Derivability and Admissibility of Inference Rules in Abstract Hilbert Systems

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Abstract

In this report we collect some general results about the notions of derivability and admissibility of inference rules in Hilbert-style proof systems. For this purpose we introduce, by analogy with abstract rewrite systems, a general framework for Hilbert systems, in which it is abstracted from the syntax of formulas and the operational content of rules. In these "abstract Hilbert systems" a rule is a set of inference steps that is endowed with a premise and a conclusion function. We adapt the notions of rule derivability and admissibility to abstract Hilbert systems, propose two variants of rule derivability, s-derivability and m-derivability, and investigate how these four notions are related. Furthermore, we consider relations that compare abstract Hilbert systems with respect to rule derivability and admissibility and with respect to (relative) formula derivability. We study the interrelations, for all abstract Hilbert systems $\mathcal{S}_1$ and $\mathcal{S}_2$, of assertions like "$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same derivable rules", "the rules of $\mathcal{S}_1$ are admissible in $\mathcal{S}_2$, and vice versa" or "$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems" and give 'interrelation prisms', i.e. diagrams in the form of a prism that capture the relationship between statements of this kind. And lastly, we explore what consequences derivability or admissibility of a rule R in an abstract Hilbert system $\mathcal{S}$ has for the possibility to eliminate applications of R from derivations in $\mathcal{S}$.

Keywords: Derivable rule, derived rule, admissible rule, Hilbert system, proof system.

1 Introduction

The notions of derivability and admissibility of inference rules are usually studied in the context of concrete systems of formal logic. A new rule R is generally called 'derivable' (or 'derived') with respect to a formal system $\mathcal{S}$ if its 'operational behaviour', i.e. the possibility R offers to produce certain conclusions when certain premises are given, can always be, in some sense, 'modeled' or 'mimicked' by appropriate derivations in $\mathcal{S}$. And a rule R is understood to be 'admissible' in a formal

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system $\mathcal{S}$ if the collection of theorems of $\mathcal{S}$ is closed under applications of R. Contrary to rule derivability, the notion of admissibility of rules also has a meaning for 'logics' that are defined semantically, i.e. for sets of theorems consisting of all those formulas defined according to a given syntax that are 'true' in a considered class of semantical structures.

There are at least two conspicuous reasons why precise definitions for these notions are mostly stipulated only for specific classes of formal systems: Firstly, the definition of rule derivability in a system $\mathcal{S}$ requires an exact explanation of when a rule R is 'modeled' by a derivation in $\mathcal{S}$, and such an explanation may hinge substantially on the formal concepts of $\mathcal{S}$ or even on the intentional notions formalized[2] by $\mathcal{S}$. But more fundamentally, both notions presuppose the concept of inference rule, formalizations of which are often very specific to concrete formal systems because they usually depend on special features of, for instance, the syntax of formulas (see, for example, the schematic definition of rules explained in Section 2).

In this report, we collect a number of basic results about rule derivability and admissibility that are applicable to all Hilbert-style systems of the simplest kind. By this we mean systems, sometimes[3] just called 'axiomatic systems', in which each rule application α within a derivation $\mathcal{D}$ is the inference of a single conclusion from a finite sequence of premises; each such rule application α does furthermore not depend on the presence or absence of assumptions in subderivations of $\mathcal{D}$ leading to α. As a framework for all such systems, we introduce the notion of "abstract Hilbert system", in which formulas are considered as unspecified objects and every rule application is treated as an inference step from which only its premises and its conclusion are relevant.

In Section 2, we first expound the abstract notion of rule that provides the basis for abstract Hilbert systems, which are defined next in a version with axiom and rule names (n-AHS's) and without (AHS's). We proceed with stipulations for the notions of derivation, theorem, consequence relation and relative derivability statement in an AHS. And furthermore, we fix some terminology about certain extensions of AHS's. In Section 3, we adapt known definitions for rule derivability and admissibility to our framework, propose two variant notions of rule derivability, and collect basic facts about the interrelations of these four introduced notions. For this, we start from an adaptation to our setting of a lemma given by Hindley and Seldin in [3]. Also, we will present some characterizations of rule derivability and admissibility in terms of the respective other notion.

In Section 4 we consider, inspired by another lemma in [3], relations that compare AHS's with respect to rule admissibility, the three introduced notions of rule derivability and with respect to the relative derivability statements that hold in

[2] For example, in the context of a system $\mathcal{S}$ of linear logic where an assumption usually gets 'consumed' by an inference in which it is used and in this case can be used only once, it seems generally reasonable to demand that an application α of a rule R is 'modeled' by a derivation $\mathcal{D}_\alpha$ in $\mathcal{S}$ *only if* there exists a bijective mapping between the premises of α and the assumptions of $\mathcal{D}_\alpha$.

[3] In [1], Avron calls these systems 'axiomatic systems' or 'Hilbert-systems for theoremhood'.

them. That is, we are interested in questions of the following kind: Is there a relationship, and if so then what kind of a relationship, between, for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, assertions like "every rule of $\mathcal{S}_1$ is derivable in $\mathcal{S}_2$, and vice versa" and "$\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems". We will introduce twelve "inclusion relations" between AHS's and twelve "mutual inclusion relations" that are induced by respective inclusion relations, and we will then present two theorems that describe the results of a systematic study concerning the interrelations of these relations. Although n-AHS's will not be considered, the results of this section can be carried over to these systems as well.

In Section 5 we will consider, in the general framework of abstract Hilbert systems, a question that has stimulated our interest in the notions of rule admissibility and derivability in the first place: What consequences does the fact that a rule R is derivable or admissible in a Hilbert-style proof system $\mathcal{S}$ have for the possibility to eliminate applications of R from derivations in $\mathcal{S}$? For this purpose, we will introduce in Subsection 5.1 four abstract notions of rule elimination in AHS's and n-AHS's. We will then show a direct correspondence between three of the 4 concepts and respective notions of rule derivability and admissibility (in the fourth case only a weaker connection will be established). Among the results of Subsection 5.2 it will be established that, for all n-AHS's $\mathcal{S}$, if a rule R is derivable in $\mathcal{S}$, but not a rule of $\mathcal{S}$, the applications of R can be eliminated from arbitrary derivations $\mathcal{D}$ by easy transformation steps. And what is more, it will be proved for this situation that every derivation $\mathcal{D}$ in $\mathcal{S}$ can be transformed into a derivation $\mathcal{D}'$ in $\mathcal{S}$ without R-applications and with the same conclusion as $\mathcal{D}$ by performing a sufficiently long sequence of such elimination steps *in an arbitrary order*.

Due to space limitations, proofs are generally omitted from this paper. However, they can be found in respective technical appendices A, B, C and D to each of the sections 2, 3, 4 and 5.[4]

2 Abstract Hilbert Systems

In this paper we will study properties of inference rules only in the most basic kind of proof systems. In the literature, these systems occur under a variety of names, among them "formal systems", "axiom(atic) systems" and "Hilbert(-style) systems". For instance, in [8] Shoenfield uses the term "formal system" in the sense of formal axiom systems: Every formal system contains as its parts a language, axioms, and rules of inference; its theorems are defined inductively from axioms and rules. Similarly, in [1] Avron describes a "formal system" in traditional understanding as containing the following components:

1. A formal language L with several syntactic categories, one of which is the category of 'well-formed formulae' (wff).

[4] These appendices will not be included in all printed versions of this report. However, the full version of this paper is available at `http://www.cs.vu.nl/~clemens/dairahs.ps` or can be obtained via `http://www.cs.vu.nl/~clemens` , the author's homepage.

2. An effective set of wff called 'axioms'.
3. An effective set of rules (called 'inference rules') for derivating theorems from the axioms.

And then, "the set of 'theorems' is usually taken to be the minimal set of wff which includes all the axioms and is closed under the rules of inference."

We will only consider proof systems that, with respect to the characterization of Hilbert systems in the sequent-style formalization due to Avron in [1], correspond to "pure", single-conclusioned "Hilbert-type systems for consequence". In Appendix E this characterization from [1] is recalled and correspondences are given between three consequence relation defined on Hilbert systems as studied here and sequent-style Hilbert systems à la Avron. However, these interrelations can only be formulated after we have explained our concept of Hilbert system and introduced the mentioned consequence relations. For this reason, we first give a rather more informal outline of the class of Hilbert system in which we are interested.

The Hilbert-style proof systems of the sort considered here will be restricted to those in which applications of rules have an outer appearance as follows: A rule application is either of the form

$$\frac{}{A} \tag{2.1}$$

(where A is a formula) with no premise and with a single conclusion (here the formula A), or of the form

$$\frac{A_1 \quad \ldots \quad A_n}{A} \tag{2.2}$$

with a finite sequence of premises (here the sequence of formulas $A_1, \ldots, A_n$) and again a single conclusion (here the formula A). And as a further condition on the Hilbert-style proof systems considered here, we demand that rule applications in derivations do not depend on occurrences of assumptions in immediate subderivations. For spelling out this condition rather more formally, we consider, as we will do throughout this report, derivations as 'prooftrees', i.e. as trees that are labelled by formulas, in place of sequences of formulas (as is done so in many traditional textbooks on logic). We will only investigate such Hilbert-style proof systems here in which the following holds: Suppose that R is a rule in a Hilbert-style proof system $\mathcal{S}$. And suppose that, for some formulas $A, A_1, \ldots, A_n$ of $\mathcal{S}$, (2.2) is an application of R; and let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be derivations (which may contain unproven assumptions) with respective conclusions $A_1, \ldots, A_n$. Then the derivation $\mathcal{D}$ of the form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array}}{A} \tag{2.3}$$

is also a derivation in $\mathcal{S}$ irrespectively of whether or not assumptions occur in $\mathcal{D}_1, \ldots, \mathcal{D}_n$. That is, the inference (2.2) at the bottom of (2.3) does not depend on the presence or absence of assumptions in subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of $\mathcal{D}$.

The precise definition of an inference rule with applications of the form (2.1) or (2.2) in a formal system $\mathcal{S}$ is usually based in an essential way on the syntax of the formula language Fo of $\mathcal{S}$. Frequently, an extension Fo' of Fo with formulas that may contain formula variables is considered[5] and rules are given in a schematic way: A rule R is based on a pair of the form $\langle\langle A_1, \ldots, A_n\rangle, A\rangle$, where $A_1, \ldots, A_n, A \in Fo'$. The "instances" or "applications" of R are then defined to be all inferences of the form $\bar{\sigma}(A_1), \ldots, \bar{\sigma}(A_n)/\bar{\sigma}(A)$ with premises $\bar{\sigma}(A_1), \ldots, \bar{\sigma}(A_n)$ and with conclusion $\bar{\sigma}(A)$, where for an arbitrary substitution σ that assigns formulas of Fo to formula variables, $\bar{\sigma}$ is a homomorphic extension of σ to a function from Fo' into Fo.

However, for a study of general properties of the notions of rule derivability and admissibility it is desirable to abstract away from the language-specific details concerned with substitution when rules are introduced as schemes, because these details may be hard to capture in a general framework. And furthermore, it is conceivable that, in some formal systems, rules could be defined in a different, though still 'mechanizable' way. One possible abstract view on how rules may be described in a formal system is taken in [3]. There, rules are considered to be determined by extensional 'rule descriptions' as follows: Given a set Fo of formulas and a natural number $n \in \omega\setminus\{0\}$, every n-ary partial function $\Phi : (Fo)^n \rightharpoonup Fo$, a *rule description*, is understood to determine an n-premise rule $R(\Phi)$ for formal systems with formula set Fo. The instances of $R(\Phi)$ are defined to be all those inferences $A_1, \ldots, A_n/A$ with premises $A_1, \ldots, A_n$ and conclusion A for which $\Phi(A_1, \ldots, A_n) = A$ holds.

A notion of inference rule that is based on such rule descriptions may be considered as too limited, for two reasons: Firstly, it does not allow for rules that enable to infer different possible conclusions from the same premise(s) such as, for example, the $\forall$-elimination rule

$$\frac{\forall x\, A}{A[t/x]}\ \forall\mathrm{E}$$

in a natural-deduction system for predicate logic. And secondly, this formalization tends to identify rules that have the same extensional behaviour, but that might be defined operationally in a different way. Yet, whereas the first objection could clearly be met by allowing 'rule descriptions' of n-premise rules to be of the form $\Phi : (Fo)^n \rightharpoonup \mathcal{P}(Fo)$, (or equivalently, to be relations on $(Fo)^{n+1}$), the second seems to call for a different formal framework.

Our formalization of a general class of Hilbert-style proof systems, to be given below, draws on the notion *abstract rewrite system* (ARS) in the notation of van Oostrom and de Vrijer in [11] and in [9, p. 317]: There, an ARS $\mathcal{A}$ is defined as a quadruple $\langle A, \Phi, \mathsf{src}, \mathsf{tgt}\rangle$ in which A and Φ are sets whose members are respectively called *objects* and *steps*, and $\mathsf{src}, \mathsf{tgt} : \Phi \to A$ are the *source* and the *target functions* of $\mathcal{A}$. The authors of [11] emphasize that this definition "is in concordance with many papers on abstract rewriting in general and residuals in particular". Interestingly, an analogous notion ("indexed 1-complexes") occurs already in the seminal paper [5] by Newman. This and other related notions are also mentioned in [11]:

> "The terminology used in connection with abstract rewrite systems varies throughout the literature, depending on the intended application area. For instance, our abstract rewrite systems are called *indexed 1-complexes* in [5], *reduction complexes*

[5] No extension is needed if formula variables are already allowed to occur in formulas of $\mathcal{S}$.

in [7], and *graphs* in [4]. Objects are called *(1)-cells*, *transitions*, *moves*, *events*, *edges* or *arrows*. Source and target are called *domain* and *codomain*, *start* and *terminus*, or *initial* and *final*."

For the definition of useful notions in relation to ARS's, see also Appendix D.2.

Before giving the formal definition of our abstract notion of rules, we need the following prerequisites: For an arbitrary set X, we denote by $Seqs_{\mathrm{f}}(X)$ the set of all finite-length sequences of elements of X, that is, we let

$$Seqs_{\mathrm{f}}(X) = \{()\} \cup \{\, (x_1, \ldots, x_n) \mid n \in \omega,\, x_1, \ldots, x_n \in X \,\}\ ,$$

where $()$ denotes a sequence of length 0; by $lg : Seqs_{\mathrm{f}}(X) \to \omega$ we designate the function which to every sequence $\sigma \in Seqs_{\mathrm{f}}(X)$ assigns its *length* $lg(\sigma)$ (for example, $lg(()) = 0$ and $lg((x_1, x_2, x_3, x_4)) = 4$). For all sets X, $\sigma \in Seqs_{\mathrm{f}}(X)$ and $x \in X$, we say that x *occurs in* σ iff $i \in \omega$ exists such that $\sigma = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $x = x_i$. It is clear what we mean by saying that, for some set X, an element x of X *occurs in* a sequence $\sigma \in Seqs_{\mathrm{f}}(X)$ n *times* or that x *occurs* in such a sequence σ *precisely* n *times*.

Definition 2.1 (An abstract notion of rule). Let Fo be a set. An *AHS-rule* (*a rule for an abstract Hilbert system*—which notion will be defined below) is a triple of the form $\langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$, where

- $Apps$ is a set, the members of which are called the *applications of* R, and
- $\mathsf{prem} : Apps \to Seqs_{\mathrm{f}}(Fo)$ and $\mathsf{concl} : Apps \to Fo$ are the *premise* and *conclusion* functions of R.

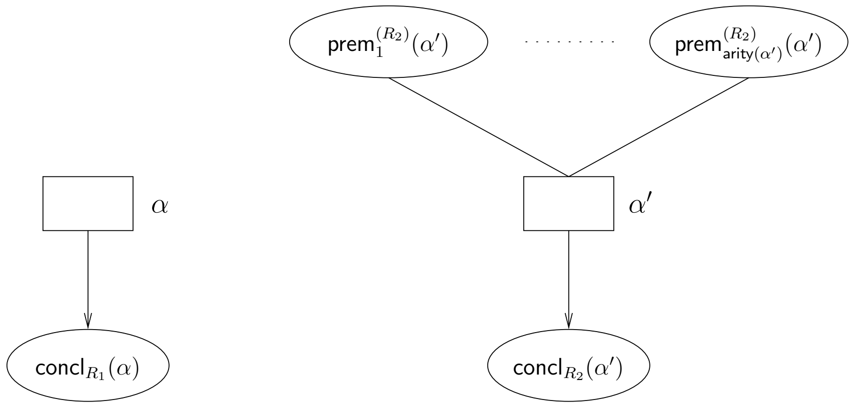
We will use the symbolic denotations $Apps_R$, prem_R and concl_R, whenever we want to refer directly to the application set, the premise and conclusion functions of a rule R, respectively. And we will use α as a syntactical variable for applications.

In addition to the functions prem and concl associated with a rule, we will now define the functions arity and prem_i, for auxiliary purposes. For every set Fo of formulas and for every rule $R = \langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$ on Fo, we introduce

$$\begin{array}{rll} \text{the function} & \mathsf{arity} : Apps \to \omega & \text{and} \\ \text{the partial functions} & \mathsf{prem}_i : Apps \rightharpoonup Fo & (\text{for all } i \in \omega\setminus\{0\}) \end{array}$$

as follows: arity assigns to every application α of R the number of its premises, i.e. the length $lg(\mathsf{prem}(\alpha))$ of the formula sequence $\mathsf{prem}(\alpha)$. And for all $i \in \omega\setminus\{0\}$, $\mathsf{prem}_i(\alpha)$ assigns to every application α of R its i-th premise, i.e. the i-th formula A_i in the sequence $\mathsf{prem}(\alpha) = (A_1, \ldots, A_{i-1}, A_i, \ldots)$, whenever this exists; otherwise $\mathsf{prem}_i(\alpha)$ is undefined. We will use the denotations arity_R and $\mathsf{prem}_i^{(R)}$, whenever we want to make the dependence of arity and prem_i upon R explicit. Using these definitions, a visualization as graph 'hyperedges' of two kinds of rule applications α and α', with no premises and with a finite, non-zero number of premises, respectively, is given in Figure 1.

Figure 1 Visualization as graph 'hyperedges' of two kinds of rule applications of AHS-rules: Of a zero premise application α of a rule R_1 and of an application α' of a rule R_2 such that α' has a finite, non-zero number of premises.



Definition 2.1 allows rule applications to possess any finite arity. Also, rules may have applications of different arities. Contrary to traditional formal systems, axioms are nowadays frequently avoided in Hilbert-style proof systems for the sake of technical convenience and are modeled by zero-premise rule applications instead. However, we have decided not to follow this practice here. In our notion of abstract Hilbert system, which will now be defined, the appearance of axioms as well as of rules with zero-premise applications will be allowed.

Definition 2.2 (Abstract Hilbert Systems). An *abstract Hilbert system* (an AHS) $\mathcal{S}$ is a triple $\langle Fo, Ax, \mathcal{R} \rangle$ consisting of sets Fo, Ax and $\mathcal{R}$ such that

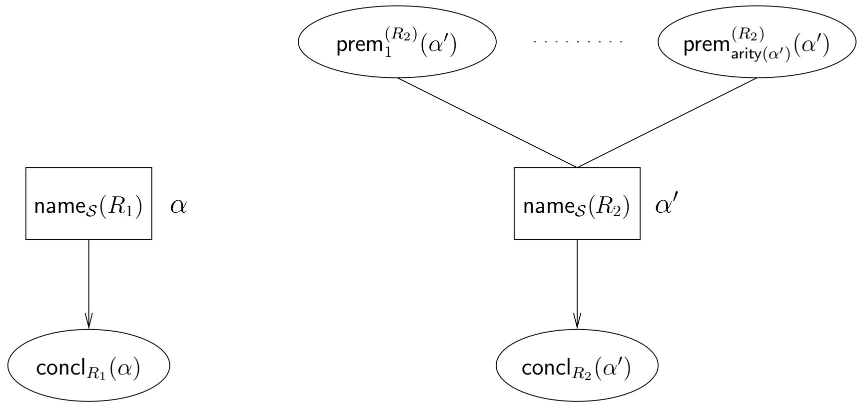
- the elements of Fo, Ax and $\mathcal{R}$ are respectively called the *formulas*, the *axioms* and the *rules* of $\mathcal{S}$,
- $Ax \subseteq Fo$ holds, i.e. all axioms of $\mathcal{S}$ are formulas of $\mathcal{S}$, and
- every rule $R \in \mathcal{R}$ is an AHS-rule on Fo.

We denote by $\mathfrak{H}$ the class of all abstract Hilbert systems. If, for a some abstract Hilbert system $\mathcal{S}$, we want to refer to its set of formulas, its set of axioms or its set of rules, then we will use the symbolic denotations $Fo_{\mathcal{S}}$, $Ax_{\mathcal{S}}$ or $\mathcal{R}_{\mathcal{S}}$, respectively.

In most usually encountered formal systems, names are given to the axioms and rules of the system. Apart from being useful identifiers for presenting proof systems and for reasoning about them, names for axioms and rules are helpful devices for making formal derivations easier understandable, at least for the human reader: Frequently, each inference in a derivation is labeled by the respective rule of which the inference is an instance (or, also called, an application). In practice however, name labels in derivations are often dropped at inferences for which it is easily recognizable to which rule they belong.

We have not taken up the concept of names for axioms and rules into our framework of AHS because it introduces a formal technicality, which often makes the formulation of definitions and theorems rather more complicated than necessary. In

Figure 2 Visualization as graph 'hyperedges' of two kinds of rule applications in an n-AHS $\mathcal{S}$: Of a zero premise application α of a rule R_1 and of an application α' of a rule R_2 such that α' has a finite, non-zero number of premises.



fact, names are inessential for the development of the notions of rule derivability and admissibility as well as for most of the results presented in this paper. However, they will play a role in Section 5 where we are concerned with a question about the possibility to eliminate applications of a considered rule from a given derivation. There, it will be important to have a notion of derivation at hand with the property that each inference in a derivation carries a label which denotes the rule according to an application of which the inference has been formed. For this purpose, we introduce also the following extended notion of abstract Hilbert system, in which names are assigned to axioms and rules.

Definition 2.3 (Abstract Hilbert Systems with names). An *abstract Hilbert system with names (for axioms and rules)* (an n-AHS) $\mathcal{S}$ is a quintupel of the form $\langle Fo, Ax, \mathcal{R}, Na, \mathsf{name}\rangle$ such that:

- Fo, Ax, $\mathcal{R}$ and Na are sets, the elements of which are respectively called the *formulas*, the *axioms*, the *rules* and the *names of* $\mathcal{S}$.
- $\langle Fo, Ax, \mathcal{R}\rangle$ is an AHS, which is called the *underlying AHS* of $\mathcal{S}$.
- $\mathsf{name} : Ax \cup \mathcal{R} \to Na$ is the *name function of* $\mathcal{S}$, which is injective on the subset $\mathcal{R}$ of its domain $Ax \cup \mathcal{R}$. The function name assigns a *name* to every axiom of $\mathcal{S}$ and a unique *name* to every rule of $\mathcal{S}$.

We denote by $\mathfrak{Hn}$ the class of all abstract Hilbert systems with names. If, for some n-AHS $\mathcal{S}$, we want to refer to its set of formulas, its set of axioms, its set of rules or to its name function, then we will use the symbolic denotations $Fo_{\mathcal{S}}$, $Ax_{\mathcal{S}}$, $\mathcal{R}_{\mathcal{S}}$, $Na_{\mathcal{S}}$ or $\mathsf{name}_{\mathcal{S}}$, respectively.

A visualization as 'graph hyperedges' of two kinds of rule applications in an n-AHS, with no premises and respectively with a finite number of premises, is given in Figure 2. Clearly, every AHS $\mathcal{S} = \langle Fo, Ax, \mathcal{R}\rangle$ can be extended to an n-AHS $\mathcal{S}'$ in a trivial way by letting $\mathcal{S}' = \langle Fo, Ax, \mathcal{R}, Ax \cup \mathcal{R}, \mathrm{id}_{Ax\cup\mathcal{R}}\rangle$, where $\mathrm{id}_{Ax\cup\mathcal{R}}$ is the identity function on $Ax \cup \mathcal{R}$; hereby we use the axioms and rules of $\mathcal{S}$ as names for themselves in $\mathcal{S}'$. The fact that in an AHS names are assigned not only to rules, but also to axioms, makes it possible to formalize the notion of axiom scheme: An *axiom scheme* $\mathcal{A}x$ in an n-AHS $\mathcal{S}$ is a non-empty set of the form $\{A \in Fo \mid \mathsf{name}_{\mathcal{S}}(A) = na\}$ for some $na \in Na_{\mathcal{S}}$.

