a normal form n (with respect to $\to$) of r such that t is the result.) – Then it holds, that $SN((Ter(\Sigma), \to_{\text{n}}))$, that is, that the reduction-relation $\to_{\text{n}}$ is terminating on $Ter(\Sigma)$.

[This follows from the fact, that the two measure-functions m_1 from (2) and m_2 from (3) are also measure-functions with respect to the reduction-relation $\to_{\text{n}}$.]

Additionally, because obviously $\to_{\text{inn}} \subseteq \to_{\text{n}}$ is the case, it follows that every normal form with respect to $\to_{\text{n}}$ is also normal form with respect to $\to_{\text{inn}}$.

In item (2) of the above remark we have referred to a proposition, for the proof of which we will need the following lemma.

Lemma. *Let $\mathcal{T} = (\Sigma, \mathcal{R})$ be an arbitrary term-rewriting system with induced rewrite relation $\to$.*

Then it holds that

$$(\forall\, r \in Ter(\Sigma))\; \left[\, \begin{pmatrix} r \textit{ is a redex, that does} \\ \textit{not contain other redexes} \end{pmatrix} \;\Rightarrow\; WN(r) \,\right] \;\Longrightarrow\; WN(\mathcal{T}) \;. \tag{5}$$

Proof. Let $\mathcal{T} = (\Sigma, \mathcal{R})$ be a term-rewriting system with the property, that for every redex r in $\mathcal{T}$, that does not properly contain another redex, $WN(r)$ holds, i.e. that every such redex r has a normal form. We will show that then also $WN(\mathcal{T})$ holds.

We let $t \in Ter(\Sigma)$ be arbitrary. It follows from the generalization of the exercise in item (2) of the above remark, that $SN(\rightarrow_{\text{inn}})$ holds for the reduction-relation $\rightarrow_{\text{inn}}$ on $Ter(\Sigma)$ defined as in (1). We can hence choose a normal form n of t with respect to $\rightarrow_{\text{inn}}$, i.e. a normal form n with respect to $\rightarrow_{\text{inn}}$ such that $t \twoheadrightarrow_{\text{inn}} n$. Because $\rightarrow_{\text{inn}} \subseteq \twoheadrightarrow$ and hence also $\twoheadrightarrow_{\text{inn}} \subseteq \twoheadrightarrow$ holds, we find that also $t \twoheadrightarrow n$ is the case. But we can now see that n must also be a normal form with respect to $\rightarrow$: If this were not the case, then there would exist a redex r in n (namely any of its innermost redexes), which does not contain any other redex properly and which does not have a normal form with respect to $\rightarrow$. But the existence of such a redex would contradict our assumption about $\mathcal{T}$. Hence we have shown that $t \twoheadrightarrow n$ holds for some normal form n with respect to $\rightarrow$ and thus that t is in fact weakly normalizing with respect to $\rightarrow$. – And because we have let $t \in Ter(\Sigma)$ be arbitrarily chosen at the beginning of the argument, we have now succeeded in proving $WN(\mathcal{T})$. ☐

For the formulation of the below lemma we will fix the following notation: For any two reduction-relations $\rightarrow_1$ and $\rightarrow_2$ on $Ter(\Sigma)$ we let the formula $Snf(\rightarrow_1, \rightarrow_2)$ be defined as

$$(\forall t \in Ter(\Sigma)) \left[t \text{ is a normal form w.r.t. } \rightarrow_1 \iff \right. \\ \left. \iff t \text{ is a normal form w.r.t. } \rightarrow_2 \right] . \tag{6}$$

$Snf(\rightarrow_1, \rightarrow_2)$ expresses that $\rightarrow_1$ and $\rightarrow_2$ possess the *same normal forms*.

Proposition. *Let $\mathcal{T} = (\Sigma, \mathcal{R})$ be a(n arbitrary) term-rewriting system with rewrite-relation $\rightarrow$. Furthermore let the reduction-relation $\rightarrow_{inn}$ for all $s, t \in Ter(\Sigma)$ be again defined by* (1).