For the next definition, we will need the notion of finite multisets over some given set. For an arbitrary set X, we let

$$\mathcal{M}_{\mathrm{f}}(X) = \{M : X \to \omega \mid M(x) \neq 0 \text{ for only finitely many } x \in X\}$$

the set of *finite multisets over* X; we say that $x \in X$ occurs in $M \in \mathcal{M}_{\mathrm{f}}(X)$ iff $M(x) \neq 0$, and that, for all $n \in \omega$, $x \in X$ *occurs* n *times in* M iff $M(x) = n$. The union of two finite multisets $M_1, M_2 \in \mathcal{M}_{\mathrm{f}}(X)$ over a set X is defined by

$$\begin{aligned} M_1 \uplus M_2 :\; & X \to \omega \\ & x \mapsto M_1 \uplus M_2 = M_1(x) + M_2(x)\,. \end{aligned}$$

(More notation about finite multisets is introduced in Appendix D.3 where basic results about the multiset ordening are gathered). For given sets X, we denote by $\mathcal{P}(X)$ the set of all subsets of X, i.e. the powerset of X, and by $\mathcal{P}_{\mathrm{f}}(X)$ the set of all *finite* subsets of X. Here and later we will use Γ, Δ as syntactical variables for multisets of formulas and Σ, Ξ for sets of formulas.

We will now introduce derivations in an AHS or n-AHS $\mathcal{S}$ as *prooftrees* in the sense of [10]: These are trees in which the nodes are labelled by formulas and in which the edges make part of rule applications and are not drawn, but are replaced by horizontal lines that represent applications. Axioms and assumptions appear as top nodes and lower nodes are formed by applications of rules. In the case of n-AHS's, which we will treat first, occurrences of axioms and of inferences that correspond to rule applications will furthermore be labeled by the names of the respective axioms or rules.

Definition 2.4 (Derivations in abstract Hilbert systems with names). Let $\mathcal{S} = \langle Fo, Ax, \mathcal{R}, Na, \mathsf{name}\rangle$ be an n-AHS. A *derivation* $\mathcal{D}$ *in* $\mathcal{S}$ is a prooftree that is the result of carrying out a finite number of construction steps of the three kinds detailed below. Simultaneously with this inductive definition, we define for all derivations $\mathcal{D}$ in $\mathcal{S}$ its multiset $\mathsf{assm}(\mathcal{D})$ of *assumptions* (where $\mathsf{assm}(\mathcal{D}) \in \mathcal{M}_{\mathrm{f}}(Fo)$), its *conclusion* $\mathsf{concl}(\mathcal{D})$ (with $\mathsf{concl}(\mathcal{D}) \in Fo$) and its *(rule application) depth* $|\mathcal{D}|$:

(i) For every axiom $A \in Ax$, the prooftree $\mathcal{D}$ of the form

$$\frac{(\mathsf{name}(A))}{A} \tag{2.4}$$

is a derivation in $\mathcal{S}$ with conclusion $\mathsf{concl}(\mathcal{D}) = A$ and with no assumptions, i.e. $\mathsf{assm}(\mathcal{D}) = \emptyset$ holds. Its depth is defined as $|\mathcal{D}| = 0$.

(ii) For all formulas $A \in Fo$, the prooftree $\mathcal{D}$ consisting only of the formula

$$A \tag{2.5}$$

is a derivation in $\mathcal{S}$ with assumptions $\mathsf{assm}(\mathcal{D}) = \{A\}$ and with conclusion $\mathsf{concl}(\mathcal{D}) = A$. It has depth $|\mathcal{D}| = 0$.

(iii) Let a rule $R \in \mathcal{R}$ and an application $\alpha \in \mathit{Apps}_R$ be given. We distinguish two cases concerning the arity of α:

Case 1. $\mathsf{arity}_R(\alpha) = 0$: Given that $\mathsf{concl}_R(\alpha) = A$, the prooftree

$$\frac{}{A}\,\mathsf{name}(R) \tag{2.6}$$

is a derivation $\mathcal{D}$ in $\mathcal{S}$ that has conclusion $\mathsf{concl}(\mathcal{D}) = A$ and no assumptions, i.e. $\mathsf{assm}(\mathcal{D}) = \emptyset$ holds. Its depth is $|\mathcal{D}| = 1$.

Case 2. $\mathsf{arity}_R(\alpha) = n \in \omega \setminus \{0\}$: Given that $\mathsf{prem}^{(R)}(\alpha) = (A_1, \ldots, A_n)$ and that $\mathsf{concl}_R(\alpha) = A$, and given further that $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are derivations in $\mathcal{S}$ with respective conclusions $A_1, \ldots, A_n$, the prooftree of the form

$$\frac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n\end{matrix}}{A}\,\mathsf{name}(R) \tag{2.7}$$

is a derivation $\mathcal{D}$ in $\mathcal{S}$ with conclusion $\mathsf{concl}(\mathcal{D}) = A$ and with assumptions and depth defined by

$$\mathsf{assm}(\mathcal{D}) = \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_i) \quad \text{and} \quad |\mathcal{D}| = 1 + \max\{\, |\mathcal{D}_i| \mid i \in \omega,\, 1 \le i \le n \,\} \;.$$

We denote by $\mathit{Der}(\mathcal{S})$ the set of all derivations in $\mathcal{S}$ and let the assumption function $\mathsf{assm} : \mathit{Der}(\mathcal{S}) \to \mathit{Seqs}_{\mathrm{f}}(\mathit{Fo})$ and the conclusion function $\mathsf{concl} : \mathit{Der}(\mathcal{S}) \to \mathit{Fo}$ on derivations of $\mathcal{S}$ be defined as described above. For arbitrary rules $R \in \mathcal{R}$, applications $\alpha \in \mathit{Apps}_R$ and derivations $\mathcal{D} \in \mathit{Der}(\mathcal{S})$, we designate by $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ the derivation in $\mathcal{S}$ consisting of an (one-step) *inference* of the form

$$\frac{}{\mathsf{concl}_R(\alpha)}\,\mathsf{name}(R) \qquad \text{or} \qquad \frac{\mathsf{prem}^{(R)}_1(\alpha) \;\; \ldots \;\; \mathsf{prem}^{(R)}_{\mathsf{arity}(\alpha)}(\alpha)}{\mathsf{concl}_R(\alpha)}\,\mathsf{name}(R) \;,$$

given that $\mathsf{arity}_R(\alpha) = 0$ holds in the left and $\mathsf{arity}_R(\alpha) \in \omega$ in the right case, and will call $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ *the derivation of* $\mathcal{S}$ *corresponding to the application* α *of* R. We will also allow to speak of occurrences of inferences $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ (for all $R \in \mathcal{R}$ and $\alpha \in \mathit{Apps}_R$) in derivations $\mathcal{D} \in \mathit{Der}(\mathcal{S})$ as of *applications of* R.

Derivations in AHS's will now be defined as analogous prooftrees in which inferences corresponding to rule applications are not labeled.

Definition 2.5 (Derivations in abstract Hilbert systems). Let $\mathcal{S}$ be an AHS. A *derivation* $\mathcal{D}$ *in* $\mathcal{S}$ is the result of carrying out a finite number of three kinds of construction steps that arise from the ones numbered (i), (ii) and (iii) in Definition 2.4 by replacing each of the prooftrees (2.4) and (2.6), for a respective formula

A occurring there, by the respective prooftree

$$\overline{\ A\ }$$

and by replacing the prooftrees of the form (2.7), for respective formulas $A, A_1, \ldots, A_n$ and derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$, by ones of the respective form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array}}{A}$$

(for respective formulas $A, A_1, \ldots, A_n$ and derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$). Simultaneously with this inductive definition, for all derivations $\mathcal{D}$ in $\mathcal{S}$ the multiset $\mathsf{assm}(\mathcal{D})$ of *assumptions* (where $\mathsf{assm}(\mathcal{D}) \in \mathcal{M}_{\mathrm{f}}(Fo)$), its *conclusion* $\mathsf{concl}(\mathcal{D})$ (where it holds $\mathsf{concl}(\mathcal{D}) \in Fo$) and its *(rule application) depth* $|\mathcal{D}|$ are defined by identical stipulations as in the respective construction steps (i), (ii) and (iii) of Definition 2.4. Again, we denote by $Der(\mathcal{S})$ the set of all derivations in $\mathcal{S}$ and let the assumption function $\mathsf{assm} : Der(\mathcal{S}) \to Seqs_{\mathrm{f}}(Fo)$ and the conclusion function $\mathsf{concl} : Der(\mathcal{S}) \to Fo$ on derivations of $\mathcal{S}$ be defined analogously as in Definition 2.4.

For arbitrary rules $R \in \mathcal{R}$ and applications $\alpha \in Apps_R$, we again designate by $\mathcal{D}_{(\alpha,R,\mathcal{S})} \in Der(\mathcal{S})$ the derivation consisting of an (one-step) *inference* of the form

$$\frac{}{\mathsf{concl}_R(\alpha)} \quad \text{or} \quad \frac{\mathsf{prem}_1^{(R)}(\alpha) \ \ldots \ \mathsf{prem}_{\mathsf{arity}(\alpha)}^{(R)}(\alpha)}{\mathsf{concl}_R(\alpha)} \ ,$$

given that respectively $\mathsf{arity}_R(\alpha) = 0$ and $\mathsf{arity}_R(\alpha) \in \omega$ hold with respect to the left and the right inference, and will call $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ *the derivation of* $\mathcal{S}$ *corresponding to the application* α *of* R. Again, we will allow to refer to occurrences of inferences $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ (for all $R \in \mathcal{R}$ and $\alpha \in Apps_R$) in derivations $\mathcal{D} \in Der(\mathcal{S})$ as *applications of* R.

There is obviously an immediate relationship between derivations in an n-AHS $\mathcal{S}$ and derivations in the AHS $\mathcal{S}_0$ underlying $\mathcal{S}$: Every derivation $\mathcal{D} \in Der(\mathcal{S})$ can be transformed into a derivation $\mathcal{D}_0 \in Der(\mathcal{S}_0)$ with the same multiset of assumptions and the same conclusion as $\mathcal{D}$ by simply dropping all labels for axioms and rules from the prooftree $\mathcal{D}$; formally this can be shown by induction on the depth $|\mathcal{D}|$ of a derivation. And conversely, from every derivation $\mathcal{D}_0 \in Der(\mathcal{S}_0)$ a derivation $\mathcal{D} \in Der(\mathcal{S})$ with the same multiset of assumptions and the same conclusion as $\mathcal{D}_0$ can be built by labeling every occurrence of an axiom in $\mathcal{D}_0$ by its name in $\mathcal{S}$ and by labeling every inference in $\mathcal{D}_0$ by the name in $\mathcal{S}$ of a rule according to an application of which the inference is formed.

For the purpose of formally stating some consequences of these easy transformations, we define, for every n-AHS $\mathcal{S}$ and its underlying AHS $\mathcal{S}_0$, the function

$$\check{\ } : Der(\mathcal{S}) \to Der(\mathcal{S}_0)$$

which maps every derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S})$ to the derivation $\check{\mathcal{D}} \in \mathit{Der}(\mathcal{S}_0)$ that arises from $\mathcal{D}$ by removing all labels for names and rules from the prooftree $\mathcal{D}$. We have sketched a proof for the following lemma in the previous paragraph.

Lemma 2.6 *Let $\mathcal{S}$ be an n-AHS and $\mathcal{S}_0$ its underlying AHS. Then it holds that*

$$\mathit{Der}(\mathcal{S}_0) = \left\{ \check{\mathcal{D}} \mid \mathcal{D} \in \mathit{Der}(\mathcal{S}) \right\} .$$

As an immediate consequence of this lemma, we find the following statement.

Proposition 2.7 *Let $\mathcal{S}$ be an n-AHS and $\mathcal{S}_0$ its underlying AHS.*

(i) For every derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S})$ there exists a derivation $\mathcal{D}_0 \in \mathit{Der}(\mathcal{S}_0)$ with the same multiset of assumptions, the same conclusion and the same depth as $\mathcal{D}$, i.e. such that it holds:

$$\mathsf{assm}(\mathcal{D}) = \mathsf{assm}(\mathcal{D}_0), \;\; \mathsf{concl}(\mathcal{D}) = \mathsf{concl}(\mathcal{D}_0) \quad \text{and} \;\; |\mathcal{D}| = |\mathcal{D}_0| \,. \tag{2.8}$$

(ii) For every derivation $\mathcal{D}_0 \in \mathit{Der}(\mathcal{S}_0)$ there exists a derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S})$ with the property (2.8).

For use in the next definition as well as in later ones, we define, for arbitrary sets X, the functions

$$\begin{aligned} \mathrm{set} : \mathcal{M}_{\mathrm{f}}(X) \cup \mathit{Seqs}_{\mathrm{f}}(X) &\to \mathcal{P}_{\mathrm{f}}(X) \\ \mathrm{mset} : \mathcal{P}_{\mathrm{f}}(X) \cup \mathit{Seqs}_{\mathrm{f}}(X) &\to \mathcal{M}_{\mathrm{f}}(X) \end{aligned}$$

in the following way: The function $\mathrm{set}(\cdot)$ assigns to every multiset $M \in \mathcal{M}_{\mathrm{f}}(X)$ the finite *set* $\mathrm{set}(M)$ of all elements of X that occur in M, and to every sequence $\sigma \in \mathit{Seqs}_{\mathrm{f}}(X)$ the finite *set* $\mathrm{set}(\sigma)$ of all elements of X that occur in σ. And the function $\mathrm{mset}(\cdot)$ assigns to every finite subset Y of X the finite *multiset* $\mathrm{mset}(X)$ in which every element of Y occurs precisely once and no other elements of X occur, and to every sequence $\sigma \in \mathit{Seqs}_{\mathrm{f}}(X)$ the finite *multiset* $\mathrm{mset}(\sigma)$ in which every element of X occurs precisely as often as in σ and no other elements of X occur.

We are now going to associate with every AHS three consequence relations that differ by specific stipulations for how the assumptions occurring in a derivation are counted or for in which sense derivations are allowed to make use of assumptions from a respectively given set or multiset of formulas.

Definition 2.8 (Three consequence relations and three kinds of relative derivability statements in an AHS or n-AHS). Let $\mathcal{S}$ be an AHS or n-AHS with formula set Fo. We define the consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$, where $\vdash_{\boldsymbol{\mathcal{S}}}, \vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}} \subseteq \mathcal{P}(\mathit{Fo}) \times \mathit{Fo}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}} \subseteq \mathcal{M}_{\mathrm{f}}(\mathit{Fo}) \times \mathit{Fo}$ by stipulating for all $A \in \mathit{Fo}$, *<u>s</u>ets* $\Sigma \in \mathcal{P}(\mathit{Fo})$ and *<u>m</u>ultisets* $\Gamma \in \mathcal{M}_{\mathrm{f}}(\mathit{Fo})$:

$$\begin{aligned} \langle \Sigma, A \rangle \in \vdash_{\boldsymbol{\mathcal{S}}} \quad &\Longleftrightarrow \quad (\exists\, \mathcal{D} \in \mathit{Der}(\mathcal{S})) \big[\, \mathrm{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big] \,, \\ \langle \Sigma, A \rangle \in \vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}} \quad &\Longleftrightarrow \quad (\exists\, \mathcal{D} \in \mathit{Der}(\mathcal{S})) \big[\, \mathrm{set}(\mathsf{assm}(\mathcal{D})) = \Sigma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big] \,, \\ \langle \Gamma, A \rangle \in \vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}} \quad &\Longleftrightarrow \quad (\exists\, \mathcal{D} \in \mathit{Der}(\mathcal{S})) \big[\, \mathsf{assm}(\mathcal{D}) = \Gamma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A \,\big] \,. \end{aligned}$$

With respect to each of these three consequence relations we consider *relative derivability statements* of the respective form $\Sigma \vdash_{\mathcal{S}} A$, $\Sigma \vdash^{(s)}_{\mathcal{S}} A$ and $\Gamma \vdash^{(m)}_{\mathcal{S}} A$, and we define what will be meant by saying that such a statement *holds*: For all $A \in Fo$ and $\Sigma \in \mathcal{P}(Fo)$, the statements $\Sigma \vdash_{\mathcal{S}} A$ and $\Sigma \vdash^{(s)}_{\mathcal{S}} A$ hold if and only if, respectively, $\langle \Sigma, A\rangle \in \vdash_{\boldsymbol{\mathcal{S}}}$ and $\langle \Sigma, A\rangle \in \vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$. And analogously, for all $\Gamma \in \mathcal{M}_{\mathrm{f}}(Fo)$ and $A \in Fo$, the statement $\Gamma \vdash^{(m)}_{\mathcal{S}} A$ holds if and only if $\langle \Gamma, A\rangle \in \vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$.

The consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$ and the induced notions of holding relative derivability statement correspond to different degrees of 'resource-consciousness' in derivations. For every AHS or n-AHS $\mathcal{S}$ and for all formulas $A \in Fo_{\mathcal{S}}$, sets Σ and multisets Γ of formulas in $Fo_{\mathcal{S}}$, the relative derivability statements $\Sigma \vdash_{\mathcal{S}} A$, $\Sigma \vdash^{(s)}_{\mathcal{S}} A$ and $\Gamma \vdash^{(m)}_{\mathcal{S}} A$ respectively assert the following: That the formula A is derivable in $\mathcal{S}$ from a *subset* of the assumptions in Σ (using every assumption as often as needed), that A is derivable from the *set* Σ of assumptions (using every assumption one or more times) or that A is derivable in $\mathcal{S}$ from *precisely* the assumptions in the *multiset* Γ of assumptions (and thereby using every assumption exactly once).

In Appendix E a close connection is established between the three kinds of consequence relations defined above and sequent-style "Hilbert systems for consequence" (HSC's) à la Avron[6]. In particular, natural correspondences are established between the class of AHS's and the subclass of HSC's consisting of all "pure" and single-conclusioned systems. These correspondences yield the statement that, for every AHS, the consequence relations $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}, \vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}, \vdash_{\boldsymbol{\mathcal{S}}}$ can be "axiomatized" by respective "pure", single-conclusioned HSC's $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$; hereby $\mathcal{H}_1$ does not contain any structural rules, whereas $\mathcal{H}_2$ is the extension of $\mathcal{H}_1$ with contraction, and $\mathcal{H}_3$ is the extension of $\mathcal{H}_1$ with weakening and contraction. Certainly through this result it becomes apparant that a fourth kind of consequence relation on AHS's has not been defined in Definition 2.8, namely one that can be "axiomatized" by a HSC that contains weakening, but not contraction. This consequence relation $\vdash^{\mathbf{(mw)}}$ is also introduced in Appendix E, but it is not treated otherwise in this report.

Furthermore it is proven in Appendix E that, for some set Fo, a relation $\Vdash$ with $\Vdash \subseteq \mathcal{P}_{\mathrm{f}}(Fo) \times Fo$ (i.e. a relation between *sets* on Fo and Fo), or of the form $\Vdash \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$ (i.e. a relation between *multisets* on Fo and Fo) is "naturally axiomatizable" by a HSC if and only if it is one of the consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$, $\vdash^{\mathbf{(mw)}}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$ on an AHS $\mathcal{S}$ with formula set Fo. Due to this result it can be said that in this report we study three of the four consequence relations on AHS's which can be "naturally axiomatized" by pure, single conclusioned "Hilbert systems for consequence" à la Avron.

The following proposition is an obvious consequence of Definition 2.8.

Proposition 2.9 *Let $\mathcal{S}$ be an AHS or an n-AHS with set Fo of formulas. Then for all formulas $A \in Fo$, for all sets $\Sigma \in \mathcal{P}(Fo)$ and for all multisets $\Gamma \in \mathcal{M}_f(Fo)$ it holds:*

$$\Gamma \vdash^{(m)}_{\mathcal{S}} A \;\Rightarrow\; \mathit{set}(\Gamma) \vdash^{(s)}_{\mathcal{S}} A\ , \quad \textit{and} \quad \Sigma \vdash^{(s)}_{\mathcal{S}} A \;\Rightarrow\; \Sigma \vdash_{\mathcal{S}} A\ .$$

[6] For the original definition of "Hilbert-type systems for consequence" see [1, p.26].

It follows immediately from Definition 2.8 and from Proposition 2.7 that, for every n-AHS $\mathcal{S}$ with underlying AHS $\mathcal{S}_0$, the consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$ coincide respectively with the consequence relations $\vdash_{\boldsymbol{\mathcal{S}_0}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}_0}}$ and $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}_0}}$; as a consequence also the respective three kinds of holding relative derivability statements in $\mathcal{S}$ and in $\mathcal{S}_0$ are in agreement. In a similar way, also the notions formalized in the next definition can be seen to coincide for every n-AHS and its underlying AHS.

Definition 2.10 (Theorems, theory of an AHS or n-AHS). Let $\mathcal{S}$ be an AHS or an n-AHS. A formula $A \in Fo_{\mathcal{S}}$ is a *theorem* of $\mathcal{S}$ if and only if $\emptyset \vdash_{\mathcal{S}} A$, i.e. iff there exists a derivation $\mathcal{D}$ in $\mathcal{S}$ from the empty set of assumptions and with conclusion A; in this case we write $\vdash_{\mathcal{S}} A$ for $\emptyset \vdash_{\mathcal{S}} A$. The *theory of* $\mathcal{S}$ is the set $Th(\mathcal{S}) = \{A \in Fo_{\mathcal{S}} \mid \vdash_{\mathcal{S}} A\}$ of theorems of $\mathcal{S}$.

Extensions of systems of formal logic are often defined in the following way (see, for example, the special case of "first-order theories" treated in [8, p. 41]): A formal system $\mathcal{S}'$ is called an *extension* of a formal system $\mathcal{S}$ if the language of $\mathcal{S}'$ is an extension of the language of $\mathcal{S}$ and if every theorem of $\mathcal{S}$ is also a theorem of $\mathcal{S}'$. Since we will mainly be interested in extensions of formal systems that arise by adding new formulas, axioms and/or rules, we introduce a particular name for such extensions.

Definition 2.11 (Extensions by enlargement of AHS's and n-AHS's). We consider AHS's and n-AHS's separately from each other in the two items below.

(i) Let $\mathcal{S} = \langle Fo, Ax, \mathcal{R}\rangle$ and $\mathcal{S}' = \langle Fo', Ax', \mathcal{R}'\rangle$ be two AHS's. We say that $\mathcal{S}'$ is an *extension by enlargement* of $\mathcal{S}$ or that $\mathcal{S}$ is a *sub-AHS of* $\mathcal{S}'$ if and only if $Fo \subseteq Fo'$, $Ax \subseteq Ax'$ and $\mathcal{R} \subseteq \mathcal{R}'$, i.e. iff the set of formulas, axioms and rules of $\mathcal{S}$ are respectively contained in the sets of formulas, axioms and rules of $\mathcal{S}'$.

(ii) Let $\mathcal{S} = \langle Fo, Ax, \mathcal{R}, Na, \mathsf{name}\rangle$ and $\mathcal{S}' = \langle Fo', Ax', \mathcal{R}', Na', \mathsf{name}'\rangle$ be n-AHS's. Then we call $\mathcal{S}'$ an *extension by enlargement* of $\mathcal{S}$, or we call $\mathcal{S}$ is a *sub-n-AHS of* $\mathcal{S}'$ if and only if it holds that $Fo \subseteq Fo'$, $Ax \subseteq Ax'$, $\mathcal{R} \subseteq \mathcal{R}'$ and $Na \subseteq Na'$, and if furthermore
$$\mathsf{name}'|_{Ax\cup\mathcal{R}} = \mathsf{name}$$
(where $\mathsf{name}'|_{Ax\cup\mathcal{R}}$ is the restriction of name' to the set $Ax \cup \mathcal{R}$) is the case, i.e. if the name function of $\mathcal{S}'$ assigns the same names to the axioms and rules of $\mathcal{S}$ as the name function of $\mathcal{S}$.

The following lemma formulates a special feature of the notion "extension by enlargement": For every AHS or n-AHS $\mathcal{S}$, a relative derivability statement that holds in $\mathcal{S}$ does also hold in every extension by enlargement of $\mathcal{S}$.

Lemma 2.12 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two AHS's, or two n-AHS's, such that $\mathcal{S}_2$ is an extension by enlargement of $\mathcal{S}_1$. Then it holds for all $A \in Fo_{\mathcal{S}_1}$, $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ and $\Gamma \in \mathcal{M}_f(Fo_{\mathcal{S}_1})$ that*

$$\begin{aligned}\Sigma \vdash_{\mathcal{S}_1} A \quad &\Longrightarrow \quad \Sigma \vdash_{\mathcal{S}_2} A \;, \\ \Sigma \vdash^{(s)}_{\mathcal{S}_1} A \quad &\Longrightarrow \quad \Sigma \vdash^{(s)}_{\mathcal{S}_2} A \;,\end{aligned}$$

$$\Gamma \vdash^{(m)}_{\mathcal{S}_1} A \quad \Longrightarrow \quad \Gamma \vdash^{(m)}_{\mathcal{S}_2} A \ .$$

For extensions by enlargement of AHS's that arise by addition of new axioms and/or rules, but not of new formulas, we introduce the following notation.

Definition 2.13 (Adding new axioms or rules to an AHS). Let $\mathcal{S}$ be an AHS of the form $\mathcal{S} = \langle Fo, Ax, \mathcal{R} \rangle$, let $\Sigma \subseteq Fo$ and let $\mathcal{R}_{\text{new}}$ be a set of rules over Fo.

(i) By $\mathcal{S}+\Sigma$ we denote the extension by enlargement $\langle Fo, Ax \cup \Sigma, \mathcal{R} \rangle$ of $\mathcal{S}$.

(ii) By $\mathcal{S}+\mathcal{R}_{\text{new}}$ we denote the AHS $\langle Fo, Ax, \mathcal{R} \cup \mathcal{R}_{\text{new}} \rangle$. If $\mathcal{R}_{\text{new}} = \{R\}$, then we will allow to write $\mathcal{S}+R$ for $\mathcal{S}+\mathcal{R}_{\text{new}}$.

An easy, later important relationship between relative derivability statements in an AHS $\mathcal{S}$ and in extensions by enlargement of $\mathcal{S}$ is formalized in the lemma below.

Lemma 2.14 *Let $\mathcal{S}$ be an AHS with set Fo of formulas. Then it holds for all $A \in Fo$ and $\Delta, \Sigma \in \mathcal{P}(Fo)$:*

$$\Delta \vdash_{\mathcal{S}+\Sigma} A \quad \Longleftrightarrow \quad \Delta \cup \Sigma \vdash_{\mathcal{S}} A \ .$$

Extensions by enlargement of n-AHS's that arise by adding new axioms from a set Σ or by adding new rules from a set $\mathcal{R}^*$ cannot be denoted uniquely in the same convenient way as $\mathcal{S}+\Sigma$ and $\mathcal{S}+\mathcal{R}^*$ were defined for AHS's Σ because in extensions by enlargement of n-AHS's also unique names have to be assigned to the new axioms and rules. We choose here not to introduce a specific notation for such extensions of n-AHS's, but define a notation for n-AHS's that arise from other n-AHS's by removing a set of rules instead.

Definition 2.15 (Removing rules from n-AHS's and AHS's).

(i) Let $\mathcal{S} = \langle Fo, Ax, \mathcal{R}, Na, \mathsf{name} \rangle$ be an n-AHS and let $\mathcal{R}^*$ be a set of rules on Fo. Then we denote by $\mathcal{S}-\mathcal{R}^*$ the n-AHS $\langle Fo, Ax, \mathcal{R} \setminus \mathcal{R}^*, Na, \mathsf{name}|_{Ax \cup (\mathcal{R} \setminus \mathcal{R}^*)} \rangle$. If $\mathcal{R}^* = \{R\}$ for some rule R on Fo, then we allow to write $\mathcal{S}-R$ for $\mathcal{S}-\mathcal{R}^*$.

(ii) Similarly, we define, for all AHS's $\mathcal{S} = \langle Fo, Ax, \mathcal{R} \rangle$ and all sets $\mathcal{R}^*$ of rules on Fo, the AHS $\mathcal{S}-\mathcal{R}^*$ as $\langle Fo, Ax, \mathcal{R} \setminus \mathcal{R}^* \rangle$; also in this case, we allow to write again $\mathcal{S}-R$ for $\mathcal{S}-\mathcal{R}^*$ if $\mathcal{R}^* = \{R\}$ for some rule R on Fo.

It is clearly the case for all n-AHS's or AHS's $\mathcal{S}$ and all sets $\mathcal{R}^*$ of rules on $Fo_{\mathcal{S}}$ that $\mathcal{S}$ is an extension by enlargement of $\mathcal{S}-\mathcal{R}^*$.

3 Rule derivability and admissibility in AHS's

In this section we will formally define the notions of rule derivability and admissibility in abstract Hilbert systems, propose two variants of rule derivability, and gather a number of immediate consequences of these definitions. We will also give two results about the interdependence between rule derivability and admissibility that we did not encounter in the literature (Proposition 3.4 and Theorem 3.5 below).

Our focus is on AHS's instead of on n-AHS's here, since names for rules and axioms do not play a relevant part in the definitions of rule derivability and admissibility. But for later use, we will give these definitions also with respect to n-AHS's, and moreover, we will state most results for n-AHS's, too. In some statements below, the notation specific to AHS's for adding new axioms or rules to abstract Hilbert systems will be used, and therefore these assertions will only apply to AHS's. However, such results can always be generalized to n-AHS's by reformulating them into formally somewhat more complicated statements in which the respective extensions of n-AHS's by new axioms or rule are described precisely. We will not do so here in the desire to keep necessary technicalities to a miminum.

The informal stipulation for the notion of rule admissibility given the Introduction, according to which a rule R is called admissible with respect to a formal system $\mathcal{S}$ iff the theory of $\mathcal{S}$ is closed under applications of R, has a precise meaning in the framework of abstract Hilbert systems. It therefore leads to a single formal notion of rule admissibility in AHS's, which will be defined in Definition 3.1 below.

The situation is somewhat different, however, for rule derivability: As it was laid out in the Introduction, a rule R is usually called derivable with respect to a formal system $\mathcal{S}$ iff every application of R can be 'modeled' or 'mimicked' by a derivation in $\mathcal{S}$. Therefore we need to formalize the precise circumstances under which a derivation does actually 'mimic' a rule application in an AHS before the definition of rule derivability in AHS's can be given. We shall now propose three clarifications for the term "mimicking derivation" for an application in an AHS, one for each of the three kinds of consequence relations defined in Definition 2.8. This will lead to three different notions of rule derivability in AHS's.

Let $\mathcal{S}$ be an AHS, $\mathcal{D}$ a derivation in $\mathcal{S}$, R a rule on $Fo_{\mathcal{S}}$, and α an application of R. We say that the derivation $\mathcal{D}$ *mimics* α *with respect to* $\vdash_{\boldsymbol{\mathcal{S}}}$, or that $\mathcal{D}$ is a *mimicking derivation for* α *in* $\mathcal{S}$ *with respect to* $\vdash_{\boldsymbol{\mathcal{S}}}$ if and only if

$$\mathrm{set}(\mathsf{assm}(\mathcal{D})) \subseteq \mathrm{set}(\mathsf{prem}(\alpha)) \quad \& \quad \mathsf{concl}(\mathcal{D}) = \mathsf{concl}(\alpha) \tag{3.1}$$

holds, i.e. iff the set of assumptions of $\mathcal{D}$ is contained in the set of assumptions of α, and if $\mathcal{D}$ has the same conclusion as α. And similarly, by demanding equal *sets* or equal *mulitsets* of assumptions as well as same conclusions, we say that $\mathcal{D}$ *mimics* α *with respect to* $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$, and that $\mathcal{D}$ *mimics* α *with respect to* $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}$ if and only if the assertion (3.2), and respectively, if (3.3) holds:

$$\mathrm{set}(\mathsf{assm}(\mathcal{D})) = \mathrm{set}(\mathsf{prem}(\alpha)) \quad \& \quad \mathsf{concl}(\mathcal{D}) = \mathsf{concl}(\alpha) \tag{3.2}$$

$$\mathsf{assm}(\mathcal{D}) = \mathrm{mset}(\mathsf{prem}(\alpha)) \quad \& \quad \mathsf{concl}(\mathcal{D}) = \mathsf{concl}(\alpha) \tag{3.3}$$

In this way three formal notions of rule derivability arise for AHS's: In Definition 3.1 below, we will agree, for each consequence relation $\vdash^{(\cdot)}_{\boldsymbol{\mathcal{S}}} \in \left\{ \vdash_{\boldsymbol{\mathcal{S}}}, \vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}, \vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}} \right\}$, on a formal clause that amounts to the stipulation

$$\begin{aligned} R \text{ is } \vdash^{(\cdot)}_{\boldsymbol{\mathcal{S}}}\text{-derivable in } \mathcal{S} \iff & (\forall \alpha \in Apps_R)(\exists \mathcal{D} \in Der(\mathcal{S})) \\ & \left[\mathcal{D} \text{ mimics } \alpha \text{ with respect to } \vdash^{(\cdot)}_{\boldsymbol{\mathcal{S}}} \right] . \end{aligned} \tag{3.4}$$

However, the use of the concept "mimicking derivation" will actually be avoided. And for every AHS $\mathcal{S}$, we will speak of derivability, s-derivability and m-derivability in $\mathcal{S}$ rather than of $\vdash_{\boldsymbol{\mathcal{S}}}$-, $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$- and $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}$-derivability in $\mathcal{S}$.

Definition 3.1 (Rule derivability and admissibility in AHS's and n-AHS's). Let $\mathcal{S}$ be an AHS or an n-AHS and let $R = \langle Apps_R, \mathsf{prem}, \mathsf{concl}\rangle$ be a rule on $Fo_{\mathcal{S}}$.

(i) The rule R is *derivable in* $\mathcal{S}$ if and only if

$$(\forall \alpha \in Apps_R)\,\big[\,\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)\,\big] \tag{3.5}$$

holds. Similarly, we say that R is *s-derivable* or that R is *m-derivable* if and only if, respectively, the assertions (3.6) and (3.7) hold:

$$(\forall \alpha \in Apps_R)\,\big[\,\mathrm{set}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{s})}_{\mathcal{S}} \mathsf{concl}(\alpha)\,\big]\ , \tag{3.6}$$

$$(\forall \alpha \in Apps_R)\,\big[\,\mathrm{mset}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{m})}_{\mathcal{S}} \mathsf{concl}(\alpha)\,\big]\ . \tag{3.7}$$

(ii) The rule R is *admissible in* $\mathcal{S}$ if and only if it holds that

$$(\forall \alpha \in Apps_R)\,\big[\,(\forall A \in \mathrm{set}(\mathsf{prem}(\alpha)))\,[\,\vdash_{\mathcal{S}} A\,] \implies \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)\,\big]\ . \tag{3.8}$$

As a consequence of the fact noted in Section 2 that, for every n-AHS $\mathcal{S}$ and its underlying AHS $\mathcal{S}_0$, the consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$, $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}$ and $\vdash_{\boldsymbol{\mathcal{S}_0}}$, $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_0}}$, $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}_0}}$ coincide respectively, we find that also the notions of derivability, s-derivability, m-derivability and admissibility of rules are respectively the same for every n-AHS and its underlying AHS.

A number of easy consequences of Definition 3.1 are gathered in the following Proposition that is an adaptation to our framework of AHS's and slight reformulation[7] of Lemma 6.14 on p. 70 in the book [3] by Hindley and Seldin.

Proposition 3.2 *Let $\mathcal{S}$ be an AHS and let R be a rule on the set of formulas of $\mathcal{S}$. Then the following statements holds:*

(i) *R is admissible in $\mathcal{S}$ iff the AHS $\mathcal{S}+R$ does not possess more theorems than $\mathcal{S}$.*

(ii) *If R is derivable in $\mathcal{S}$, then R is also admissible in $\mathcal{S}$. The implication in the opposite direction does not hold in general.*

(iii) *If R is derivable in $\mathcal{S}$, then R is derivable in any extension $\mathcal{S}'$ of $\mathcal{S}$ that is obtained from $\mathcal{S}$ by adding new formulas, axioms and/or rules, that is, then R is derivable in every extension by enlargement of $\mathcal{S}$.*

Proposition 3.2 (i) can be viewed to provide further justification, in addition to the informal stipulation for rule admissibility explained above, for the use of the term "*admissible* rule": If a rule R is admissible in an AHS $\mathcal{S}$, then R is not only "admissible" in $\mathcal{S}$ in the sense that the theory of $\mathcal{S}$ is closed under applications of

[7] This concerns item (iii) of Proposition 3.2 and item (iii) of Lemma 6.14 on p. 70 in [3]: Hindley and Seldin do not consider extensions that arise by extending the set of formulas of a formal system. However, they consider extensions that result by introducing new axioms and/or new rules.

$\mathcal{S}$, but also in the sense that *allowing* R as an additional rule for an extension $\mathcal{S}+R$ of $\mathcal{S}$ does not lead to an AHS with more theorems than $\mathcal{S}$.

In the following proposition the relationship between the three introduced notions of rule derivability is explained.

Proposition 3.3 *Let $\mathcal{S}$ be an AHS or an n-AHS, and let R be a rule on the set of formulas of $\mathcal{S}$. If R is m-derivable in $\mathcal{S}$, then it is also s-derivable in $\mathcal{S}$. And if R is s-derivable in $\mathcal{S}$, then it is also derivable in $\mathcal{S}$. Neither of the implications in the opposite direction holds in general. Furthermore the assertions in (ii) and (iii) of Proposition 3.2 hold also if "derivable" is replaced by "s-derivable" and "m-derivable", respectively.*

The following proposition contains a characterization of the exact circumstances under which a rule is admissible, but not derivable in an AHS. And it furthermore asserts that the property of a rule R to be admissible in an AHS $\mathcal{S}$ is equivalent to the property of a certain 'restriction' R_0 of R to be derivable in $\mathcal{S}$.

Proposition 3.4 *Let $\mathcal{S}$ be an AHS or an n-AHS and let $R = \langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$ be a rule on $Fo_{\mathcal{S}}$.*

(i) Suppose that R is admissible in $\mathcal{S}$. Then it holds that:

$$\begin{aligned}
&R \textit{ is not derivable in } \mathcal{S} \iff \\
&\qquad \iff (\exists\, \alpha \in Apps)\,\Big[\,\big((\exists A \in set(\mathsf{prem}(\alpha)))[\,\nvdash_{\mathcal{S}} A\,]\big)\ \ \& \\
&\qquad\qquad\qquad \&\ \ set(\mathsf{prem}(\alpha)) \nvdash_{\mathcal{S}} \mathsf{concl}(\alpha)\,\Big]\,. \qquad (3.9)
\end{aligned}$$

(ii) Let R_0 be the rule that arises by restricting the applications of R to all those that exclusively have theorems of $\mathcal{S}$ as premises, i.e. let $R_0 = \langle Apps_0, \mathsf{prem}_0, \mathsf{concl}_0\rangle$, where

$$Apps_0 = \{\alpha \in Apps \mid (\forall A \in set(\mathsf{prem}(\alpha)))\,[\vdash_{\mathcal{S}} A\,]\} \qquad (3.10)$$

and where prem_0 and concl_0 are the respective restrictions of prem and concl to the set $Apps_0$. Then it holds that

$$R \textit{ is admissible in } \mathcal{S} \iff R_0 \textit{ is derivable in } \mathcal{S}\,. \qquad (3.11)$$

We conclude this section by stating a theorem that establishes a link between the assertions of items (ii) and (iii) in Proposition 3.2. It gives, for all AHS's $\mathcal{S}$, two closely related characterizations of rule derivability in $\mathcal{S}$ in terms of rule admissibility in certain extensions of $\mathcal{S}$.

Theorem 3.5 *Let $\mathcal{S}$ be an AHS with set Fo as its set of formulas, and let R be a rule on Fo. Then the following three statements are equivalent:*

(i) R is derivable in $\mathcal{S}$.

(ii) R is admissible in every AHS $\mathcal{S}+\Sigma$ with $\Sigma \in \mathcal{P}(Fo)$ arbitrary.

(iii) R is admissible in every extension by enlargement of $\mathcal{S}$.

Following [3], we extend the notions of derivability and admissibility of rules also to formulas.

Definition 3.6 (Derivability and admissibility of formulas in AHS's). Let $\mathcal{S}$ be an AHS on an n-AHS with formula set Fo. We call a formula $A \in Fo$ both *admissible* and *derivable in* $\mathcal{S}$ as well as *s-derivable* and *m-derivable in* $\mathcal{S}$, if and only if $\vdash_{\mathcal{S}} A$ holds, i.e. iff A is a theorem of $\mathcal{S}$.

The reason for stipulating the notions of admissibility, derivability, s-derivability and m-derivability to coincide on formulas consists in the following easy observation: Within derivations, axioms of an AHS have the same 'behaviour' as zero-premise rule applications. In fact, by formulating axiom schemes as zero-premise rules, every AHS $\mathcal{S}$ can be transformed into an AHS $\mathcal{S}'$ with an empty set of axioms such that the same relative derivability statements are true in $\mathcal{S}$ and in $\mathcal{S}'$; in this way axioms of an AHS $\mathcal{S}$ can be viewed as zero-premise rules in a closely related AHS $\mathcal{S}'$. On zero-premise rules, however, there is agreement between the notions of admissibility, derivability, s-derivability and m-derivability, as an obvious consequence of Definition 3.1.

4 Relations between abstract Hilbert systems

In this section we consider relations that compare AHS's with respect to earlier introduced concepts such as relative derivability statements and the notions of rule derivability and admissibility. For this we have drawn inspiration from a lemma in [3] (cf. Remark 4.14 below for details). First, we will introduce a total of twenty-four relations between AHS's, and then we will state results about their interrelations.

We do not consider n-AHS's in this section since names for axioms and rules do not play any role in the concepts developed below. It should be mentioned, however, that all results given here about interrelations between relations on AHS's hold also as statements about analogous interrelations between analogously defined relations on n-AHS's (except for the case of Corollary 4.15 which must be reformulated for n-AHS's, see a remark below).

The kind of relations between AHS's that we will consider are "inclusion relations" $\preceq_{P,Q}$, which are respectively based on properties P and Q of objects in AHS's such as formulas, theorems, rules and relative derivability statements. As properties we will hereby use, for example, "is theorem in the considered system", "is admissible rule on the formulas of the considered system" and similar ones. For given properties P and Q of this sort, the definition of the *inclusion relation* $\preceq_{P,Q}$ with respect to P and Q will always be of the form that for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$

$$\mathcal{S}_1 \sim_{P,Q} \mathcal{S}_2 \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \text{Every formula in } \mathcal{S}_1 \text{ is also a formula of } \mathcal{S}_2, \text{ and} \\ \text{every object } x \text{ in } \mathcal{S}_1 \text{ having the property } P \text{ does} \\ \text{also appear in } \mathcal{S}_2 \text{ as an object with the property } Q. \end{array} \right\}$$

is stipulated; a relation $\preceq_{P,Q}$ defined in this way will always be considered as a subclass of $\mathfrak{H} \times \mathfrak{H}$. And furthermore we will fix, for every inclusion relation $\preceq_{P,Q}$, a *mutual inclusion relation* $\sim_{P,Q} \subseteq \mathfrak{H} \times \mathfrak{H}$ with respect to P and Q that *is induced* *by* $\preceq_{P,Q}$: For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, we define

$$\mathcal{S}_1 \sim_{P,Q} \mathcal{S}_2 \quad \Longleftrightarrow \quad \mathcal{S}_1 \preceq_{P,Q} \mathcal{S}_2 \ \ \& \ \ \mathcal{S}_2 \preceq_{P,Q} \mathcal{S}_1 \ . \tag{4.1}$$

By an inclusion or mutual inclusion relation with respect to only a single property P we will mean the inclusion or mutual inclusion relation $\preceq_{P,P}$ or $\sim_{P,P}$, respectively.

The concrete inclusion and mutual inclusion relations defined below, will actually *not* be denoted in the form $\preceq_{P,Q}$ or $\sim_{P,Q}$ using the properties P and Q with respect to which they are defined, but they will be denoted by the symbols $\preceq$ and $\sim$ with attached sub- and superscripts that abbreviate the name of the respective relation. We start by defining inclusion and mutual inclusion relations with respect to the property "is theorem" and with respect to "is relative derivability statement"; the latter property consists of three variants, one for each of the three consequence relations defined in Definition 2.8.

Definition 4.1 (The relations $\preceq_{th}$ and $\preceq_{rth}$, $\preceq^{(\mathbf{s})}_{rth}$, $\preceq^{(\mathbf{m})}_{rth}$ between AHS's).

(i) We define the relation $\preceq_{th}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$:

$$\mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \ \Longleftrightarrow \ Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \ \& \ (\forall A \in Fo_{\mathcal{S}_1})[\,(\vdash_{\mathcal{S}_1} A) \ \Rightarrow \ (\vdash_{\mathcal{S}_2} A)\,] \ .$$

(ii) We define the relations $\preceq_{rth}$ and $\preceq^{(\mathrm{m})}_{rth}$ on $\mathfrak{H}$ by stipulating for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$

$$\begin{aligned}
\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2 \ \Longleftrightarrow \ & Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \ \& \\
& \& \ (\forall \Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1}))\,(\forall A \in Fo_{\mathcal{S}_1})\,\big[\,(\Sigma \vdash_{\mathcal{S}_1} A) \ \Rightarrow \ (\Sigma \vdash_{\mathcal{S}_2} A)\,\big] \ , \\
\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2 \ \Longleftrightarrow \ & Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \ \& \\
& \& \ (\forall \Gamma \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}))\,(\forall A \in Fo_{\mathcal{S}_1})\big[\,(\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A) \ \Rightarrow \ (\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_2} A)\,\big] \ .
\end{aligned}$$

The relation $\preceq^{(\mathrm{s})}_{rth}$ on $\mathfrak{H}$ is defined in an analogous way to $\preceq_{rth}$ involving relative derivability statements with respect to $\vdash^{(\mathrm{s})}_{\mathcal{S}_1}$ and $\vdash^{(\mathrm{s})}_{\mathcal{S}_2}$ instead of with respect to $\vdash_{\mathcal{S}_1}$ and $\vdash_{\mathcal{S}_2}$.

By adapting the notion of extension of a formal system (explained preceding Definition 2.11 above) to abstract Hilbert systems, we also define: For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}_2$ is an *extension* of $\mathcal{S}_1$ if and only if $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ holds.

Definition 4.2 (The relations $\sim_{th}$ and $\sim_{rth}$, $\sim^{(\mathbf{s})}_{rth}$, $\sim^{(\mathbf{m})}_{rth}$ between AHS's). The relations $\sim_{th}$, $\sim_{rth}$, $\sim^{(\mathrm{s})}_{rth}$ and $\sim^{(\mathrm{m})}_{rth}$ are the mutual inclusion relations induced by the inclusion relations $\preceq_{th}$, $\preceq_{rth}$, $\preceq^{(\mathrm{s})}_{rth}$ and $\preceq^{(\mathrm{m})}_{rth}$, respectively.

For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ such that $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ holds, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equivalent* or *theorem equivalent*. If, with $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, $\mathcal{S}_1 \sim_{rth} \mathcal{S}_2$ holds, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equivalent with respect to relative theoremhood*.