Then it holds that

$$\mathcal{T} \vDash Snf(\rightarrow, \rightarrow_{inn}) \iff \mathcal{T} \vDash WN(\rightarrow) . \tag{7}$$

Proof. We let $\mathcal{T} = (\Sigma, \mathcal{R})$ be an arbitrary term-rewriting system with rewriting-relation $\rightarrow$. We let the reduction-relation $\rightarrow_{\text{inn}}$ on $Ter(\Sigma)$ be defined as in (1). We will demonstrate the two directions of the logical equivalence (7).

"$\Leftarrow$": We assume that $\mathcal{T}$ is *WN* and will show that $\rightarrow$ and $\rightarrow_{\text{inn}}$ possess the same normal forms.

For this it suffices to prove that every normal form with respect to $\rightarrow_{\text{inn}}$ is also a normal form with respect to $\rightarrow$ (it is obvious from the definition of $\rightarrow_{\text{inn}}$ in (1), that $\rightarrow_{\text{inn}} \subseteq \rightarrow^+$ holds, and hence, that every normal form with respect to $\rightarrow$ is also normal form with respect to $\rightarrow_{\text{inn}}$).

Let n be a normal form in $\mathcal{T}$ with respect to $\to_{\mathrm{inn}}$. We consider the two—at this stage thinkable—cases for n: (a) n does not contain any redexes with respect to $\to$, and (b) n does contain $\to$-redexes: In case (a) n is also normal form with respect to $\to$. But in case (b) it must hold for all innermost redexes r (of which there is then at least one) in n, that no normal form with respect to $\to$ exists for r. But the existence of such a term r with $\neg\, \mathit{WN}(r)$ would be a contradiction with our assumption, that $\mathcal{T}$ is in fact weakly normalizing. Thus case (b) cannot occur at all here and we have shown that n is in fact also a normal form with respect to $\to$.

"$\Rightarrow$": We will argue by contraposition. We will assume $\mathcal{T} \not\models \mathit{WN}$ and prove that then there are more normal forms with respect to $\to_{\mathrm{inn}}$ than with respect to $\to$.

We assume $\mathcal{T} \not\models \mathrm{WN}$, i.e. $\neg\, \mathit{WN}(\mathcal{T})$. Then it follows from the lemma above, that there exists a redex r exists in $\mathcal{T}$, which does not contain another redex properly and which is not weakly normalizing, i.e. such that $\neg\, \mathit{WN}(r)$ holds. We choose such a redex r. This redex is then a normal form with respect to $\twoheadrightarrow_{\mathrm{inn}}$, but it is not a normal form with respect to $\to$. Hence we conclude that $\mathcal{T} \not\models \mathit{Snf}(\to, \to_{\mathrm{inn}})$.

□

Remark. The formally slightly weaker statement than that of the Lemma, namely, that

$$(\forall\, r \in \mathit{Ter}(\Sigma))\ [\ r \text{ is a redex } \Rightarrow\ \mathit{WN}(r)\ ] \quad\Longrightarrow\quad \mathit{WN}(\mathcal{T}) \tag{8}$$

holds for all TRS's $\mathcal{T}$ (since (5) is true for all TRS's $\mathcal{T}$, clearly (5) and (8) are equivalent for all TRS's $\mathcal{T}$), might be compared with the analogous assertion, that also

$$(\forall\, r \in \mathit{Ter}(\Sigma))\ [\ r \text{ is a redex } \Rightarrow\ \mathit{SN}(r)\ ] \quad\Longrightarrow\quad \mathit{SN}(\mathcal{T}) \tag{9}$$

holds for all TRS's $\mathcal{T}$. This statement and the slightly stronger formulation of it, namely, that

$$(\forall\, r, s \in \mathit{Ter}(\Sigma))\ \left[\ \begin{pmatrix} r \text{ is a redex} \\ \text{with contractum } s \end{pmatrix} \ \Rightarrow\ \mathit{SN}(s)\ \right] \quad\Longrightarrow\quad \mathit{SN}(\mathcal{T}) \tag{10}$$