Here are two examples of alternative formulations for these definitions: For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, the assertion $\mathcal{S}_1 \preceq^{(\mathrm{s})}_{rth} \mathcal{S}_2$ expresses that all formulas of $\mathcal{S}_1$ are also formulas of $\mathcal{S}_2$, and that all relative derivability statements in $\mathcal{S}_1$ with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_1}}$

are also relative derivability statements in $\mathcal{S}_2$ with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_2}}$; and $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ means that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same formulas and the same theorems. Some basic properties of the relations defined above are stated in the following proposition.

Proposition 4.3 *The relations $\preceq_{th}$, $\preceq_{rth}$, $\preceq^{(s)}_{rth}$ and $\preceq^{(m)}_{rth}$ are pre-order relations, i.e. they are reflexive and transitive, and the relations $\sim_{th}$, $\sim_{rth}$, $\sim^{(s)}_{rth}$ and $\sim^{(m)}_{rth}$ are equivalence relations.*

Using the equivalence relation $\sim_{th}$, the assertion (i) of Proposition 3.2 can be reformulated as: For all AHS's $\mathcal{S}$ and all rules R on $Fo_{\mathcal{S}}$, it holds that

$$R \text{ is admissible in } \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{S}{+}R \sim_{th} \mathcal{S} \ . \tag{4.2}$$

Since clearly $\mathcal{S} \preceq_{th} \mathcal{S}{+}R$ holds for all AHS's $\mathcal{S}$ and rules R on $Fo_{\mathcal{S}}$, the right side of the logical equivalence in (4.2) can be weakened to $\mathcal{S}{+}R \preceq_{th} \mathcal{S}$ such that the resulting assertion is equivalent with (4.2). We will see in Corollary 4.15 below that, similar to the characterization of rule admissibility in (4.2), there exist characterizations for two of the three notions of rule derivability in terms of mutual inclusion relations from Definition 4.2.

In close connection with Theorem 3.5, the following proposition formulates characterizations of the relations $\preceq_{rth}$ and $\sim_{rth}$ in terms of the relations $\preceq_{th}$ and $\sim_{th}$, respectively.

Proposition 4.4 *For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ the following two logical equivalences hold:*

$$\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2 \iff (\forall\, \Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1}))\,[\, \mathcal{S}_1{+}\Sigma \preceq_{th} \mathcal{S}_2{+}\Sigma \,] \ , \tag{4.3}$$

$$\mathcal{S}_1 \sim_{rth} \mathcal{S}_2 \iff (\forall\, \Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1}))\,[\, \mathcal{S}_1{+}\Sigma \sim_{th} \mathcal{S}_2{+}\Sigma \,] \ . \tag{4.4}$$

We continue by defining inclusion and mutual inclusion relations between AHS's with respect to the property "is admissible rule" and, respectively, with respect to the properties "is derivable rule", "is s-derivable rule" and "is m-derivable rule".

Definition 4.5 (The relations $\preceq_{adm}$ and $\preceq_{der}$, $\preceq^{(\mathrm{s})}_{der}$, $\preceq^{(\mathrm{m})}_{der}$ between AHS's).

(i) We define the relation $\preceq_{adm}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$:

$$\begin{aligned} \mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \iff & Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \ \& \\ & \& \ (\forall A \in Fo_{\mathcal{S}_1})\big[\, A \text{ is adm. in } \mathcal{S}_1 \ \Rightarrow \ A \text{ is adm. in } \mathcal{S}_2 \big] \ \& \\ & \& \ (\forall R,\, R \text{ is rule on } Fo_{\mathcal{S}_1}) \\ & \qquad \big[R \text{ is admissible in } \mathcal{S}_1 \ \Rightarrow \ R \text{ is admissible in } \mathcal{S}_2 \big] \ . \end{aligned}$$

(ii) We define the relations $\preceq_{der}$, $\preceq^{(\mathrm{s})}_{der}$ and $\preceq^{(\mathrm{m})}_{der}$ on $\mathfrak{H}$ by analogous stipulations as for $\preceq_{adm}$ that rely respectively on the notions of derivability, s-derivability and m-derivability of rules instead of on the notion of rule admissibility: For example, we stipulate that $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2$ holds if and only if the following three conditions are met: The formulas of $\mathcal{S}_1$ are contained among the formulas of

$\mathcal{S}_2$, every formula of $\mathcal{S}_1$ that is m-derivable in $\mathcal{S}_1$ is also m-derivable in $\mathcal{S}_2$, and every rule on $Fo_{\mathcal{S}_1}$ that is m-derivable in $\mathcal{S}_1$ is also m-derivable in $\mathcal{S}_2$.

Definition 4.6 (The relations $\sim_{adm}$ and $\sim_{der}$, $\sim^{(\mathbf{s})}_{der}$, $\sim^{(\mathbf{m})}_{der}$ between AHS's). The relations $\sim_{adm}$, $\sim_{der}$, $\sim^{(\mathrm{s})}_{der}$ and $\sim^{(\mathrm{m})}_{der}$ are the mutual inclusion relations induced by the inclusion relations $\preceq_{adm}$, $\preceq_{der}$, $\preceq^{(\mathrm{s})}_{der}$ and $\preceq^{(\mathrm{m})}_{der}$, respectively.

For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ such that $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$ ($\mathcal{S}_1 \sim_{der} \mathcal{S}_2$, $\mathcal{S}_1 \sim^{(\mathrm{s})}_{der} \mathcal{S}_2$, $\mathcal{S}_1 \sim^{(\mathrm{m})}_{der} \mathcal{S}_2$) holds, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *equivalent with respect to rule admissibility* (*with respect to rule derivability*, *w.r.t. rule s-derivability*, *w.r.t. rule m-derivability*).

As an example for an alternative verbal formalization of a stipulation in Definition 4.6, we give the following: For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ the assertion $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$ expresses that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same formulas and the same admissible formulas and rules. The following proposition is as easy a consequence of the last two definitions as Proposition 4.3 was one of Definition 4.1 and Definition 4.2.

Proposition 4.7 *The relations $\preceq_{adm}$, $\preceq_{der}$, $\preceq^{(s)}_{der}$ and $\preceq^{(m)}_{der}$ are pre-order relations, i.e. they are reflexive and transitive, and the relations $\sim_{adm}$, $\sim_{der}$, $\sim^{(s)}_{der}$ and $\sim^{(m)}_{der}$ are equivalence relations.*

As formulated in the following lemma, there exist respective immediate connections between, on the one hand, the relations $\preceq_{adm}$ and $\sim_{adm}$, and on the other hand, the notions "conservative extension" and "extension" of AHS's.

Lemma 4.8 (i) *For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ it holds*

$$\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \iff Th(\mathcal{S}_2) \supseteq Fo_{\mathcal{S}_1} \;\vee\; \mathcal{S}_2 \textit{ is conservative extension of } \mathcal{S}_1 \,. \quad (4.5)$$

(ii) *For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ with the same set of formulas, i.e. with $Fo_{\mathcal{S}_1} = Fo_{\mathcal{S}_2}$, it holds*

$$\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{th} \mathcal{S}_2 \;\vee\; \mathcal{S}_2 \textit{ is inconsistent}\,.$$

(iii) *For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ it holds*

$$\mathcal{S}_1 \sim_{adm} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{th} \mathcal{S}_2 \,. \quad (4.6)$$

Finally, we will define inclusion and mutual inclusion relations with respect to the pair (P, Q) of properties, where P stands for "is rule *of* the considered system" and Q denotes "is admissible rule *in* the considered system", and with respect to three analogous pairs of properties that arise by replacing "admissible" in Q by "derivable", "s-derivable" or "m-derivable", respectively.

Definition 4.9 (The relations $\preceq_{r/adm}$ and $\preceq_{r/der}$, $\preceq^{(\mathbf{s})}_{r/der}$, $\preceq^{(\mathbf{m})}_{r/der}$ on AHS's).

(i) We define the relation $\preceq_{r/adm}$ on the class $\mathfrak{H}$ by stipulating for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$:

$$\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \iff Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} \;\&\; (\forall A \in Ax_{\mathcal{S}_1})\big[\, A \text{ is admissible in } \mathcal{S}_2 \,\big] \;\&\; \;\&\; (\forall R \in \mathcal{R}_{\mathcal{S}_1})\big[\, R \text{ is admissible in } \mathcal{S}_2 \,\big]\,.$$

(ii) We define the relations $\preceq_{r/der}$, $\preceq^{(\mathrm{s})}_{r/der}$ and $\preceq^{(\mathrm{m})}_{r/der}$ on $\mathfrak{H}$ by analogous stipulations as for $\preceq_{r/adm}$ that rely respectively on the notions of derivability, s-derivability and m-derivability of rules instead of on the notion of admissibility: For example, we let $\mathcal{S}_1 \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}_2$ be true if and only if it holds that: All formulas of $\mathcal{S}_1$ are also formulas of $\mathcal{S}_2$, and all axioms and rules of $\mathcal{S}_1$ are s-derivable in $\mathcal{S}_2$.

Definition 4.10 (The relations $\sim_{r/adm}$ and $\sim_{r/der}$, $\sim^{(\mathbf{s})}_{r/der}$, $\sim^{(\mathbf{m})}_{r/der}$ on AHS's). The relations $\sim_{r/adm}$, $\sim_{r/der}$, $\sim^{(\mathrm{s})}_{r/der}$ and $\sim^{(\mathrm{m})}_{r/der}$ are the mutual inclusion relations induced by the inclusion relations $\preceq_{r/adm}$, $\preceq_{r/der}$, $\preceq^{(\mathrm{s})}_{r/der}$ and $\preceq^{(\mathrm{m})}_{r/der}$, respectively.

For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1 \sim_{r/adm} \mathcal{S}_2$ ($\mathcal{S}_1 \sim_{r/der} \mathcal{S}_2$, $\mathcal{S}_1 \sim^{(\mathrm{s})}_{r/der} \mathcal{S}_2$, $\mathcal{S}_1 \sim^{(\mathrm{m})}_{r/der} \mathcal{S}_2$) holds, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *rule equivalent with respect to rule admissibility* (*rule equivalent with respect to rule derivability*, *rule equivalent w.r.t. rule s-derivability*, *rule equivalent w.r.t. rule m-derivability*).

The mutual inclusion relations $\sim_{r/der}$ and $\sim_{r/adm}$ correspond to the two notions "rule-equivalence" and "theorem-equivalence" as defined by Hindley and Seldin in [3, p. 71]. Since in Definition 4.2 we have already defined a notion of theorem equivalence as a relation that is not designated symbolically in [3] and that is intended to reflect the meaning of this term more directly, we have chosen to use longer explicit names for the mutual inclusion relations in Definition 4.10.

For the 8 relations defined in the two definitions above, transitivity is no longer obvious. In fact, it does not hold for $\preceq_{r/adm}$ (see Example C.1 in Appendix C for a counterexample), nor for $\preceq^{(\mathrm{s})}_{r/der}$ and $\sim^{(\mathrm{s})}_{r/der}$ (see Example C.2 for a counterexample applicable in these two cases). However, we clearly find the following statement.

Proposition 4.11 *The relations $\preceq_{r/adm}$, $\preceq_{r/der}$, $\preceq^{(s)}_{r/der}$ and $\preceq^{(m)}_{r/der}$ are reflexive, and the relations $\sim_{r/adm}$, $\sim_{r/der}$, $\sim^{(s)}_{r/der}$ and $\sim^{(m)}_{r/der}$ are reflexive and symmetric.*

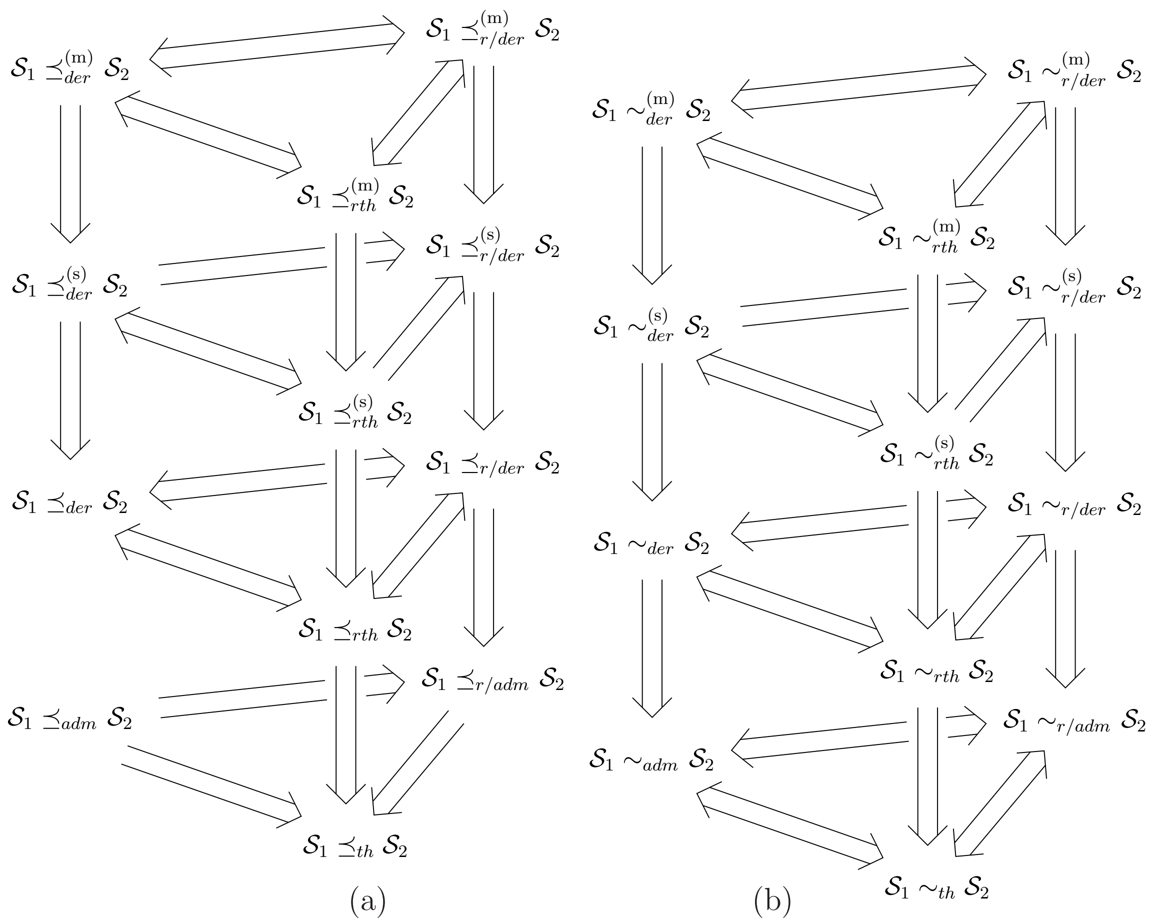
Our main theorem, which we will give now, settles the question of how the twelve introduced inclusion relations between abstract Hilbert systems are interrelated.

Theorem 4.12 (Interrelations between introduced inclusion relations). *For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, the implications and logical equivalences shown in Figure 3 (a) hold between assertions of the form $\mathcal{S}_1 \preceq \mathcal{S}_2$, where $\preceq$ is an inclusion relation defined in Definitions 4.1, Definition 4.5 or Definition 4.9. Implication arrows appearing in Figure 3 (a) that are not inverted indicate that the respective implication in the opposite direction does not hold in general.*

Quite obviously, Theorem 4.12 also describes the precise relationships towards each other of the twelve inclusion relations introduced above. For example, Theorem 4.12 implies directly that the relations $\preceq_{rth}$, $\preceq_{der}$ and $\preceq_{r/der}$ coincide, and that only two of the three relations $\preceq^{(\mathrm{s})}_{rth}$, $\preceq^{(\mathrm{s})}_{der}$ and $\preceq^{(\mathrm{s})}_{r/der}$ coincide, namely $\preceq^{(\mathrm{s})}_{rth}$ and $\preceq^{(\mathrm{s})}_{der}$, whereas both are properly contained in $\preceq^{(\mathrm{s})}_{r/der}$.

The following theorem states that the interrelations asserted by Theorem 4.12

Figure 3 Two 'interrelation prisms': (a) Interrelations stated, for arbitrary AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, by Theorem 4.12 between assertions involving the twelve defined inclusion relations, and (b) interrelations stated, for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, by Theorem 4.13 between assertions involving the twelve defined *mutual* inclusion relations.



between the twelve introduced inclusion relations carry over to analogous interrelations between the twelve respectively induced mutual inclusion relations.

Theorem 4.13 (Interrelations between introduced mutual incl. relations). *For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, the implications and logical equivalences shown in Figure 3 (b) hold between assertions of the form $\mathcal{S}_1 \sim \mathcal{S}_2$, where $\sim$ is a mutual inclusion relation defined in Definitions 4.2, Definition 4.6 or Definition 4.10. Again in this situation, implication arrows appearing in Figure 3 (b) that are not inverted indicate that the respective implication in the opposite direction does not hold in general.*

Theorem 4.13 is mainly a corollary to Theorem 4.12: Except for four additional implication arrows in Figure 3 (b) that involve the relations $\sim_{adm}$, $\sim_{r/adm}$, $\sim_{th}$, and except for the clause asserting that not inverted implications in Figure 3 (b) do not hold in general, Theorem 4.13 follows in a direct way from Theorem 4.12: This is because every implication arrow in Figure 3 (a) 'induces' a corresponding implication arrow in Figure 3 (b) due to (4.1), the way in which a mutual inclusion relation is linked via its definition to the inclusion relation by which it is induced.

Remark 4.14 Theorem 4.13 was actually inspired by and is a substantial generalization of Lemma 6.16 in [3, p. 71], which if adapted to our framework and terminology asserts that for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\mathcal{S}_1 \sim_{r/adm} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{th} \mathcal{S}_2 \tag{4.7}$$

holds; this is also stated by Theorem 4.13. We extended the equivalence (4.7) to a further logical equivalence with the relation $\sim_{adm}$, which does not appear in [3]. Furthermore the arising triangle of implications has been carried over to analogous triangles of implications involving respectively the relations $\sim_{r/der}$, $\sim_{der}$ and $\sim_{rth}$ as well as the relations $\sim^{(\mathrm{m})}_{r/der}$, $\sim^{(\mathrm{m})}_{der}$ and $\sim^{(\mathrm{m})}_{rth}$; and we found a weaker relationship between the relations $\sim^{(\mathrm{s})}_{r/der}$, $\sim^{(\mathrm{s})}_{der}$ and $\sim^{(\mathrm{s})}_{rth}$ (of these additional relations only $\sim_{r/der}$ appears also in [3], and that is under the name of "rule-equivalence").

We conclude this section with an easy consequence of Theorem 4.13: For the notions of derivability and m-derivability of rules, similar characterizations as the one stated in (4.2) for rule admissibility can be given. For s-derivability, however, only a weaker statement holds.

Corollary 4.15 *Let $\mathcal{S}$ be an AHS and let R be a rule on the set of formulas of $\mathcal{S}$. Then the following implications and equivalences hold:*

$$R \textit{ is admissible in } \mathcal{S} \iff \mathcal{S}{+}R \sim_{th} \mathcal{S}\ , \tag{4.8}$$

$$R \textit{ is derivable in } \mathcal{S} \iff \mathcal{S}{+}R \sim_{rth} \mathcal{S}\ , \tag{4.9}$$

$$R \textit{ is s-derivable in } \mathcal{S} \Longleftarrow \mathcal{S}{+}R \sim^{(s)}_{rth} \mathcal{S}\ , \tag{4.10}$$

$$R \textit{ is m-derivable in } \mathcal{S} \iff \mathcal{S}{+}R \sim^{(m)}_{rth} \mathcal{S}\ . \tag{4.11}$$

For a counterexample to the implication "$\Rightarrow$" in (4.10) see Example C.10 in Appendix C. As we have remarked at the start of this section, Corollary 4.15 is the only statement in this section whose statement cannot immediately be transferred to n-AHS's. This is due to the fact that we have not defined n-AHS's of the form $\mathcal{S}{+}\mathcal{R}$ for an n-AHS $\mathcal{S}$ and a set of rules $\mathcal{R}$ on $Fo_{\mathcal{S}}$. However, Corollary 4.15 can quite obviously be reformulated in the following way:

Corollary 4.16 *Let $\mathcal{S}$ be an AHS and let R be a rule of $\mathcal{S}$. Then the following implications and equivalences hold:*

$$R \textit{ is admissible in } \mathcal{S}{-}R \iff \mathcal{S} \sim_{th} \mathcal{S}{-}R\ , \tag{4.12}$$

$$R \textit{ is derivable in } \mathcal{S}{-}R \iff \mathcal{S} \sim_{rth} \mathcal{S}{-}R\ , \tag{4.13}$$

$$R \textit{ is s-derivable in } \mathcal{S}{-}R \Longleftarrow \mathcal{S} \sim^{(s)}_{rth} \mathcal{S}{-}R\ , \tag{4.14}$$

$$R \textit{ is m-derivable in } \mathcal{S}{-}R \iff \mathcal{S} \sim^{(m)}_{rth} \mathcal{S}{-}R\ . \tag{4.15}$$

And we want to close this section by stating that the statement of Corollary 4.16 is also true for n-AHS's $\mathcal{S}$ and relations $\sim_{th}$, $\sim_{rth}$, $\sim_{rth}^{(s)}$ and $\sim_{rth}^{(m)}$ between n-AHS's that are defined analogously as the respective, here introduced relations between AHS's.

5 How can derived and admissible rules be eliminated from derivations in abstract Hilbert systems?

In this section we investigate the question what consequences the fact that a rule R is admissible, derivable, s-derivable or m-derivable in an AHS $\mathcal{S}$ has for the possibility of eliminating applications of R from derivations in $\mathcal{S}{+}R$; also, we will consider an analogous question for n-AHS's. First we will introduce, in Subsection 5.1, four abstract notions of rule elimination: Roughly, we will stipulate that a rule R can be eliminated from a derivation $\mathcal{D}$ in an AHS or n-AHS $\mathcal{S}$, if $\mathcal{D}$ 'can be replaced' by a derivation $\mathcal{D}'$ in $\mathcal{S}$ without applications of R. For this, $\mathcal{D}'$ must of course demonstrate all relative derivability statements in $\mathcal{S}$ that are demonstrated by $\mathcal{D}$. Recall that for every derivation $\mathcal{D}$ in an AHS or n-AHS $\mathcal{S}$ at least one relative derivability statement arises, which relates the multiset of assumptions of $\mathcal{D}$, the set of assumptions of $\mathcal{D}$, or a superset of the set of assumptions of $\mathcal{D}$ with the conclusion of $\mathcal{D}$. The multiset of assumptions of a derivation can be thought of as the 'input' of the derivation; the conclusion as the 'output' of the derivation. In analogy with the three concepts of "mimicking derivation" *for rule applications* defined in Section 3, we will propose three formalizations of the concept "mimicking derivation" *for derivations* in AHS's or n-AHS's; we will hereby stipulate when two derivations have the same or a 'similar' 'input/output-behaviour'. Based on these definitions, we will formulate different abstract notions of rule elimination and show a number of characterizations for them in terms of rule derivability and admissibility. This part of the present section will apply to both AHS's and n-AHS's.

And secondly, in Subsection 5.2, we will turn to a study of effective rule elimination. This will only be carried out in n-AHS's because the involved notions of reduction lean themselves much better to a formalization in abstract Hilbert systems with names. Among similar questions, we will be concerned with what consequences the property of a rule R to be derivable in an n-AHS $\mathcal{S}$ has for the possibility to eliminate applications of R from derivations in $\mathcal{S}$ *effectively*. For this, we will introduce abstract rewrite systems of "rule elimination by mimicking steps" using the fact that considered applications of a rule R that is derivable in an n-AHS $\mathcal{S}$ can always be eliminated from derivations in $Der(\mathcal{S})$ by replacing them through mimicking derivations. Our main finding in this respect will be that, for derivable rules, abstract rewrite systems of "rule elimination by mimicking steps" are strongly normalizing. And we will give similar results with respect to the variant notions of rule derivability, s-derivability and m-derivability.

5.1 Rule elimination in derivations of AHS's and n-AHS's

For the purpose of defining abstract notions of rule elimination, we need to fix three binary relations between derivations that formalize notions of 'similarity' between the assumptions and conclusions of derivations. These three relations on derivations will be reminiscent of the notions of "mimicking derivations" for rule applications defined at the beginning of in Section 3, and if viewed suitably, they can be seen to be proper extensions of respective relations defined there.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's or n-AHS's, more precisely, let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H} \cup \mathfrak{Hn}$, and let $\mathcal{D}_1 \in \mathit{Der}(\mathcal{S}_1)$ and $\mathcal{D}_2 \in \mathit{Der}(\mathcal{S}_2)$ be derivations. We say that $\mathcal{D}_1$ *mimics* $\mathcal{D}_2$ (symbolically denoted by $\mathcal{D}_1 \precsim \mathcal{D}_2$), or that $\mathcal{D}_2$ *is mimicked by* $\mathcal{D}_1$, if and only if

$$\mathrm{set}(\mathsf{assm}(\mathcal{D}_1)) \subseteq \mathrm{set}(\mathsf{assm}(\mathcal{D}_2)) \quad \& \quad \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \tag{5.1}$$

holds, i.e. iff $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same conclusion and the set of assumptions of $\mathcal{D}_1$ is contained among the set of assumptions of $\mathcal{D}_2$.

Furthermore, we say that $\mathcal{D}_1$ *s-mimics* $\mathcal{D}_2$ (abbreviated by $\mathcal{D}_1 \simeq^{(\mathrm{s})} \mathcal{D}_2$), or that $\mathcal{D}_2$ *is s-mimicked by* $\mathcal{D}_1$, if and only if it

$$\mathrm{set}(\mathsf{assm}(\mathcal{D}_1)) = \mathrm{set}(\mathsf{assm}(\mathcal{D}_2)) \quad \& \quad \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \tag{5.2}$$

holds, i.e. iff $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same conclusion and the same set of assumptions. And similarly, we stipulate that $\mathcal{D}_1$ *m-mimics* $\mathcal{D}_2$ (denoted by $\mathcal{D}_1 \simeq^{(\mathrm{m})} \mathcal{D}_2$), or that $\mathcal{D}_2$ *is m-mimicked by* $\mathcal{D}_1$, if and only if the assertion

$$\mathsf{assm}(\mathcal{D}_1) = \mathsf{assm}(\mathcal{D}_2) \quad \& \quad \mathsf{concl}(\mathcal{D}_1) = \mathsf{concl}(\mathcal{D}_2) \tag{5.3}$$

holds, i.e. thus if $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same conclusion and the same multiset of assumptions. Clearly, $\precsim$, $\simeq^{(\mathrm{s})}$ and $\simeq^{(\mathrm{m})}$ define binary relations on the class

$$\mathfrak{Der} = \bigcup \{\, \mathit{Der}(\mathcal{S}) \mid \mathcal{S} \in \mathfrak{H} \cup \mathfrak{Hn} \} \tag{5.4}$$

of derivations in an AHS or n-AHS, i.e. these relations are classes of pairs that are formed from derivations in an AHS or n-AHS. They have the following easy verifiable properties.

Proposition 5.1 (i) *The relation $\precsim$ on the class $\mathfrak{Der}$ is reflexive and transitive. And the relations $\simeq^{(s)}$ and $\simeq^{(m)}$ on $\mathfrak{Der}$ are equivalence relations.*

(ii) *It holds that $\simeq^{(m)} \subseteq \simeq^{(s)} \subseteq \precsim$.*

As mentioned above, the here introduced notions of mimicking, s-mimicking and m-mimicking derivations can be viewed as generalizations *for derivations* of the notions of mimicking derivation with respect to $\vdash_{\mathcal{S}}$, with respect to $\vdash^{(\mathbf{s})}_{\mathcal{S}}$ and with respect to $\vdash^{(\mathbf{m})}_{\mathcal{S}}$ *for applications of rules* (in some AHS or n-AHS $\mathcal{S}$), which have been defined in Section 3. To make this precise, we recall that every application α of a rule R in an AHS $\mathcal{S}$ corresponds to a derivation $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ consisting of the (one-step) inference between the sequence of premises and the conclusion of α; such derivations $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ have been defined in Definition 2.2 and in Definition 2.3. Using this correspondence, we find that for one-step inference derivations the notions of mimicking derivation coincide with respective earlier definitions. More precisely, for all AHS's $\mathcal{S}$, rules R on $Fo_{\mathcal{S}}$, applications $\alpha \in Apps_R$ and derivations $\mathcal{D} \in Der(\mathcal{S})$, it is obvious to see that the following three equivalences hold:

$$\mathcal{D} \text{ mimics } \alpha \text{ w.r.t. } \vdash_{\boldsymbol{\mathcal{S}}} \iff \mathcal{D} \text{ mimics the derivation } \mathcal{D}_{(\alpha,R,\mathcal{S})} \text{ in } \mathcal{S}\,, \quad (5.5)$$

$$\mathcal{D} \text{ mimics } \alpha \text{ w.r.t. } \vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}} \iff \mathcal{D} \text{ s-mimics the derivation } \mathcal{D}_{(\alpha,R,\mathcal{S})} \text{ in } \mathcal{S}\,, \quad (5.6)$$

$$\mathcal{D} \text{ mimics } \alpha \text{ w.r.t. } \vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}} \iff \mathcal{D} \text{ m-mimics the derivation } \mathcal{D}_{(\alpha,R,\mathcal{S})} \text{ in } \mathcal{S}\,. \quad (5.7)$$

In the case of n-AHS's, (5.5), (5.6) and (5.7) hold for all n-AHS's $\mathcal{S}$, derivations $\mathcal{D} \in Der(\mathcal{S})$ and applications $\alpha \in Apps_R$ of rules R in $\mathcal{S}$. In this way we indeed recognize that the earlier defined notions of mimicking derivation for rule applications have now been extended to respective notions of mimicking derivation for more general derivations than just one-step inferences.

It is an immediate consequence of the equivalences (5.5), (5.6) and (5.7) that the definitions of rule derivability and admissibility can be restated exclusively in terms of the notions for mimicking derivation introduced here. This is because the alternative definitions in (3.4) of the three notions of rule derivability can now be reformulated appropriately. For instance in the case of rule derivability, it holds for all AHS's and n-AHS's $\mathcal{S}$ and for all rules R of $\mathcal{S}$ that

$$R \text{ is derivable in } \mathcal{S}{-}R \iff$$
$$\iff (\forall \alpha \in Apps_R)\,(\exists\, \mathcal{D}_\alpha \in Der(\mathcal{S}{-}R))\,\big[\,\mathcal{D}_\alpha \precsim \mathcal{D}_{(\alpha,R,\mathcal{S})}\,\big]\,. \quad (5.8)$$

Clearly, analogous reformulations of the definitions of rule s-derivability and m-derivability can be given by replacing $\precsim$ on the right-hand side of (5.8) by $\simeq^{(\mathrm{s})}$ or $\simeq^{(\mathrm{m})}$. And in a similar way, an alternative formulation for the definition of rule admissibility can take the form

$$R \text{ is admissible in } \mathcal{S}{-}R \iff$$
$$\iff (\forall \alpha \in Apps_R)\,\big[\,(\forall A \in \mathrm{set}(\mathsf{prem}(\alpha)))[\vdash_{\mathcal{S}} A\,] \;\Rightarrow$$
$$\Rightarrow\; (\exists\, \mathcal{D} \in Der(\mathcal{S}{-}R))\,[\,\mathcal{D}_\alpha \precsim \mathcal{D}_{(\alpha,R,\mathcal{S})}\,]\,\big]$$

for all AHS's and n-AHS's $\mathcal{S}$ and for all rules R of $\mathcal{S}$ (this logical equivalence remains valid if $\precsim$ is replaced by $\simeq^{(\mathrm{s})}$ or by $\simeq^{(\mathrm{m})}$).

Based on the three above defined notions of "mimicking derivation" for derivations, we can now stipulate three different meanings for the expression "rule elimination holds" *for derivations* in abstract Hilbert systems: For all AHS's or n-AHS's $\mathcal{S}$ and for every rule R of $\mathcal{S}$, we say that *the applications of R can*, respectively, *be eliminated*, *s-eliminated* or *m-eliminated* from a derivation $\mathcal{D} \in Der(\mathcal{S})$ if and only if a derivation $\mathcal{D}' \in Der(\mathcal{S}{-}R)$ exists that respectively mimics, s-mimics or m-mimics $\mathcal{D}$.

By this we are lead to the following definition, in which 4 different meanings are given to the term "rule elimination holds" *in abstract Hilbert systems* (with or without names). For all AHS's and n-AHS's $\mathcal{S}$ and for every rule R of $\mathcal{S}$, it will be defined that "R-elimination holds in $Der(\mathcal{S})$" of derivations of $\mathcal{S}$ "with respect to $\precsim$", "with respect to $\simeq^{(\mathrm{s})}$" or "with respect to $\simeq^{(\mathrm{m})}$", if and only if for all derivations $\mathcal{D}$ in $\mathcal{S}$ the applications of R can, respectively, be eliminated, s-eliminated or m-eliminated from $\mathcal{D}$. However, we will state these definitions directly in terms of respective notions of "mimicking derivation". And as a forth notion of rule derivability, we will stipulate, for all AHS's or n-AHS's $\mathcal{S}$ and all rules R of $\mathcal{S}$, that "R-elimination holds in $\mathcal{S}$" if and only if, for every derivation $\mathcal{D}$ in $\mathcal{S}$ *without assumptions*, the applications of R can be eliminated from $\mathcal{D}$.

Definition 5.2 (Four notions of rule elimination). Let $\mathcal{S}$ be an AHS or n-AHS, and let R be a rule of $\mathcal{S}$.

(i) We say that *R-elimination holds in* $\mathcal{S}$ if and only if every derivation $\mathcal{D}$ in $\mathcal{S}$ with no assumptions can be mimicked by a derivation $\mathcal{D}'$ in $\mathcal{S}{-}R$, i.e. iff it holds that:

$$(\forall\, \mathcal{D} \in Der(\mathcal{S})) \left[\mathrm{set}(\mathsf{assm}(\mathcal{D})) = \emptyset \;\Rightarrow\; (\exists\, \mathcal{D}' \in Der(\mathcal{S}{-}R)) \, [\, \mathcal{D}' \precsim \mathcal{D} \,] \right] . \quad (5.9)$$

(ii) We say that *R-elimination holds in* $Der(\mathcal{S})$ *with respect to* $\precsim$ if and only if every derivation $\mathcal{D}$ of $\mathcal{S}$ can be mimicked by a derivation $\mathcal{D}'$ of $\mathcal{S}{-}R$, i.e. iff

$$(\forall\, \mathcal{D} \in Der(\mathcal{S}))\, (\exists\, \mathcal{D}' \in Der(\mathcal{S}{-}R))\, [\, \mathcal{D}' \precsim \mathcal{D} \,] \quad (5.10)$$

holds. Similarly, we say that *R-elimination holds in* $Der(\mathcal{S})$ *with respect to* $\simeq^{(s)}$ or that *R-elimination holds in* $Der(\mathcal{S})$ *with respect to* $\simeq^{(m)}$ if and only if, respectively the assertions (5.11) and (5.12) hold:

$$(\forall\, \mathcal{D} \in Der(\mathcal{S}))\, (\exists\, \mathcal{D}' \in Der(\mathcal{S}{-}R))\, [\, \mathcal{D}' \simeq^{(\mathrm{s})} \mathcal{D} \,] \,, \quad (5.11)$$

$$(\forall\, \mathcal{D} \in Der(\mathcal{S}))\, (\exists\, \mathcal{D}' \in Der(\mathcal{S}{-}R))\, [\, \mathcal{D}' \simeq^{(\mathrm{m})} \mathcal{D} \,] \,. \quad (5.12)$$

As a rather easy consequence of Corollary 4.15, we give the following theorem, which contains characterizations of three of the four defined notions of rule elimination in terms of respective notions of rule admissibility or rule derivability. For rule elimination with respect to the s-mimicking relation $\simeq^{(\mathrm{s})}$ only a weaker statement holds.

Theorem 5.3 (Notions of rule elimination versus notions of rule admissibility and derivability). *Let $\mathcal{S}$ be an AHS or an n-AHS and let R be a rule of $\mathcal{S}$. Then the following logical implications and equivalences hold:*

$$\begin{array}{rcll} \textit{R-elimination holds in } \mathcal{S} & \Longleftrightarrow & R \textit{ is admissible in } \mathcal{S}\,, & (5.13)\\ \textit{R-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \precsim & \Longleftrightarrow & R \textit{ is derivable in } \mathcal{S}\,, & (5.14)\\ \textit{R-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \simeq^{(s)} & \Longrightarrow & R \textit{ is s-derivable in } \mathcal{S}\,, & (5.15)\\ \textit{R-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \simeq^{(m)} & \Longleftrightarrow & R \textit{ is m-derivable in } \mathcal{S}\,. & (5.16) \end{array}$$

The implication "$\Leftarrow$" in (5.15) does not hold in general. It is easy to see that this is an immediate consequence of the fact that the implication "$\Rightarrow$" in (4.10) of Corollary 4.15 does not hold in general (as is shown by Example C.10, which can also provide a counterexample for "$\Leftarrow$" in (5.15)).

5.2 *Effective rule-elimination by (s-, m-) mimicking steps in derivations of n-AHS's*

In this subsection it is our goal to investigate what implications the fact that, for an n-AHS $\mathcal{S}$ and a rule R of $\mathcal{S}$, R-elimination in $Der(\mathcal{S})$ is possible according to one of the above three notions of rule elimination, has for the problem of eliminating applications of R from arbitrary given derivations of $\mathcal{S}$ *in a stepwise* and *effective manner*. The reason, why we will only be interested in n-AHS's for studying this question, consists in the following fact: According to our definition of rules for abstract Hilbert systems, different rules R_1 and R_2 may possess applications $\alpha_1 \in Apps_{R_1}$ and $\alpha_2 \in Apps_{R_2}$ with the same sequence of premises and with the same conclusion, i.e. with $\mathsf{prem}(\alpha_1) = \mathsf{prem}(\alpha_2)$ and with $\mathsf{concl}(\alpha_1) = \mathsf{concl}(\alpha_2)$.[8] Therefore the concrete problem of eliminating all applications of a rule R of $\mathcal{S}$ from a derivation $\mathcal{D}$ in an AHS $\mathcal{S}$ is not well-formulated in general because the expression "$\mathcal{D}$ contains applications of R" does not necessarily always have an unambiguous meaning: It may be the case that $\mathcal{D} \in Der(\mathcal{S}{-}R)$, but that $\mathcal{D}$ contains applications of R, i.e. that $\mathcal{D}$ contains inferences $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ for applications α of R (every such inference $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ must then also be an inference $\mathcal{D}_{(\alpha',R',\mathcal{S})}$ for an application α' of a rule R' in $Der(\mathcal{S}{-}R)$). In n-AHS's however, this problem does not occur since rules possess unique names and all one-step inferences within derivations are labeled by the name of a rule according to an application of which the inference is formed. As a consequence, for every derivation $\mathcal{D}$ and for every rule R of an n-AHS $\mathcal{S}$ it can be determined whether $\mathcal{D}$ contains applications of R or not.

For motivating abstract rewrite systems of rule elimination "by mimicking steps", let $\mathcal{S}$ be an arbitrary n-AHS and let R be a rule of $\mathcal{S}$ such that R-elimination holds in $Der(\mathcal{S})$. Then it follows from Theorem 5.3 that R is actually derivable in $\mathcal{S}$. This entails that every application α of R can be mimicked by at least one derivation $\mathcal{D}_\alpha$ in $\mathcal{S}{-}R$. And hence it is possible to "replace" arbitrary applications of R within a considered derivation of $\mathcal{S}$ by a respective mimicking derivation. More precisely, in every derivation $\mathcal{D} \in Der(\mathcal{S})$ that contains R-applications *mimicking steps* of one of the two forms explained below are possible in which an application α of R is eliminated by being replaced through a mimicking derivation $\mathcal{D}_\alpha$ for α. Firstly, elimination steps ϕ for zero-premise R-applications that are of the form

$$\phi \;:\; \begin{array}{c} \overline{\quad(A)\quad}\,\mathsf{name}(R) \\ \mathcal{D}_0 \end{array} \quad \to^{(R)}_{\mathrm{mim}} \quad \begin{array}{c} \mathcal{D}_\alpha \\ (A) \\ \mathcal{D}_0 \end{array} \tag{5.17}$$

where

[8] It is also the case that one rule may possess two different applications with the same sequence of premises and with the same conclusion.

- $\mathcal{D}_0 \in Der(\mathcal{S})$, $\mathcal{D}_\alpha \in Der(\mathcal{S}{-}R)$, $A \in Fo$, a particular occurrence of the assumption A in $\mathcal{D}_0$ has been singled out by the expression (A);
- there exists an application $\alpha \in Apps_R$ with $\mathsf{prem}(\alpha) = ()$ and $\mathsf{concl}(\alpha) = A$ such that
  - the derivation $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ that corresponds to α in $\mathcal{S}$ is substituted into the singled out occurrence (A) of the assumption A in $\mathcal{D}_0$ with the result of the derivation on the left-hand side in (5.17), and
  - $\mathcal{D}_\alpha$ mimics α in $\mathcal{S}{-}R$ and is substituted into the singled out occurrence (A) of A in $\mathcal{D}_0$ with the result of the derivation on the right hand side in (5.17).

And secondly, elimination steps ϕ for R-applications with a non-zero number of premises that are of the form

$$\phi \;:\quad \begin{array}{c} \begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array} \\ \hline (A) \\ \mathcal{D}_0 \end{array}\,\mathsf{name}(R) \quad \to^{(R)}_{\text{mim}} \quad \begin{array}{c} \begin{array}{ccc} \mathcal{D}_{i_1} & & \mathcal{D}_{i_k} \\ (A_{i_1}) & \ldots & (A_{i_k}) \end{array} \\ \mathcal{D}_\alpha \\ (A) \\ \mathcal{D}_0 \end{array} \tag{5.18}$$

whereby

- $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n \in Der(\mathcal{S})$, $A_1, \ldots, A_n, A \in Fo_{\mathcal{S}}$, $\mathcal{D}_\alpha \in Der(\mathcal{S}{-}R)$, $n \in \omega \setminus \{0\}$ and $k \in \omega$ are natural numbers (or zero); furthermore we denote by $\mathcal{D}$ and $\mathcal{D}'$ the derivations on the left and on the right side of (5.18), respectively;
- a particular occurrence of the assumption A in $\mathcal{D}_0$ has been singled out by the expression (A) such that $\mathcal{D}$ is the result of substituting the prooftree drawn above (A), which ends with an application of R, into this particular occurrence of the assumption A in $\mathcal{D}_0$; and accordingly, $\mathcal{D}'$ is the result of substituting the prooftree drawn above (A) in $\mathcal{D}_0$ into this particular occurrence of the assumption A in $\mathcal{D}_0$;
- there exists $\alpha \in Apps_R$ such that $\mathsf{prem}^{(R)}(\alpha) = (A_1, \ldots, A_n)$ and $\mathsf{concl}_R(\alpha) = A$ such that
  - the derivation $\mathcal{D}_\alpha \in Der(\mathcal{S}{-}R)$ mimics α,
  - $\mathsf{assm}(\mathcal{D}_\alpha) = \text{mset}((A_{i_1}, A_{i_2}, \ldots, A_{i_k}))$ for some $i_1, \ldots, i_k \in \omega$ with the property $1 \leq i_1, i_2, \ldots, i_k \leq n$. The expressions $(A_{i_1}), \ldots, (A_{i_k})$ at the top of $\mathcal{D}_\alpha$ in $\mathcal{D}'$ represent single occurrences of the assumptions $A_{i_1}, \ldots, A_{i_k}$ in $\mathcal{D}_\alpha$, which together make up all assumptions of $\mathcal{D}_\alpha$. Within $\mathcal{D}'$ the derivations $\mathcal{D}_{i_j}$ are substituted into the occurrences (A_{i_j}) of the assumptions A_{i_j} of $\mathcal{D}_\alpha$ for $j \in \{1, \ldots, k\}$ respectively (if $\mathsf{assm}(\mathcal{D}_\alpha) = \emptyset$, there is no need, and also no possibility, for substituting some of the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ above assumptions of $\mathcal{D}_\alpha$).

These kinds of steps give rise to the *ARS* $\to^{(R)}_{mim}(\mathcal{S})$ *of R-elimination by mimicking steps* that is defined by

$$\to^{(R)}_{\text{mim}}(\mathcal{S}) = \langle Der(\mathcal{S}), \Phi^{(R)}_{\text{mim}}(\mathcal{S}), \mathsf{src}, \mathsf{tgt} \rangle \;,$$

where the steps $\Phi^{(R)}_{\text{mim}}(\mathcal{S})$ of $\rightarrow^{(R)}_{\text{mim}}(\mathcal{S})$ are formally given by

$$\Phi^{(R)}_{\text{mim}}(\mathcal{S}) = \Big\{ \langle \mathcal{D}, \begin{matrix}(A)\\ \mathcal{D}_0\end{matrix}, \mathcal{D}_\alpha \rangle \;\Big|\; \mathcal{D}, \mathcal{D}_0, \mathcal{D}' \in Der(\mathcal{S}),\ \mathcal{D}_\alpha \in Der(\mathcal{S}{-}R),\ \mathcal{D} \text{ is the left-hand side of a transition shown in (5.17) or (5.18), } \mathcal{D}_0 \text{ is an end-part of } \mathcal{D} \text{ with a singled out occurrence of the assumption } A \text{ in it, which in } \mathcal{D} \text{ corresponds to the conclusion of an } R\text{-application, and the subderivation } \mathcal{D}_\alpha \text{ of } \mathcal{D}' \textit{ mimics} \text{ the displayed application, called } \alpha, \text{ of } R \text{ in } \mathcal{D} \Big\} \tag{5.19}$$

and the source and target functions of $\rightarrow^{(R)}_{\text{mim}}(\mathcal{S})$ by

$$\begin{aligned} \mathsf{src} : \Phi^{(R)}_{\text{mim}}(\mathcal{S}) &\longrightarrow Der(\mathcal{S}) & \qquad \mathsf{tgt} : \Phi^{(R)}_{\text{mim}}(\mathcal{S}) &\longrightarrow Der(\mathcal{S}) \\ \langle \mathcal{D}, \begin{matrix}(A)\\ \mathcal{D}_0\end{matrix}, \mathcal{D}_\alpha \rangle &\longmapsto \mathsf{src}(\langle \ldots \rangle) = \mathcal{D}\,, & \langle \mathcal{D}, \begin{matrix}(A)\\ \mathcal{D}_0\end{matrix}, \mathcal{D}_\alpha \rangle &\longmapsto \mathsf{tgt}(\langle \ldots \rangle) = \mathcal{D}'\,, \end{aligned}$$

where $\mathcal{D}'$ stands for the right-hand side in (5.17) or (5.18) of a respective step of $\rightarrow^{(R)}_{\text{mim}}(\mathcal{S})$ given by $\langle \mathcal{D}, \begin{matrix}(A)\\ \mathcal{D}_0\end{matrix}, \mathcal{D}_\alpha \rangle$.

Due to our assumption that R is derivable in $\mathcal{S}{-}R$, mimicking steps (5.17) and (5.18) of the ARS $\rightarrow^{(R)}_{\text{mim}}(\mathcal{S})$ are always possible in derivations of $\mathcal{S}$ that contain R-applications and are not possible in $\mathcal{S}$-derivations that do not contain R-applications. It follows that

$$\mathcal{NF}(\rightarrow^{(R)}_{\text{mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R)\,, \tag{5.20}$$

where we have used the notation to denote the set of normal forms of an ARS $\mathcal{A}$ by $\mathcal{NF}(\mathcal{A})$ (we will use this notation again, below; for its more formal definition, see Appendix D.2).

And furthermore, every derivation $\mathcal{D} \in Der(\mathcal{S})$ can be transformed into a derivation $\mathcal{D} \in Der(\mathcal{S}{-}R)$ by a finite number of successive mimicking steps in which always topmost occurrences of applications of R are considered and eliminated: No new applications of R are introduced by a mimicking step for the elimination of a topmost occurrence of R, and therefore $\mathcal{D}'$ can be reached from $\mathcal{D}$ by precisely n mimicking steps, where n is the number of applications of R in $\mathcal{D}$. These arguments sketch the proof of the following lemma.