(which statement is in fact also equivalent to (9) for all term-rewriting systems $\mathcal{T}$) holds for all TRS's $\mathcal{T}$ are (equivalent by logical contrapositions to) the assertions of *Exercise 2.2.19* on p. 25 in the chapter about "Term Rewriting Systems" by J.W. Klop in the "*Handbook of Logic in Computer Science*", Volume 2, edited by S. Abramsky, Dov M. Gabbay and T.S.E. Maibaum, Oxford Science Publications, Clarendon Press, Oxford, 1992).

It can also be observed, that direct proofs for (8) and (9) have a very similar flavour and can actually be formalized in manners analogous to each other.

A direct proof of (8) could be sketched in this way: An arbitrary term t in a TRS $\mathcal{T}$, in which every redex is weakly normalizing, can be normalized by repeating on it finitely often the operation of picking an arbitrary redex r and replacing it by a normal form n (with respect to $\rightarrow$) of r; the existence of a normal form n for every redex r with respect to $\rightarrow$ is thereby always guaranteed by the assumption on $\mathcal{T}$. The reason, why this describes an effective normalization-procedure, consists in the facts, that (1) the above defined reduction $\rightarrow_{\mathrm{n}}$ is strongly normalizing and that (2) in the assumed case, where all redexes in $\mathcal{T}$ are known to be weakly normalizing, every normal form with respect to $\rightarrow_{\mathrm{n}}$ is also a normal form with respect to $\rightarrow$.

A direct proof of (9) can be given in the following way: Let $\mathcal{T}$ be a term-rewrite system, in which every redex is strongly normalizing. Suppose now, that we have given an infinite reduction-sequence σ (with respect to the reduction-relation $\rightarrow$) in $\mathcal{T}$ of the form

$$\sigma :\; t_0 \xrightarrow[p_1]{r_1} t_1 \xrightarrow[p_2]{r_2} t_2 \xrightarrow[p_3]{r_3} \;\ldots\; \xrightarrow[p_n]{r_n} t_n \xrightarrow[p_{n+1}]{r_{n+1}} t_{n+1} \xrightarrow[p_{n+2}]{r_{n+2}} \;\ldots\;,$$

where in the i-th reduction-step $t_i \xrightarrow[p_{i+1}]{r_{i+1}} t_{i+1}$ a redex r_{i+1} at position $p_{i+1} \in Pos(t_i)$ in t_i (i.e. it holds that $r_{i+1} \equiv t_i|_{p_{i+1}}$) is reduced. Now let

$$P_{i1} := \{\, p \mid p \in Pos(t_i) \;\wedge\; (\exists\, 0 < j \leq i)\,[\, p = p_j \;\wedge\; \neg(\exists\, 0 < k \leq i)\,[\, p_k < p\,]\,]\,\} \qquad (11)$$

for all $i \in \mathbb{N}_0$ be the set of the upmost positions in t_i, at which redexes have been reduced during the first i reductions in σ (hereby positions are understood to be ordered by the prefix-order $<$ on the words in $\{0,1\}^*$). It is clear that $P_{i1} \subseteq Pos(t_0)$ holds for all $i \in \mathbb{N}_0$; since all redexes in $\mathcal{T}$ are SN, it is implied that $SN(t_i|_p)$ holds for all $p \in P_{i1}$ and $i \in \mathbb{N}_0$. Now let the set P_{i2} be defined for all $i \in \mathbb{N}_0$ as

$$P_{i2} := \{\, p \mid p \in Pos(t_i) \;\wedge\; (\forall \tilde{p} \in P_{i1})\,[\, p \,||\, \tilde{p}\,] \;\wedge\; t_i|_p \text{ is redex}\} \;. \qquad (12)$$