Lemma 5.4 *Let $\mathcal{S}$ be an n-AHS, and R be a rule of $\mathcal{S}$ that is derivable in $\mathcal{S}{-}R$.*

(i) *A derivation of $\mathcal{S}$ is a normal form of $\rightarrow^{(R)}_{\textit{mim}}(\mathcal{S})$ if and only if it does not contain applications of R, that is,* (5.20) *holds.*

(ii) *The ARS $\rightarrow^{(R)}_{\textit{mim}}(\mathcal{S})$ of R-elimination by mimicking steps is weakly normalizing.*

Furthermore, we define *the ARS $\rightarrow^{(R)}_{\textit{s-mim}}(\mathcal{S})$ of R-elimination by s-mimicking steps* and *the ARS $\rightarrow^{(R)}_{\textit{s-mim}}(\mathcal{S})$ of R-elimination by s-mimicking steps* respectively by

$$\to^{(R)}_{\text{s-mim}}(\mathcal{S}) = \langle \mathit{Der}(\mathcal{S}), \Phi^{(R)}_{\text{s-mim}}(\mathcal{S}), \mathsf{src}', \mathsf{tgt}' \rangle \,, \tag{5.21}$$

$$\to^{(R)}_{\text{m-mim}}(\mathcal{S}) = \langle \mathit{Der}(\mathcal{S}), \Phi^{(R)}_{\text{m-mim}}(\mathcal{S}), \mathsf{src}'', \mathsf{tgt}'' \rangle \,, \tag{5.22}$$

where the set of steps $\Phi^{(R)}_{\text{s-mim}}(\mathcal{S})$ of $\to^{(R)}_{\text{s-mim}}(\mathcal{S})$ and $\Phi^{(R)}_{\text{m-mim}}(\mathcal{S})$ of $\to^{(R)}_{\text{m-mim}}(\mathcal{S})$ are defined by stipulations analogous to (5.19) in which the word "mimics" is respectively replaced by "s-mimics" and "m-mimics".

The two following lemmas can be proven analogously as Lemma 5.4 above.

Lemma 5.5 *Let $\mathcal{S}$ be an n-AHS, and R be a rule of $\mathcal{S}$ that is s-derivable in $\mathcal{S}{-}R$.*

(i) *A derivation $\mathcal{D}$ of $\mathcal{S}$ is a normal form of $\to^{(R)}_{\textit{s-mim}}(\mathcal{S})$ if and only if it does not contain applications of R, i.e. iff $\mathcal{D} \in \mathit{Der}(\mathcal{S}{-}R)$ holds.*

(ii) *The ARS $\to^{(R)}_{\textit{s-mim}}(\mathcal{S})$ of R-elimination by s-mimicking steps is weakly normalizing.*

Lemma 5.6 *Let $\mathcal{S}$ be an n-AHS, and R be a rule of $\mathcal{S}$ that is m-derivable in $\mathcal{S}{-}R$.*

(i) *A derivation of $\mathcal{S}$ is a normal form of $\to^{(R)}_{\textit{m-mim}}(\mathcal{S})$ if and only if it does not contain applications of R, i.e. iff $\mathcal{D} \in \mathit{Der}(\mathcal{S}{-}R)$ holds.*

(ii) *The ARS $\to^{(R)}_{\textit{m-mim}}(\mathcal{S})$ of R-elimination by m-mimicking steps is weakly normalizing.*

In our motivation for an ARS of the form $\to^{(R)}_{\text{mim}}(\mathcal{S})$, where $\mathcal{S}$ is an n-AHS and R is a rule of $\mathcal{S}$, we have argued that if R-elimination holds in $\mathcal{S}$ with respect to $\precsim$ then mimicking steps for the elimination of R-applications are always possible in such derivations of $\mathcal{S}$ that actually contain R-applications. Similarly, it is easy to see that, for an n-AHS $\mathcal{S}$ and a rule R of $\mathcal{S}$, if R-elimination holds in $\mathcal{S}$ with respect to $\simeq^{(\text{s})}$, or if R-elimination holds in $\mathcal{S}$ with respect to $\simeq^{(\text{m})}$, then eliminations of R-applications by s-mimicking steps, or respectively by m-mimicking steps, are always possible in derivations of $\mathcal{S}$ that contain R-applications.

But until now we have not considered the question whether rule elimination by (s-, m-) mimicking steps is actually "correct" in relation to the earlier defined notions of rule elimination with respect to $\precsim$ ($\simeq^{(\text{s})}$, $\simeq^{(\text{m})}$). That is, we have not asked for an n-AHS $\mathcal{S}$, a rule R of $\mathcal{S}$ and a derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S})$: Given that a derivation $\mathcal{D}'$ in $\mathcal{S}$ without R-applications has been reached from $\mathcal{D}$ as the result of a finite sequence of successively applied mimicking steps, does the derivation $\mathcal{D}'$ actually mimick $\mathcal{D}$? And thus: Can $\mathcal{D}'$ in this case really be considered to be the result of R-elimination with respect to $\precsim$?

For a formal investigation of this question, we define a respective notion of "correctness" for each of the three kinds of ARS's of rule elimination by mimicking steps defined above. Let $\mathcal{S}$ be an n-AHS and R be a rule of $\mathcal{S}$. We say that "R-elimination by mimicking steps in $\mathit{Der}(\mathcal{S})$ is *correct* with respect to $\precsim$" if and only if for all

$\mathcal{D}, \mathcal{D}' \in Der(\mathcal{S})$

$$(\exists\, \phi \in (\Phi^{(R)}_{\text{mim}}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{\text{mim}} \mathcal{D}' \;\;\&\;\; \mathcal{D}' \in Der(\mathcal{S}{-}R) \,\big] \implies \implies \mathcal{D}' \precsim \mathcal{D} \qquad (5.23)$$

holds. And similarly, we agree to say that "R-elimination by s-mimicking steps (by m-mimicking steps) in $Der(\mathcal{D})$ is *correct* with respect to $\simeq^{(\text{s})}$ (with respect to $\simeq^{(\text{m})}$)" if and only if for all $\mathcal{D}, \mathcal{D}' \in Der(\mathcal{S})$ (5.24) holds (or respectively, iff (5.25) holds):

$$(\exists\, \phi \in (\Phi^{(R)}_{\text{s-mim}}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{\text{s-mim}} \mathcal{D}' \;\;\&\;\; \mathcal{D}' \in Der(\mathcal{S}{-}R) \,\big] \implies \implies \mathcal{D}' \simeq^{(\text{s})} \mathcal{D} \qquad (5.24)$$

$$(\exists\, \phi \in (\Phi^{(R)}_{\text{m-mim}}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{\text{m-mim}} \mathcal{D}' \;\;\&\;\; \mathcal{D}' \in Der(\mathcal{S}{-}R) \,\big] \implies \implies \mathcal{D}' \simeq^{(\text{m})} \mathcal{D} \qquad (5.25)$$

The following lemma gathers important statements about the correctness of rule elimination by (s-, m-) mimicking steps with respect to the relations $\precsim$ and $\simeq^{(\text{m})}$.

It is easy to verify that, for all mimicking steps ϕ in $\to^{(R)}_{\text{mim}}(\mathcal{S})$ (where $\mathcal{S}$ is an n-AHS and R is a rule of $\mathcal{S}$), the target $\mathsf{tgt}(\phi)$ of ϕ mimics the source $\mathsf{src}(\phi)$ of ϕ; and that similarly, for all m-mimicking steps, the target $\mathsf{tgt}(\phi)$ of ϕ m-mimics the source $\mathsf{src}(\phi)$ of ϕ. These observations are generalized to the respective reflexive-transitive closures of $\to^{(R)}_{\text{mim}}(\mathcal{S})$ and $\to^{(R)}_{\text{m-mim}}(\mathcal{S})$ in the next lemma.

Lemma 5.7 *Let $\mathcal{S}$ be an n-AHS and R be a rule of $\mathcal{S}$. Furthermore let $\xrightarrow{*}{}^{(R)}_{mim}$, $\xrightarrow{*}{}^{(R)}_{s\text{-}mim}$ and $\xrightarrow{*}{}^{(R)}_{m\text{-}mim}$ be the respective reflexive-transitive closures of the three ARS's $\to^{(R)}_{mim}(\mathcal{S})$, $\to^{(R)}_{s\text{-}mim}(\mathcal{S})$ and $\to^{(R)}_{m\text{-}mim}(\mathcal{S})$.*

Then for all $\mathcal{D}, \mathcal{D}' \in Der(\mathcal{S})$ it holds that:

$$(\exists\, \phi \in (\Phi^{(R)}_{mim}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{mim} \mathcal{D}' \,\big] \implies \mathcal{D}' \precsim \mathcal{D}\ , \qquad (5.26)$$

$$(\exists\, \phi \in (\Phi^{(R)}_{s\text{-}mim}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{s\text{-}mim} \mathcal{D}' \,\big] \implies \mathcal{D}' \precsim \mathcal{D}\ , \qquad (5.27)$$

$$(\exists\, \phi \in (\Phi^{(R)}_{m\text{-}mim}(\mathcal{S}))^*)\, \big[\, \phi : \mathcal{D} \xrightarrow{*}{}^{(R)}_{m\text{-}mim} \mathcal{D}' \,\big] \implies \mathcal{D}' \simeq^{(m)} \mathcal{D}\ . \qquad (5.28)$$

However, for s-mimicking steps ϕ in an ARS $\to^{(R)}_{\text{s-mim}}(\mathcal{S})$, where $\mathcal{S}$ is an n-AHS and R is a rule of $\mathcal{S}$, it does not hold in general that the derivation $\mathsf{tgt}(\phi)$ s-mimics the derivation $\mathsf{src}(\phi)$, although $\mathsf{tgt}(\phi)$ still always mimics $\mathsf{src}(\phi)$. This means that $\precsim$ cannot be replaced by $\simeq^{(\text{s})}$ in (5.27). And what is more, (5.27) in Lemma 5.7 cannot be replaced by (5.24).

Lemma 5.8 *There exist n-AHS's $\mathcal{S}$ such that, for some rules R of $\mathcal{S}$, R-elimination by s-mimicking steps in $Der(\mathcal{S})$ is not correct with respect to $\simeq^{(s)}$.*

The questions about the correctness of ARS's by (s-, m-) mimicking steps with respect to the mimicking relations $\precsim$ ($\simeq^{(\mathrm{s})}$ and $\simeq^{(\mathrm{m})}$) are now settled by the following theorem, which is an immediate consequence of Lemma 5.7 and Lemma 5.8.

Theorem 5.9 (Correctness of rule elimination by (s-, m-) mimicking steps with respect to $\precsim$ (with respect to $\precsim$, with respect to $\simeq^{(\mathrm{m})}$)). *Let $\mathcal{S}$ be an n-AHS and R be a rule of $\mathcal{S}$. Then it holds:*

(i) *R-elimination by mimicking steps in $Der(\mathcal{S})$ is correct with respect to $\precsim$.*

(ii) *R-elimination in $Der(\mathcal{S})$ by s-mimicking steps is correct with respect to $\precsim$; but it is not in general also correct with respect to $\simeq^{(s)}$.*

(iii) *R-elimination in $Der(\mathcal{S})$ by m-mimicking steps is correct with respect to $\simeq^{(m)}$.*

We are now going to state a lemma that is of central importance for our results about the introduced ARS's of rule elimination: It asserts that for n-AHS's $\mathcal{S}$ and rules R of $\mathcal{S}$ that are derivable in $\mathcal{S}{-}R$, R-elimination by mimicking steps does always terminate for derivations $\mathcal{D}$ of $\mathcal{S}$: This means that no matter in what order R-elimination by mimicking steps takes place starting with a derivation $\mathcal{D} \in Der(\mathcal{S})$, a normal form of $\to^{(R)}_{\mathrm{mim}}(\mathcal{S})$, and by Lemma 5.4 consequently a derivation in $\mathcal{S}$ without R-applications, is reached in a finite number of steps.

Lemma 5.10 *Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$ that is derivable in $\mathcal{S}$. Then the ARS $\to^{(R)}_{mim}(\mathcal{S})$ of R-elimination in $Der(\mathcal{S})$ by mimicking steps is strongly normalizing.*

The proof of this lemma proceeds by reducing the termination problem of an arbitrary ARS of rule elimination by mimicking steps to the well-foundedness of the multiset ordening on $\mathcal{M}_{\mathrm{f}}(\omega)$. It is given in Appendix D and the necessary formal prerequisites about ARS's, the multiset ordening and the method for reducing the termination problem of one ARS to the termination problem of another, are given in three earlier sections of this appendix.

Since sub-ARS's of strongly normalizing ARS's are again strongly normalizing, and since, for all AHS's and n-AHS's $\mathcal{S}$ and for all rules R of $\mathcal{S}$, each of the ARS's $\to^{(R)}_{\text{s-mim}}(\mathcal{S})$ and $\to^{(R)}_{\text{m-mim}}(\mathcal{S})$ are sub-ARS's of $\to^{(R)}_{\mathrm{mim}}(\mathcal{S})$, the following lemma is an immediate consequence of Lemma 5.10.

Lemma 5.11 *Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$.*

(i) *If R is s-derivable in $\mathcal{S}{-}R$, then the ARS $\to^{(R)}_{s\text{-}mim}(\mathcal{S})$ of R-elimination in $Der(\mathcal{S})$ by s-mimicking steps is strongly normalizing.*

(ii) *If R is m-derivable in $\mathcal{S}{-}R$, then the ARS $\to^{(R)}_{m\text{-}mim}(\mathcal{S})$ of R-elimination in $Der(\mathcal{S})$ by m-mimicking steps is strongly normalizing.*

Based on above definitions, we are now going to introduce three notions of "strong rule elimination": For all n-AHS's $\mathcal{S}$ and all rules R of $\mathcal{S}$, we will agree on a stipulation to the effect that "strong R-eliminations by mimicking steps holds in $\mathcal{S}$" if and only if R-elimination by mimicking steps terminates on every derivation $\mathcal{D}$ of $\mathcal{S}$ with the result of a derivation $\mathcal{D}'$ in $\mathcal{S}$ without R-applications. And in an analogous way the notions of “strong rule elimination by s-mimicking steps” and of “strong rule elimination by m-mimicking steps” will be defined.

Definition 5.12 (Strong rule elimination by (s-, m-) mimicking steps). Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$.

We say that *strong R-elimination by mimicking steps holds in Der*($\mathcal{S}$) iff

$$\to^{(R)}_{\text{mim}}(\mathcal{S}) \text{ is strongly normalizing, and } \quad \mathcal{NF}(\to^{(R)}_{\text{mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R) \tag{5.29}$$

holds. And similarly, we say that *strong R-elimination by s-mimicking steps holds in Der*($\mathcal{S}$), and respectively, that *strong R-elimination by m-mimicking steps holds in Der*($\mathcal{S}$) if and only if, respectively, (5.30) and (5.31) holds:

$$\to^{(R)}_{\text{s-mim}}(\mathcal{S}) \text{ is strongly normalizing, and } \quad \mathcal{NF}(\to^{(R)}_{\text{s-mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R)\ , \tag{5.30}$$

$$\to^{(R)}_{\text{m-mim}}(\mathcal{S}) \text{ is strongly normalizing, and } \quad \mathcal{NF}(\to^{(R)}_{\text{m-mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R)\ . \tag{5.31}$$

Our main theorem below of the second part of this section characterizes each of the three notions of strong rule elimination defined above in terms of a corresponding notion of rule derivability.

Theorem 5.13 (Strong rule elimination by (s-, m-) mimicking steps). *Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$. Then the following three logical equivalences hold:*

$$\textit{Strong R-elimination by mimicking steps holds in } Der(\mathcal{S}) \iff \iff R \textit{ is derivable in } \mathcal{S}{-}R\ , \tag{5.32}$$

$$\textit{strong R-elimination by s-mimicking steps holds in } Der(\mathcal{S}) \iff \iff R \textit{ is s-derivable in } \mathcal{S}{-}R\ , \tag{5.33}$$

$$\textit{strong R-elimination by m-mimicking steps holds in } Der(\mathcal{S}) \iff \iff R \textit{ is m-derivable in } \mathcal{S}{-}R\ . \tag{5.34}$$

As a corollary to this theorem and to Theorem 5.3 we find the following connections between notions introduced in Subsection 5.1 and in the present subsection.

Corollary 5.14 (Notions of strong rule elimination versus notions of rule elimination). *Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$. Then the following three statements hold:*

$$\textit{Strong R-elimination by mimicking steps holds in } Der(\mathcal{S}) \iff \iff R\textit{-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \precsim\ , \tag{5.35}$$

$$\textit{strong R-elimination by s-mimicking steps holds in } Der(\mathcal{S}) \Longleftarrow \Longleftarrow R\textit{-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \simeq^{(s)}\ , \tag{5.36}$$

$$\textit{strong R-elimination by m-mimicking steps holds in } Der(\mathcal{S}) \iff \iff R\textit{-elimination holds in } Der(\mathcal{S}) \textit{ w.r.t. } \simeq^{(m)}\ . \tag{5.37}$$

We conclude this section by emphasizing a practical consequence of Corollary 5.14 and of Theorem 5.9. Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$. Then R-elimination by mimicking steps in $Der(\mathcal{S})$ is "complete" *for* the notion of R-elimination in $Der(\mathcal{S})$ with respect to $\precsim$ from Subsection 5.1 in the following sense: If R-elimination holds in $Der(\mathcal{S})$ with respect to $\precsim$, then for every $\mathcal{D} \in Der(\mathcal{S})$ a derivation $\mathcal{D}'$ with

$$\mathcal{D}' \in Der(\mathcal{S}{-}R) \quad \& \quad \mathcal{D}' \precsim \mathcal{D}\,, \tag{5.38}$$

i.e. an R-application-free derivation in $\mathcal{S}$ that mimics $\mathcal{D}$, can be found effectively as the result of applying a long enough sequence of arbitrary, but composable mimicking steps to $\mathcal{D}$. Or in other words, if R-elimination holds in $Der(\mathcal{S})$ with respect to $\precsim$, then for every $\mathcal{D} \in Der(\mathcal{S})$ a derivation $\mathcal{D}'$ with (5.38) can be produced by performing the following iteration as often as necessary until termination: If the reached derivation $\tilde{\mathcal{D}}$ contains applications of R, carry out an arbitrary mimicking step to $\tilde{\mathcal{D}}$ (under our assumption this is always possible because of (5.35) and Definition 5.12), thereby eliminating a particular application of R; if $\tilde{\mathcal{D}}$ does not contain applications of R, terminate. After finitely many iterations a derivation $\mathcal{D}'$ with (5.38) will always be reached. Termination follows hereby from our assumption because of (5.35) and the definition of "strong R-elimination by mimicking steps holds in $Der(\mathcal{S})$"; correctness, i.e. the fact that the resulting derivation $\mathcal{D}'$ mimics $\mathcal{D}$, follows from Theorem 5.9.

And in an analogous sense, R-elimination by m-mimicking steps in $Der(\mathcal{S})$ can be viewed to be "complete" *for* the notion of R-elimination in $Der(\mathcal{S})$ with respect to the m-mimicking relation $\simeq^{(\mathrm{m})}$ from Subsection 5.1.

6 Conclusion

Our aim in this report was to investigate those general aspects of the notions of rule derivability and admissibility in Hilbert-style proof systems that can be studied independently from assumptions about the syntax of formulas, about how rules are given formally and in which way they determine inferences. Hereby we have only considered Hilbert-style systems of the simplest kind, where inferences in derivations do not depend assumptions that have (probably) been made in subderivations. In Section 2 we introduced, by analogy with abstract rewrite systems as defined in [9], the framework of abstract Hilbert systems (AHS's), and the variant framework of abstract Hilbert systems with names for rules and axioms (n-AHS's).

In Section 3, we adapted known definitions for rule derivability and admissibility to abstract Hilbert systems. In the case of rule derivability, we used three formalizations of the term "mimicking derivation" *for a rule application* and introduced three variants, rule derivability, s-derivability and m-derivability by stipulations that amount to: A rule R is derivable (s-derivable, m-derivable) in an abstract Hilbert system $\mathcal{S}$ if and only if for every application of R there exists a mimicking (s-mimicking, m-mimicking) derivation in $\mathcal{S}$. Next we gathered basic facts about the interrelations of these (together with rule admissibility) four notions and gave two results about characterizations of rule derivability and admissibility in terms of the respective other notion.

In Section 4, we looked at relations that compare AHS's with respect to their admissible, derivable, s-derivable or m-derivable rules and with respect to their relative derivability statements. For this purpose, we introduced twelve inclusion relations and twelve mutual inclusion relations that are induced by respective inclusion relations. As the result of a systematic examination of the relationship of these relations towards each other, we then gave two theorems: These describe the logical implications and equivalences that hold in general, and respectively that do not hold in general, between statements that compare two AHS's with respect to one of the introduced inclusion or mutual inclusion relations. And we used pictures for 'interrelation prisms' as a means to formulate these theorems as well as to visualize them.

In the last section, Section 5, we investigated what general consequences the fact that a rule R is admissible, derivable, s-derivable or m-derivable in an AHS or n-AHS $\mathcal{S}$ has for the possibility to eliminate R-eliminations from derivations in $\mathcal{S}$. We introduced four different abstract notions of "rule elimination". For this, we first defined three formalizations of the term "mimicking derivation" *for a derivation* and then, for three of the four notions, used stipulations of a form like: For a rule R in an AHS or n-AHS $\mathcal{S}$, R-elimination holds for derivations in $\mathcal{S}$ if and only if for every derivation in $\mathcal{S}$ there exists a mimicking derivation that does not contain R-applications. As an easy consequence of the results obtained in Section 4, we showed that there exists a direct correspondence between three of the four notions of rule elimination with either rule admissibility or with a respective notion of rule derivability (in the fourth case, involving s-derivability, only a weaker connection holds).

And in the second part of Section 5, Subsection 5.2, we proved that if a rule R is derivable in an n-AHS $\mathcal{S}$ then R-elimination for derivations in $\mathcal{S}$ can be performed in an effective way: For a considered derivation $\mathcal{D}$ in $\mathcal{S}$, pick an arbitrary application of R in $\mathcal{D}$ and replace it (its part in $\mathcal{D}$) by mimicking derivation; carry out such *mimicking steps* repeatedly until no further applications of R are present. We show correctness and termination for this nondeterministic procedure. And we find that in n-AHS's a similar result holds also for m-derivable rules.

Further Work

It seems straightforward to transfer our results about rule derivability and admissibility in abstract Hilbert systems to similar abstractions of Gentzen systems, i.e. of sequent-style proof systems. This is because, from an outside view, Gentzen systems are Hilbert-style formalisms: A Gentzen system $\mathcal{G}$ can be 'modeled' by an abstract Hilbert system $\mathcal{S}_{\mathcal{G}}$ that contains the sequents of $\mathcal{G}$ as its formulas.

The situation is likely to be different, however, for 'usual formalizations' of natural-deduction systems that rely on the concept of assumption-discharging[9], because of the following: It is easy to give examples for a natural-deduction system

[9] Cf., for example, the descriptions of such systems in [10].

$\mathcal{S}$ and a Hilbert system inference rule R on the formulas of $\mathcal{S}$, i.e. a rule R applications of which do neither depend on nor discharge open assumptions, such that the characterization of rule admissibility given in Proposition 3.2 (i), which can be restated as

$$Th(\mathcal{S}) \text{ is closed under applications of } R \quad \Longleftrightarrow \quad \mathcal{S}{+}R \sim_{th} \mathcal{S}\ , \tag{6.1}$$

does not hold (but the implication "$\Leftarrow$" of (6.1) remains valid even in such situations). This suggests that in natural-deduction systems the notion of rule admissibility might split, at least for Hilbert system rules, into a weaker and a stronger notion that arise respectively from the assertions on the left and on the right hand side of the equivalence in (6.1).

It is our intention to study the notions of rule derivability and admissibility in abstractions of natural-deduction systems, both in their 'usual formalizations' and in their sequent-style formalizations, in a subsequent paper. For this, also the dependence of reasonable notions of rule derivability and admissibility on the two kinds of formalizations of natural-deduction systems will be of particular interest.

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Appendix A: Proofs for statements in Section 2

Proof of Lemma 2.12. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two AHS's, or two n-AHS's, such that $\mathcal{S}_2$ is an extension by enlargement of $\mathcal{S}_1$. Then, due to Definition 2.5, or due to Definition 2.4, it holds that every derivation $\mathcal{D}$ in $\mathcal{S}_1$ is also a derivation in $\mathcal{S}_2$ with the same conclusion and the same multiset of assumptions. From this it follows that every relative derivability statement with respect to $\vdash_{\boldsymbol{\mathcal{S}_1}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}_1}}$ or $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}_1}}$ that holds is, accordingly, also a relative derivability statement with respect to $\vdash_{\boldsymbol{\mathcal{S}_2}}$, $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}_2}}$ or $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}_2}}$ that holds.

Proof of Lemma 2.14. Let $\mathcal{S}$ be an AHS with set Fo of formulas, and let $A \in Fo$, $\Sigma, \Delta \in \mathcal{P}(Fo)$. The statement of the lemma is a consequence of the two following easy observations about an obvious correspondence between derivations $\mathcal{D}$ in $\mathcal{S}+\Sigma$ and derivations $\mathcal{D}'$ in $\mathcal{S}$:

- Every derivation $\mathcal{D}$ in $\mathcal{S}+\Sigma$ with $\mathsf{concl}(\mathcal{D}) = A$ and $\text{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Delta$ can be modified into a derivation $\mathcal{D}'$ in $\mathcal{S}$ with $\mathsf{concl}(\mathcal{D}') = A$ and $\text{set}(\mathsf{assm}(\mathcal{D}')) \subseteq \Delta \cup \Sigma$ by simply replacing every occurrence at the top of the prooftree $\mathcal{D}$ of an axiom C (with $C \in \Sigma$) by an occurrence of the assumption C.
- Every derivation $\mathcal{D}'$ in $\mathcal{S}$ with $\mathsf{concl}(\mathcal{D}) = A$ and $\text{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Delta \cup \Sigma$ can be transformed into a derivation $\mathcal{D}$ in $\mathcal{S}+\Sigma$ with $\mathsf{concl}(\mathcal{D}) = A$ and $\text{set}(\mathsf{assm}(\mathcal{D})) \subseteq \Delta$ by simply changing occurrences of assumptions C (with $C \in \Sigma$) at the top of $\mathcal{D}'$ into occurrences of axioms C of $\mathcal{S}+\Sigma$.

□

Appendix B: Proofs for statements in Section 3

Proof of Proposition 3.2. We will prove items (i), (ii) and (iii) of the proposition in the three items (a), (b) and (c) below, respectively. For all three cases, we let $\mathcal{S}$ be an arbitrary AHS and R an arbitrary rule on $Fo_{\mathcal{S}}$.

(a) For showing "⇒" in the assertion (i) of the proposition, we assume that R is admissible in $\mathcal{S}$. We have to show that $\mathcal{S}+R$ does not possess more theorems than $\mathcal{S}$.

Due to the definition of "R is admissible in $\mathcal{S}$", for every derivation $\mathcal{D}$ in $\mathcal{S}+R$ such that $\mathsf{assm}(\mathcal{D}) = \emptyset$ and such that only the bottommost rule application in $\mathcal{D}$ is an application of R, there exists a derivation $\mathcal{D}'$ in $\mathcal{S}$ with $\mathsf{assm}(\mathcal{D}') = \emptyset$ and with the same conclusion as $\mathcal{D}$. This has as a consequence that in arbitrary derivations $\tilde{\mathcal{D}}$ in $\mathcal{S}+R$ with $\mathsf{assm}(\tilde{\mathcal{D}}) = \emptyset$ topmost applications of R can always by eliminated by replacing them together with the subderivations leading up to them by derivations in $\mathcal{S}$ with respectively the same conclusions and without assumptions.

Due to this, it follows that for every derivation $\tilde{\mathcal{D}}$ in $\mathcal{S}+R$ with $\mathsf{assm}(\tilde{\mathcal{D}}) = \emptyset$ there exists a derivation $\tilde{\mathcal{D}}'$ in $\mathcal{S}$ with $\mathsf{assm}(\tilde{\mathcal{D}}') = \emptyset$ and with the same conclusion as $\tilde{\mathcal{D}}$: This can be shown by induction on the depth $|\tilde{\mathcal{D}}|$ of $\tilde{\mathcal{D}}$. As a consequence we find that every theorem of $\mathcal{S}{+}R$ is also a theorem of $\mathcal{S}$, because for every derivation $\tilde{\mathcal{D}}$ in $\mathcal{S}{+}R$ with $\mathsf{assm}(\tilde{\mathcal{D}}) = \emptyset$ and with $\mathsf{concl}(\tilde{\mathcal{D}}) = A$ a derivation $\tilde{\mathcal{D}}'$ in $\mathcal{S}$ with $\mathsf{assm}(\tilde{\mathcal{D}}') = \emptyset$ and with $\mathsf{concl}(\tilde{\mathcal{D}}') = A$ can be found as described above.

Thus $\mathcal{S}{+}R$ does not possess more theorems than $\mathcal{S}$.

For showing "$\Leftarrow$" in the assertion (i) of the proposition, we assume that every theorem of $\mathcal{S}{+}R$ is also a theorem of $\mathcal{S}$ and will show that R is admissible in $\mathcal{S}$. We have to prove (3.8).

For this, we let $\alpha \in \mathit{Apps}_R$ be arbitrary with the property that all premises of α are theorems of $\mathcal{S}$. We have to show that also the conclusion of α is a theorem of $\mathcal{S}$. We will first consider the case $\mathsf{arity}(\alpha) = 0$ and then the case $\mathsf{arity}(\alpha) \in \omega\backslash\{0\}$.

If $\mathsf{arity}(\alpha) = 0$, then

$$\frac{}{\mathsf{concl}(\alpha)}$$

is a derivation in $\mathcal{S}{+}R$ consisting of this zero-premise application of R, and hence $\mathsf{concl}(\alpha)$ is a theorem of $\mathcal{S}{+}R$. By our assumption that every theorem of $\mathcal{S}{+}R$ is also a theorem of $\mathcal{S}$ it follows that then also $\mathsf{concl}(\alpha)$ is a theorem of $\mathcal{S}$.

If $\mathsf{arity}(\alpha) = n \in \omega\backslash\{0\}$ is the case, we let $A, A_1, \ldots, A_n \in \mathit{Fo}_{\mathcal{S}}$ be such that $\mathsf{prem}(\alpha) = (A_1, \ldots, A_n)$ and $\mathsf{concl}(\alpha) = A$. Since by our assumption about α made above $A_1, \ldots, A_n$ are theorems of $\mathcal{S}$, there exist derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\mathcal{S}$ with $\mathsf{assm}(\mathcal{D}_i) = \emptyset$ and $\mathsf{concl}(\mathcal{D}_i) = A_i$ for all $i \in \{1, \ldots, n\}$. As a consequence the derivation $\mathcal{D}$ of the form

$$\frac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n\end{matrix}}{A}$$

(with a bottommost application of R) is a derivation in $\mathcal{S}{+}R$ with $\mathsf{assm}(\mathcal{D}) = \emptyset$ and with $\mathsf{concl}(\mathcal{D}) = A$. Hence A is a theorem of $\mathcal{S}{+}R$. Due to our assumption that every theorem of $\mathcal{S}{+}R$ is also a theorem of $\mathcal{S}$ it follows now that $\mathsf{concl}(\alpha) = A = \mathsf{concl}(\mathcal{D})$ is also a theorem of $\mathcal{S}$.

We have thereby demonstrated that for every application α of R with the property that all premises of α are theorems of $\mathcal{S}$ it also holds that the conclusion of α is a theorem of $\mathcal{S}$. Therefore $\mathit{Th}(\mathcal{S})$ is closed under applications of R, or, what is the same, (3.8) holds, which shows that R is admissible in $\mathcal{S}$.

(b) We assume that R is derivable in $\mathcal{S}$ and will show that R is also admissible in $\mathcal{S}$. Thus we know that (3.5) holds and have to prove that (3.8) holds as well. Given our knowledge of (3.5), (3.8) follows if we can prove

$$\begin{aligned}(\forall \alpha \in \mathit{Apps}_R)\, \big\{\, & \mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \quad \Longrightarrow \\ & \Longrightarrow \quad \big[\, (\forall A \in \mathrm{set}(\mathsf{prem}(\alpha)))\, [\, \vdash_{\mathcal{S}} A \,] \;\Rightarrow\; \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \,\big]\big\}\,. \end{aligned} \tag{B.1}$$

Therefore it remains to show (B.1). For this we let $\alpha \in \mathit{Apps}_R$ be arbitrary such that

$$\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \quad \& \quad (\forall A \in \mathrm{set}(\mathsf{prem}(\alpha)))\,[\,\vdash_{\mathcal{S}} A\,] \tag{B.2}$$

holds and will prove

$$\vdash_{\mathcal{S}} \mathsf{concl}(\alpha)\ . \tag{B.3}$$

In case that $\mathsf{arity}(\alpha) = 0$, there is nothing to prove since then (B.2) contains the assertion $\emptyset \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$ and hence (B.3).

We suppose now that $\mathsf{arity}(\alpha) \in \omega \backslash \{0\}$ holds and let $A, A_1, \ldots, A_n \in Fo_{\mathcal{S}}$ be such that

$$\mathsf{prem}(\alpha) = (A_1, \ldots, A_n) \quad \text{and} \quad \mathsf{concl}(\alpha) = A\ .$$

Due to (B.2), there exist derivations $\mathcal{D}_\alpha$ and $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\mathcal{S}$ such that

$$\mathsf{assm}(\mathcal{D}_\alpha) \subseteq \mathrm{set}(\mathsf{prem}(\alpha)) = \mathrm{set}((A_1, \ldots, A_n))\ , \quad \mathsf{concl}(\mathcal{D}_\alpha) = \mathsf{concl}(\alpha) = A\ ,$$
$$\mathsf{assm}(\mathcal{D}_i) = \emptyset, \quad \mathsf{concl}(\mathcal{D}_i) = A_i \quad (\text{for all } i \in \{1, \ldots, n\})$$

hold. If $\mathsf{assm}(\mathcal{D}_\alpha) = \emptyset$, then the conclusion $\mathsf{concl}(\alpha)$ of $\mathcal{D}_\alpha$ is a theorem of $\mathcal{S}$ and hence (B.3) holds. If $\mathsf{assm}(\mathcal{D}_\alpha) \neq \emptyset$, then $\mathcal{D}_\alpha$ can be represented as symbolic prooftree of the form

$$\begin{array}{c} (A_{i_1}) \;\;\ldots\;\; (A_{i_k}) \\ \mathcal{D}_\alpha \\ A \end{array}$$

for some $k \in \omega \backslash \{0\}$ and $i_1, \ldots, i_k \in \omega$ with $1 \leq i_1, \ldots, i_k \leq n$ such that

$$\mathrm{set}(\mathsf{prem}(\alpha)) = \mathrm{set}((A_{i_1}, \ldots, A_{i_k}))$$

holds and such that the expressions $(A_{i_1}), \ldots, (A_{i_k})$ denote single occurrences of assumptions in $\mathcal{D}_\alpha$ ordered from left to right. It follows that the symbolic prooftree

$$\begin{array}{c} \mathcal{D}_{i_1} \qquad\quad \mathcal{D}_{i_k} \\ (A_{i_1}) \;\;\ldots\;\; (A_{i_k}) \\ \mathcal{D}_\alpha \\ A \end{array}$$

describes a derivation $\mathcal{D}'$ in $\mathcal{S}$ that arises from $\mathcal{D}_\alpha$ by substituting the derivations $\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_k}$ respectively for the occurrences of assumptions $(A_{i_1}), \ldots, (A_{i_k})$ in $\mathcal{D}_\alpha$. For $\mathcal{D}'$ we find that

$$\mathsf{assm}(\mathcal{D}') = \bigcup_{j=1}^{k} \mathsf{assm}(\mathcal{D}_{i_j}) = \emptyset \quad \text{and} \quad \mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D}_\alpha) = A = \mathsf{concl}(\alpha)$$

holds. Hence the conclusion $\mathsf{concl}(\alpha)$, i.e. the formula A, of the $\mathcal{S}$-derivation $\mathcal{D}'$ is a theorem of $\mathcal{S}$, and thus we have shown (B.3) also in this case.

We have shown that for arbitrary $\alpha \in \mathit{Apps}_R$ with (B.2) also (B.3) holds, and hence we have proven (B.1). But that was what we needed for a demonstration of (3.8), i.e. for establishing that R is admissible in $\mathcal{S}$.

As a proof for the claim in item (ii) of the proposition that rule admissibility does not imply rule derivability in general, we give the following counterexample of a rule $R_{B.C}$ and an AHS $\mathcal{S}_A$: Let $\mathit{Fo} = \{A, B, C\}$ be a three-element set, and let $\bullet$ be an arbitrary set. We consider the two rules $R_{.A}$ and $R_{B.C}$ that have each only one application, namely

$$\frac{}{A}\,R_{.A} \qquad\qquad \text{and} \quad \frac{B}{C}\,R_{B.C}$$

(although we argue about rule derivability and admissibility in AHS's here, we have annotated these two applications, for better identification, by the "names" R_A and $R_{B.C}$ of the respective rules as if we were to consider these applications as derivations in an n-AHS). More precisely, we let $R_{.A} = \langle \{\bullet\}, \mathsf{prem}_{.A}, \mathsf{concl}_{.A} \rangle$

$$\mathsf{prem}_{.A}(\bullet) = () \quad \text{and} \quad \mathsf{concl}_{.A}(\bullet) = A\ ,$$

and we let $R_{B.C} = \langle \{\bullet\}, \mathsf{prem}_{B.C}, \mathsf{concl}_{B.C} \rangle$

$$\mathsf{prem}_{B.C}(\bullet) = (B) \quad \text{and} \quad \mathsf{concl}_{B.C}(\bullet) = C\ .$$

Furthermore, we let $\mathcal{S}_A$ be the AHS $\langle \mathit{Fo}, \{R_{.A}\} \rangle$. Then $\mathit{Th}(\mathcal{S}_A) = \{A\}$ holds, and hence $R_{B.C}$ is admissible in $\mathcal{S}_A$: This is a consequence of the fact that for the single application $\bullet$ of $R_{B.C}$ it holds that its premise B is not a theorem of $\mathcal{S}_A$. However, $R_{B.C}$ is not derivable in $\mathcal{S}_A$, since obviously $B \vdash_{\mathcal{S}} C$ is not a relative derivability statement of $\mathcal{S}_A$.

(c) For showing assertion (iii) of the proposition, let $\mathcal{S}'$ be an extension by enlargement of $\mathcal{S}$ and let R be derivable in $\mathcal{S}$. We have to show that R is also derivable in $\mathcal{S}'$. By definition of rule derivability, we find that for all applications $\alpha \in \mathit{Apps}_R$

$$\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$$

holds. By Lemma 2.12, this implies that for all $\alpha \in \mathit{Apps}_R$ also

$$\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}'} \mathsf{concl}(\alpha)$$

holds, which means that R is derivable in $\mathcal{S}'$.

□

Proof of Proposition 3.3. We will first prove that m-derivability implies s-derivability. That s-derivability implies derivability can be shown analogously.

Let $\mathcal{S}$ be an AHS, and let R be a rule on Fo that is m-derivable in $\mathcal{S}$. For the purpose of showing that R is s-derivable in $\mathcal{S}$, let $\alpha \in \mathit{Apps}_R$ be arbitrary. Since R is m-derivable in $\mathcal{S}$, it holds that

$$\mathrm{mset}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{m})}_{\mathcal{S}} \mathsf{concl}(\alpha)\ .$$

By Proposition 2.9 and the obvious fact $\mathrm{set}(\mathrm{mset}(\mathsf{prem}(\alpha))) = \mathrm{set}(\mathsf{prem}(\alpha))$, the relative derivability statement

$$\mathrm{set}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{s})}_{\mathcal{S}} \mathsf{concl}(\alpha) \tag{B.4}$$

follows. In this way we have shown that (B.4) holds for all applications $\alpha \in \mathit{Apps}_R$, and hence that R is s-derivable in $\mathcal{S}$.

As a proof for the claim of Proposition 3.3 that m-derivability does not imply s-derivability in general, and that s-derivability does not imply derivability in general, we give the following counterexamples to these two implications: Let $\mathit{Fo} = \{A, B\}$ be a two-element set, and let $\bullet$ be an arbitrary set. We consider the three rules $R_{A.B}, R_{AA.B}$ and $R_{AB.B}$ with single applications of the respective form

$$\frac{A}{B}\, R_{A.B} \qquad\qquad \frac{A \qquad A}{B}\, R_{AA.B} \qquad\qquad \frac{A \qquad B}{B}\, R_{AB.B}$$

(we did allow ourselves to use n-AHS-like "name labels" for these applications here). More precisely, we let $R_{A.B} = \langle \{\bullet\}, \mathsf{prem}_{A.B}, \mathsf{concl}_{A.B} \rangle$ with

$$\mathsf{prem}_{A.B}(\bullet) = (A) \quad \text{and} \quad \mathsf{concl}_{A.B}(\bullet) = B\ ,$$

we let $R_{AA.B} = \langle \{\bullet\}, \mathsf{prem}_{AA.B}, \mathsf{concl}_{AA.B} \rangle$

$$\mathsf{prem}_{AA.B}(\bullet) = (A, A) \quad \text{and} \quad \mathsf{concl}_{AA.B}(\bullet) = B\ ,$$

and we let $R_{AB.B} = \langle \{\bullet\}, \mathsf{prem}_{AB.B}, \mathsf{concl}_{AB.B} \rangle$ with

$$\mathsf{prem}_{AB.B}(\bullet) = (A, B) \quad \text{and} \quad \mathsf{concl}_{AB.B}(\bullet) = B\ .$$

And furthermore, we let $\mathcal{S}$ be the AHS $\langle \mathit{Fo}, \emptyset, \{R_{AA.B}\} \rangle$. Then it is easy to verify that $R_{A.B}$ is derivable and s-derivable, but not m-derivable in $\mathcal{S}$; and that $R_{AB.A}$ is derivable, but neither s-derivable nor m-derivable in $\mathcal{S}$.

The claim of Proposition 3.3 that true statements arise if in Proposition 3.2 (iii) "derivable" is replaced by "s-derivable" and "m-derivable", respectively, can be shown analogously as in item (c) of the proof of Proposition 3.2 above: Both statements follow from Lemma 2.12, which asserts that every holding relative derivability statement in an AHS $\mathcal{S}$ with respect to either $\vdash^{\mathbf{(s)}}_{\mathcal{S}}$ or $\vdash^{\mathbf{(m)}}_{\mathcal{S}}$ is also a holding relative derivability statement in every extension by enlargement $\mathcal{S}'$ of $\mathcal{S}$ with respect to $\vdash^{\mathbf{(s)}}_{\mathcal{S}'}$, or respectively, with respect to $\vdash^{\mathbf{(m)}}_{\mathcal{S}'}$.

And the claim of Proposition 3.3 that s-derivability and m-derivability imply admissibility, but that each of the implications in the opposite direction is not true in general, clearly follows by Proposition 3.2 (ii) and the above explained facts that s-derivability and m-derivability imply derivability.

□

Proof of Proposition 3.4. Let $\mathcal{S}$ be an AHS or an n-AHS, and let R be a rule on $\mathit{Fo}_{\mathcal{S}}$ with $R = \langle \mathit{Apps}, \mathsf{prem}, \mathsf{concl} \rangle$. We will show statements (i) and (ii) of the proposition in items (a) and (b) below.

(a) For statement (i) of the proposition, we assume that R is admissible in $\mathcal{S}$ and will show (3.9): In this logical equivalence, "$\Leftarrow$" is obvious, since the hypotheses of this implication asserts the existence of an application α of R that cannot be mimicked by a derivation in $\mathcal{S}$. We are left to show the implication "$\Rightarrow$". Suppose that R is not derivable in $\mathcal{S}$. Then there must exist an application $\alpha \in \mathit{Apps}_R$ such that $\text{set}(\mathsf{prem}(\alpha)) \nvdash_{\mathcal{S}} \mathsf{concl}(\alpha)$; let such an application α be chosen arbitrarily. Now we are done if we can show that among the premises of α there is at least one formula that is not a theorem of $\mathcal{S}$. This must indeed be the case because otherwise, due to the fact that R is admissible in $\mathcal{S}$, also $\mathsf{concl}(\alpha)$ would be a theorem of $\mathcal{S}$, i.e. $\vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$ and therefore also $\text{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$ would hold, in contradiction with our choice of the application α.

(b) For a proof of assertion (ii) of the proposition, we let $R_0 = \langle \mathit{Apps}_0, \mathsf{prem}_0, \mathsf{concl}_0 \rangle$, where Apps_0 is defined as in (3.10) and prem_0 and concl_0 are the respective restrictions of the premise function prem and of the conclusion function concl of R to the subset Apps_0 of the set of applications of R. We have to show (3.11). It is possible to demonstrate that this is essentially a consequence of assertion (i) of the proposition. But since such a proof uses an indirect argument, we give a direct demonstration here.

For the implication "$\Rightarrow$" in (3.11), we notice the following chain of implications and logical equivalences:

$$
\begin{aligned}
&R \text{ is admissible in } \mathcal{S} \iff \\
&\iff (\forall \alpha \in \mathit{Apps}) \left[(\forall A \in \text{set}(\mathsf{prem}(\alpha))) \, [\, \vdash_{\mathcal{S}} A \,] \;\Rightarrow\; \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\iff (\forall \alpha \in \mathit{Apps}_0) \left[\vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\implies (\forall \alpha \in \mathit{Apps}_0) \left[\text{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\iff (\forall \alpha \in \mathit{Apps}_0) \left[\text{set}(\mathsf{prem}_0(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}_0(\alpha) \right] \\
&\iff R_0 \text{ is derivable in } \mathcal{S} \,.
\end{aligned}
$$

The implications "$\Rightarrow$" are hereby respectively justified, following this chain of equivalences and implications from top to bottom, by the definition of "R is admissible in $\mathcal{S}$", by the definition of the set Apps_0 of applications of R_0 in (3.10), by the definition of the conseqence relation $\vdash_{\mathcal{S}}$, by the definition of R_0, and by the definition of "R_0 is derivable in $\mathcal{S}$", respectively.

For the implication "$\Leftarrow$" in (3.11), we argue along a slightly different chain of logical implications and equivalences:

$$
\begin{aligned}
&R_0 \text{ is derivable in } \mathcal{S} \iff \\
&\iff (\forall \alpha \in \mathit{Apps}_0) \left[\text{set}(\mathsf{prem}_0(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}_0(\alpha) \right] \\
&\iff (\forall \alpha \in \mathit{Apps}_0) \left[\text{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\implies (\forall \alpha \in \mathit{Apps}_0) \left[(\forall A \in \text{set}(\mathsf{prem}(\alpha))) \, [\, \vdash_{\mathcal{S}} A \,] \;\Rightarrow\; \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\iff (\forall \alpha \in \mathit{Apps}) \left[(\forall A \in \text{set}(\mathsf{prem}(\alpha))) \, [\, \vdash_{\mathcal{S}} A \,] \;\Rightarrow\; \vdash_{\mathcal{S}} \mathsf{concl}(\alpha) \right] \\
&\iff R \text{ is admissible in } \mathcal{S} \,.
\end{aligned}
$$

Hereby the implications "⇒" are respectively justified by the definition of "R_0 is derivable in $\mathcal{S}$", by the definitions of R_0 and R, the fact that derivability of R_0 in $\mathcal{S}$ implies admissibility of R_0 in $\mathcal{S}$ by Proposition 3.2 (ii), the definition of the set Apps_0 in (3.10), and the definition of "R is admissible in $\mathcal{S}$".

□

Proof of Proposition 3.5. Let $\mathcal{S}$ be an AHS with formula set Fo and let R be a rule on Fo. We will prove the equivalence of the statements (i), (ii) and (iii) in the proposition by showing the three implications (i) ⇒ (iii), (iii) ⇒ (ii) and (ii) ⇒ (i).