For P_{i2} also $P_{i2} \subseteq Pos(t_0)$ holds; the set P_{i2} is the set of all redex-positions in t_i, that are also positions of such redexes in t_0, at which or above which no redexes have been reduced during the first i steps in the reduction-sequence σ. Hence it holds for all $i \in \mathbb{N}_0$ for the set P_i defined as

$$P_i \;:=\; P_{i1} \cup P_{i2} \quad , \text{ that } \quad P_i \subseteq Pos(t_0) \cap Pos(t_i) \;\wedge\; (\forall p \in P_i)\,[SN(t_i|_p)] \qquad (13)$$

is the case. It follows that for every $i \in \mathbb{N}_0$ only finitely many of the reductions immediately following the i-th step in σ can take place at positions $\tilde{p}$ with $\tilde{p} \geq p$ for some $p \in P_i$. As a consequence, the sequence $\left\{ \sum_{p \in P_i} |p| \right\}_i$ of natural numbers is monotonously decreasing and it always decreases strictly after finitely many indices.

Therefore we find, that if the reduction-sequence σ does not terminate earlier (and hence is finite), there must exist a first $l \in \mathbb{N}_0$ such that $P_l = \{\epsilon\}$ (where ϵ is the empty word in $\{0,1\}^*$). If $l = 0$, then t_0 must be a redex itself[2]; if $l > 0$, then $t_{l-1} \equiv r_l$ must

[2] In this case t_0 is also an innermost redex in itself.

have been a redex. Since all redexes in $\mathcal{T}$ are by assumption strongly normalizing, it follows in both cases, that there exists an index $l_0 \in \mathbb{N}_0$ such that $SN(t_{l_0})$. Hence the reduction-sequence σ can possess only a finite number of reduction-steps after its l_0-th step. Thus we have shown, that the reduction-sequence σ must always be finite.

Now the main part of the above sketched direct proof of (8), namely, that the reduction-relation $\to_{\mathrm{n}}$ defined in (4) is strongly normalizing, can actually be reformulated in a way similar to the proof just outlined for (9): Suppose that

$$\sigma : \; t_0 \underset{p_1}{\overset{r_1}{\to}}_{\mathrm{n}} t_1 \underset{p_2}{\overset{r_2}{\to}}_{\mathrm{n}} t_2 \underset{p_3}{\overset{r_3}{\to}}_{\mathrm{n}} t_3 \; \ldots \; \underset{p_n}{\overset{r_n}{\longrightarrow}} t_n \underset{p_{n+1}}{\overset{r_{n+1}}{\to}}_{\mathrm{n}} \; t_{n+1} \underset{p_{n+2}}{\overset{r_{n+2}}{\to}}_{\mathrm{n}} \; \ldots$$

is an arbitrary reduction-sequence with respect to the reduction-relation $\to_{\mathrm{n}}$ starting at a term $t_0 \in Ter(\Sigma)$, where in the i-th step $t_{i-1} \underset{p_i}{\overset{r_i}{\to}}_{\mathrm{n}} t_i$ in σ a redex r_i in t_{i-1} at position p_i is replaced by a normal form (for all $i \in \mathbb{N}$ occuring as indices for terms t_i in σ). If we now let the sets P_{i1} and P_{i2} of positions be defined for all $i \in \mathbb{N}_0$ as in (11) and (12), then the following similar statement to (13),

$$P_i \; := \; P_{i1} \cup P_{i2} \;\; \subseteq \;\; Pos(t_0) \cap Pos(t_i) \;\; \wedge \;\; (\forall p \in P_i)\,[\; t_i|_p \text{ is n.f. } \vee \;\; WN(t_i|_p)\;] \;\; ,$$

holds for all $i \in \mathbb{N}_0$. And moreover it is the case that the sequence of natural numbers or zero $\left\{\sum_{p \in P_i} |p|\right\}_i$ is decreasing and that the sequence $\left\{\sum_{p \in P_{i1}} |p| + \sum_{p \in P_{i2}} (1 + |p|)\right\}_i$ is even strictly decreasing. From this it follows again easily, that the reduction-sequence σ must be finite.

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