(a) The implication (i) ⇒ (iii) is an immediate consequence of items (ii) and (iii) in Proposition 3.2.

(b) The implication (iii) ⇒ (ii) is obvious because every AHS of the form $\mathcal{S}+\Sigma$, where $\Sigma \in \mathcal{P}(Fo)$, is also an extension by enlargement of $\mathcal{S}$.

(c) To prove (ii) ⇒ (i), suppose that R is admissible in every extension of $\mathcal{S}$ of the form $\mathcal{S}+\Sigma$ for some $\Sigma \in \mathcal{P}(Fo)$. Let α be an arbitrary application of R, and let $\Sigma = \mathrm{set}(\mathsf{prem}(\alpha))$. Clearly, $\vdash_{\mathcal{S}+\Sigma} A$ holds for all $A \in \mathrm{set}(\mathsf{prem}(\alpha))$, and hence, since R is admissible in $\mathcal{S}+\Sigma$, it follows that $\vdash_{\mathcal{S}+\Sigma} \mathsf{concl}(\alpha)$. From this we conclude by Lemma 2.14 that $\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$. Since α was an arbitrary application of R in this argument, we have proved that $\mathrm{set}(\mathsf{prem}(\alpha)) \vdash_{\mathcal{S}} \mathsf{concl}(\alpha)$ holds for all applications of R, which shows (3.5) and hence that R is derivable in $\mathcal{S}$.

□

Appendix C: Proofs, auxiliary statements for Section 4

Proof of Proposition 4.3. Reflexivity and transitivity of $\preceq_{th}$, $\preceq_{rth}$, $\preceq^{(\mathrm{s})}_{rth}$ and $\preceq^{(\mathrm{m})}_{rth}$ follow immediately from the definitions of these relations. As a consequence, the relations $\sim_{th}$, $\sim_{rth}$, $\sim^{(\mathrm{s})}_{rth}$ and $\sim^{(\mathrm{m})}_{rth}$ are reflexive and transitive, too, and due to their definitions, obviously also symmetric.

Proof of Proposition 4.4. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be arbitrary AHS's. We will only prove (4.3), since (4.4) is an obvious consequence of (4.3) due to the way in which $\sim_{rth}$ and $\sim_{th}$ have been defined in terms of $\preceq_{rth}$ and $\preceq_{th}$.

Both directions of the implications in (4.3) are consequences of Proposition 2.9, and can be shown rather similarly.

For showing "⇒" in (4.3), we assume that $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$ holds. This means that $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ holds. We let a set $\Sigma \subseteq Fo_{\mathcal{S}_1}$ of formulas be arbitrary and need to show that $\mathcal{S}_1+\Sigma \preceq_{th} \mathcal{S}_2+\Sigma$ holds. Since we find $Fo_{\mathcal{S}_1+\Sigma} = Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2} = Fo_{\mathcal{S}_2+\Sigma}$, we only have to prove that every theorem of $\mathcal{S}_1+\Sigma$ is also a theorem of $\mathcal{S}_2+\Sigma$. Let $A \in Fo_{\mathcal{S}_1}$ be an arbitrary theorem of $\mathcal{S}_1+\Sigma$. Then we have $\vdash_{\mathcal{S}_1+\Sigma} A$, which, due to Lemma 2.14, entails $\Sigma \vdash_{\mathcal{S}_1} A$. From this also $\Sigma \vdash_{\mathcal{S}_2} A$ follows because of our assumption $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$. Using Lemma 2.14 again, we find that also $\vdash_{\mathcal{S}_2+\Sigma} A$ holds, i.e. that A is a theorem of $\mathcal{S}_2+\Sigma$.

For showing "⇐" in (4.3), we assume that that $\mathcal{S}_1{+}\Sigma \preceq_{th} \mathcal{S}_2{+}\Sigma$ holds for all $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ and will prove that $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$ holds. Since we get $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ as a consequence of $\mathcal{S}_1 \preceq_{th} \mathcal{S}_2$, it remains to be shown that every relative derivability statement in $\mathcal{S}_1$ (with respect to $\vdash_{\boldsymbol{\mathcal{S}_1}}$) is also a relative derivability statement in $\mathcal{S}_2$ (with respect to $\vdash_{\boldsymbol{\mathcal{S}_2}}$). For demonstrating this, let $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ and $A \in Fo_{\mathcal{S}_1}$ be arbitrary such that $\Sigma \vdash_{\mathcal{S}_1} A$ holds. By Lemma 2.14 we get that $\vdash_{\mathcal{S}_1+\Sigma} A$ holds. Since, due to our assumption, $\mathcal{S}_1{+}\Sigma \preceq_{th} \mathcal{S}_2{+}\Sigma$ holds, this implies $\vdash_{\mathcal{S}_2+\Sigma} A$, which, again by using Lemma 2.14, entails $\Sigma \vdash_{\mathcal{S}_2} A$. Since Σ and A have been arbitrary in this argument, we have hereby shown the assertion $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$.

□

Proof of Lemma 4.8. In (a) below we will first show that items (ii) and (iii) of the lemma follow easily from item (i), which will then be proven in (b) below.

(a) Item (ii) of the lemma follows from item (i) due to two fact that hold for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ with the same set of formulas: Firstly it is the case that $\mathcal{S}_2$ is a conservative extension of $\mathcal{S}_1$ if and only if $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems; and secondly, $Th(\mathcal{S}_2) \supseteq Fo_{\mathcal{S}_1}$ holds if and only if $\mathcal{S}_2$ is inconsistent.

Item (iii) of the lemma follows easily from item (ii): Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be arbitrary AHS's. For showing "⇐" in (4.6), we assume $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same set of formulas and hence we can apply (ii) to $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ and to $\mathcal{S}_2 \sim_{th} \mathcal{S}_1$ ($\sim_{th}$ is symmetric by Proposition 4.3). In this way we find that $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_{adm} \mathcal{S}_1$ hold. Hence we have shown $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$. For showing "⇒" in (4.6), we now assume $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$. Then $Fo_{\mathcal{S}_1} = Fo_{\mathcal{S}_2}$ is the case and hence (ii) is applicable to $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$ and to $\mathcal{S}_2 \preceq_{adm} \mathcal{S}_1$. In this way we either obtain $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ directly, or find that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are inconsistent, which again entails $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$.

(b) We show item (i) of the lemma. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be arbitrary AHS's. The implication "⇐" in (4.5) can be verified in a straightforward manner; therefore we will not demonstrate this here. For proving "⇒" in (4.5), we assume that $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$ and $Th(\mathcal{S}_2) \not\supseteq Fo_{\mathcal{S}_1}$ holds and will show that $\mathcal{S}_2$ is a conservative extension of $\mathcal{S}_1$. We first notice that $\mathcal{S}_2$ is an extension of $\mathcal{S}_1$ due to the definition of the assumption $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$. Hence it remains to be shown that the extension $\mathcal{S}_2$ of $\mathcal{S}_1$ is conservative.

Because of our second assumption, the set $Fo_{\mathcal{S}_2} \setminus Th(\mathcal{S}_1)$ is non-empty. We fix a formula B in this set for the following argument. Due to this choice, B is not a theorem of $\mathcal{S}_2$, and because $\mathcal{S}_2$ is an extension of $\mathcal{S}_1$, B is not a theorem of $\mathcal{S}_1$ either. For showing that $\mathcal{S}_2$ is conservative as an extension of $\mathcal{S}_1$, let A be an arbitrary formula of $\mathcal{S}_1$ that is a theorem of $\mathcal{S}_2$; we will demonstrate that A is then also a theorem of $\mathcal{S}_1$. Let R be the rule on $Fo_{\mathcal{S}_2}$ that has only the single application

$$\frac{A}{B} \;.$$

Since $B \in Fo_{\mathcal{S}_1}$, R is also a rule on $Fo_{\mathcal{S}_1}$. Since $\vdash_{\mathcal{S}_2} A$ and $\not\vdash_{\mathcal{S}_2} B$ hold, R is not admissible in $\mathcal{S}_2$. Due to our assumption $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$, this entails that R

cannot be admissible in $\mathcal{S}_1$ either. Since we have seen that B is not a theorem of $\mathcal{S}_1$, it follows that A must be a theorem of $\mathcal{S}_1$. In this way we have proved that every formula of $\mathcal{S}_1$ that is a theorem of $\mathcal{S}_2$ is also a theorem of $\mathcal{S}_1$. Hence we can conclude now that $\mathcal{S}_2$ is a conservative extension of $\mathcal{S}_1$.

□

Example C.1 (The relation $\preceq_{r/adm}$ is not transitive). Let $Fo = \{A, B, C\}$ be a three-element set and let $R_{A.B}$ and $R_{A.C}$ be the rules on Fo each of which has only one application, namely

$$\frac{A}{B}\,R_{A.B} \qquad\qquad \frac{A}{C}\,R_{A.C}$$

(hereby we have also used $R_{A.B}$ and $R_{A.C}$ informally as "names" of these rules to denote their respective application as if we considered derivations in n-AHS's). Furthermore let three AHS's $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be determined by their axioms and rules according to the following table:

AHS	set of axioms	set of rules
$\mathcal{S}_1$	$\emptyset$	$\{R_{A.C}\}$
$\mathcal{S}_2$	$\emptyset$	$\{R_{A.B}\}$
$\mathcal{S}_3$	$\{A\}$	$\{R_{A.B}\}$

Then it is easy to see that

$$\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \quad \& \quad \mathcal{S}_2 \preceq_{r/adm} \mathcal{S}_3 \tag{C.1}$$

holds: $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ follows from the fact that $Th(\mathcal{S}_2) = \emptyset$ (and hence the single rule $R_{A.C}$ of $\mathcal{S}_1$ is trivially admissible in $\mathcal{S}_2$) and $\mathcal{S}_2 \preceq_{r/adm} \mathcal{S}_3$ is obvious since the single rule of $\mathcal{S}_2$, $R_{A.B}$, is also a rule of $\mathcal{S}_3$. But furthermore we find that

$$\mathcal{S}_1 \not\preceq_{r/adm} \mathcal{S}_3 \tag{C.2}$$

is the case: The rule $R_{A.C}$ of $\mathcal{S}_1$ is not admissible in $\mathcal{S}_3$, because, as a consequence of $Th(\mathcal{S}_3) = \{A, B\}$, the theory of $\mathcal{S}_3$ is not closed under applications of $R_{A.C}$.

Now (C.1) and (C.2) clearly demonstrate that the relation $\preceq_{r/adm}$ on $\mathfrak{H}$ is not transitive.

■

Example C.2 (The relations $\preceq^{(s)}_{r/der}$ and $\sim^{(s)}_{r/der}$ are not transitive). We will give a counterexample against the transitivity of both the relations $\preceq^{(s)}_{r/der}$ and $\sim^{(s)}_{r/der}$. For this, we choose a 4-element set $Fo = \{A, B, C_1, C_2\}$. We will consider three AHS's that have Fo as their sets of formulas, possess no axioms, and whose rules are among the five ones $R_1, R_2, R_{3a}, R_{3b}, R_4$, which have each only one application, namely the ones of the following list:

$$\frac{C_1}{A}\,R_1 \qquad \frac{C_2}{A}\,R_2 \qquad \frac{A \quad A}{B}\,R_{3a} \qquad \frac{A}{B}\,R_{3b} \qquad \frac{C_1 \quad C_2}{B}\,R_4$$

(here we allowed to use R_1, R_2, R_{3a}, R_{3b} and R_4 as respective "names" for these rules as if we considered n-AHS-derivations respectively corresponding to these rule applications). Relying on these rules, we let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be AHS's with set Fo of formulas that possess no axioms and that are determined by their respective set of rules as described by the following table:

AHS	set of axioms	set of rules
$\mathcal{S}_1$	$\emptyset$	$\{R_1, R_2, R_{3a}, R_4\}$
$\mathcal{S}_2$	$\emptyset$	$\{R_1, R_2, R_{3a}\}$
$\mathcal{S}_3$	$\emptyset$	$\{R_1, R_2, R_{3b}\}$

Now it is easy to check that the following holds:

$$\mathcal{S}_1 \sim^{(\mathrm{s})}_{r/der} \mathcal{S}_2 \quad \& \quad \mathcal{S}_2 \sim^{(\mathrm{s})}_{r/der} \mathcal{S}_3 \ . \tag{C.3}$$

The only not entirely obvious assertion to check hereby is that the rule R_4 of $\mathcal{S}_1$ is actually s-derivable in $\mathcal{S}_2$: But this a consequence of the existence of the derivation

$$\frac{\dfrac{C_1}{A}\,R_1 \qquad \dfrac{C_2}{A}\,R_2}{B}\,R_{3a} \tag{C.4}$$

in $\mathcal{S}_2$. We furthermore notice that

$$\mathcal{S}_1 \not\precsim^{(\mathrm{s})}_{r/der} \mathcal{S}_3 \tag{C.5}$$

holds due to the fact that the rule R_4 is not s-derivable in $\mathcal{S}_3$: This is a consequence of the facts (i) that every relative derivability statement holding in $\mathcal{S}_3$ with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_3}}$ must be of the form $\{D\} \vdash^{(\mathrm{s})}_{\mathcal{S}_3} E$ for some $D, E \in Fo$, since $\mathcal{S}_3$ has only one-premise rules, whereas (ii) $\{C_1, C_2\} \vdash^{(\mathrm{s})}_{\mathcal{S}_1} B$ is a relative derivability statement that holds in $\mathcal{S}_1$ with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_1}}$ due to the presence of R_4 in $\mathcal{S}_1$.

Now from (C.3) and (C.5) it follows immediately, in view of the definition of $\sim^{(\mathrm{s})}_{r/der}$ in terms of $\precsim^{(\mathrm{s})}_{r/der}$, that both of the relations $\precsim^{(\mathrm{s})}_{r/der}$ and $\sim^{(\mathrm{s})}_{r/der}$ are not transitive.

For later reference, we also note

$$\mathcal{S}_2 \not\precsim^{(\mathrm{s})}_{rth} \mathcal{S}_3 \ , \quad \text{and hence also} \quad \mathcal{S}_2 \not\sim^{(\mathrm{s})}_{rth} \mathcal{S}_3 \ , \tag{C.6}$$

which follows from the observation (i) above, and from the fact that $\{C_1, C_2\} \vdash^{(\mathrm{s})}_{\mathcal{S}_1} B$ is also a relative derivability statement that holds in $\mathcal{S}_2$ with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_2}}$, due to the derivation (C.4) in $\mathcal{S}_2$.

■

Proof of Theorem 4.12. The statement of Theorem 4.12 consists of the assertions of Proposition C.3, Theorem C.4 and Theorem C.5, all of which are stated, and then successively proved, below. □

Proposition C.3 ('Vertical interrelations' between defined incl. relations). *The following containment assertions hold between the inclusion relations on $\mathfrak{H}$ defined in Definitions 4.1, Definition 4.5 and Definition 4.9:*

$$\preceq^{(m)}_{rth} \;\subseteq\; \preceq^{(s)}_{rth} \;\subseteq\; \preceq_{rth} \;\subseteq\; \preceq_{th} \;, \tag{C.7}$$

$$\preceq^{(m)}_{der} \;\subseteq\; \preceq^{(s)}_{der} \;\subseteq\; \preceq_{der} \;, \tag{C.8}$$

$$\preceq^{(m)}_{r/der} \;\subseteq\; \preceq^{(s)}_{r/der} \;\subseteq\; \preceq_{r/der} \;\subseteq\; \preceq_{r/adm} \;. \tag{C.9}$$

For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, these assertions justify the downwards-pointing, vertical implication arrows in an 'interrelation prism' as shown in Figure 3 (a).

Theorem C.4 ('Horizontal interrelations' between defined incl. relations).

(i) For the pre-order relations $\preceq_{th}$, $\preceq_{r/adm}$ and $\preceq_{adm}$ on the class $\mathfrak{H}$ of all abstract Hilbert systems it holds: $\preceq_{adm}$ is properly contained in $\preceq_{r/adm}$, which in turn is properly contained in $\preceq_{th}$. More formally, it holds:

$$\preceq_{adm} \;\subsetneqq\; \preceq_{r/adm} \;\subsetneqq\; \preceq_{th} \;. \tag{C.10}$$

(ii) The pre-order relations $\preceq_{r/der}$, $\preceq_{der}$ and $\preceq_{rth}$ on $\mathfrak{H}$ coincide, and the same is true for the pre-order relations $\preceq^{(m)}_{r/der}$, $\preceq^{(m)}_{der}$ and $\preceq^{(m)}_{rth}$. More formally, the following two assertions hold:

$$\preceq_{r/der} \;=\; \preceq_{der} \;=\; \preceq_{rth} \;, \tag{C.11}$$

$$\preceq^{(m)}_{r/der} \;=\; \preceq^{(m)}_{der} \;=\; \preceq^{(m)}_{rth} \;. \tag{C.12}$$

(iii) The pre-order relations $\preceq^{(s)}_{der}$ and $\preceq^{(s)}_{rth}$ on $\mathfrak{H}$ coincide. Furthermore, both relations are properly *contained in the relation $\preceq^{(s)}_{r/der}$. Thus it holds:*

$$\preceq^{(s)}_{r/der} \;\supsetneqq\; \preceq^{(s)}_{der} \;=\; \preceq^{(s)}_{rth} \;. \tag{C.13}$$

For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, the assertions in (i), (ii) and (iii) above guarantee the horizontal implications and logical equivalences in an 'interrelation prism' of the form shown in Figure 3 (a).

Theorem C.5 *Each of the containment assertions in* (C.7), (C.8) *and* (C.9) *of Proposition C.3 are proper inclusions, i.e. each inclusion symbol $\subseteq$ used there can* *be replaced by the symbol* $\subsetneq$. *And furthermore, the relation* $\preceq_{adm}$ *is not included in any of the relations* $\preceq_{der}$, $\preceq_{der}^{(s)}$ *and* $\preceq_{der}^{(m)}$, *nor are any of* $\preceq_{der}$, $\preceq_{der}^{(s)}$ *and* $\preceq_{der}^{(m)}$ *included in* $\preceq_{adm}$.

Proof of Proposition C.3. The containment assertions in (C.9) are an immediate consequence of Proposition 3.3, the facts that m-derivability implies s-derivability and that s-derivability implies derivability, and of Proposition 3.2 (ii), the fact that derivability implies admissibility.

The containment assertions in (C.7) follow easily from Proposition 2.9: As an example, we prove that $\preceq_{rth}^{(s)} \subseteq \preceq_{rth}$: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be arbitrary AHS's such that $\mathcal{S}_1 \preceq_{rth}^{(s)} \mathcal{S}_2$. For showing $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$, we let $A \in Fo_{\mathcal{S}_1}$ and $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ be arbitrary such that $\Sigma \vdash_{\mathcal{S}_1} A$. By the definitions of the consequence relations $\vdash_{\boldsymbol{\mathcal{S}_1}}$ and $\vdash_{\boldsymbol{\mathcal{S}_1}}^{\mathbf{(s)}}$, we find that $\Sigma_0 \vdash_{\mathcal{S}_1}^{(s)} A$, holds for some $\Sigma_0 \subseteq \Sigma$. Due to $\mathcal{S}_1 \preceq_{rth}^{(s)} \mathcal{S}_2$, it follows that $\Sigma_0 \vdash_{\mathcal{S}_2}^{(s)} A$ holds, which entails $\Sigma \vdash_{\mathcal{S}_2} A$ because of the definition of $\vdash_{\boldsymbol{\mathcal{S}_2}}$. Therefore, every relative derivability statement with respect to $\vdash_{\boldsymbol{\mathcal{S}_1}}$ is also a relative derivability statement with respect to $\vdash_{\boldsymbol{\mathcal{S}_2}}$. Since the assertion $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ is part of our assumption $\mathcal{S}_1 \preceq_{rth}^{(s)} \mathcal{S}_2$, we have proven $\mathcal{S}_1 \preceq_{rth} \mathcal{S}_2$.

For the containment assertions (C.8), we will only show $\preceq_{der}^{(m)} \subseteq \preceq_{der}^{(s)}$ in detail; $\preceq_{der}^{(s)} \subseteq \preceq_{der}$ can be proven in a very similar way. To show $\preceq_{der}^{(m)} \subseteq \preceq_{der}^{(s)}$, we have to demonstrate

$$\mathcal{S}_1 \preceq_{der}^{(m)} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq_{der}^{(s)} \mathcal{S}_2 \qquad \text{(for all } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}) \tag{C.14}$$

For this, we let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's with the property

$$\mathcal{S}_1 \preceq_{der}^{(m)} \mathcal{S}_2 \,. \tag{C.15}$$

and we have to show

$$\mathcal{S}_1 \preceq_{der}^{(s)} \mathcal{S}_2 \,. \tag{C.16}$$

Since $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ and the fact that every m-derivable formula of $\mathcal{S}_1$ (and thus every theorem or s-derivable formula of $\mathcal{S}_1$) is an m-derivable formula of $\mathcal{S}_2$ (and hence also a theorem or s-derivable formula of $\mathcal{S}_2$) are part of our assumption (C.15), it suffices to show that every s-derivable rule of $\mathcal{S}_1$ is also s-derivable in $\mathcal{S}_2$.

For showing this, we let R be an arbitrary rule on $Fo_{\mathcal{S}_1}$ that is s-derivable in $\mathcal{S}_1$; we have to show that R is also s-derivable in $\mathcal{S}_2$, i.e. we must show

$$(\forall \alpha \in Apps_R) \left[\text{set}(\mathsf{prem}^{(R)}(\alpha)) \vdash_{\mathcal{S}_2}^{(s)} \mathsf{concl}_R(\alpha) \right] . \tag{C.17}$$

We let $\alpha \in Apps_R$ be arbitrary. Since R is s-derivable in $\mathcal{S}_1$, we find that

$$\text{set}(\mathsf{prem}^{(R)}(\alpha)) \vdash_{\mathcal{S}_1}^{(s)} \mathsf{concl}_R(\alpha) \tag{C.18}$$

holds. And furthermore we find the following chain of equivalences:

$$
\begin{aligned}
\mathrm{set}(\mathsf{prem}^{(R)}(\alpha)) \vdash^{(\mathrm{s})}_{\mathcal{S}_1} \mathsf{concl}_R(\alpha) \;\Longrightarrow \\
\Longrightarrow\;\; & \Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_1} \mathsf{concl}_R(\alpha) && \text{for some } \Gamma \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}) \\
& && \text{with } \mathrm{set}(\Gamma) = \mathrm{set}(\mathsf{prem}^{(R)}(\alpha)) \\
\Longrightarrow\;\; & \Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_2} \mathsf{concl}_R(\alpha) && \text{for some } \Gamma \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}) \\
& && \text{with } \mathrm{set}(\Gamma) = \mathrm{set}(\mathsf{prem}^{(R)}(\alpha)) \\
\Longrightarrow\;\; & \mathrm{set}(\mathsf{prem}^{(R)}(\alpha)) \vdash^{(\mathrm{s})}_{\mathcal{S}_2} \mathsf{concl}_R(\alpha)\;. && \qquad\qquad (\mathrm{C}.19)
\end{aligned}
$$

The first and the third implication are hereby consequences of the definitions of the consequence relations $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_1}}$, $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}_1}}$ and $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}_2}}$, $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}_2}}$, respectively. The second implication follows from

$$\Delta \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A \;\Longrightarrow\; \Delta \vdash^{(\mathrm{m})}_{\mathcal{S}_2} A \quad (\text{for all } \Delta \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}) \text{ and } A \in Fo_{\mathcal{S}_1}) \qquad (\mathrm{C}.20)$$

which can be argued for in this way: Let $\Delta \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1})$ and $A \in Fo_{\mathcal{S}_1}$ be arbitrary such that $\Delta \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A$ holds. We choose $\tilde{R}$ to be a rule on $Fo_{\mathcal{S}_1}$ that has only a single application, which is of the form

$$\frac{\sigma}{A}$$

for some sequence $\sigma \in Seqs_{\mathrm{f}}(Fo_{\mathcal{S}_1})$ with the property $\mathrm{mset}(\sigma) = \Delta$. Owing to $\Delta \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A$, $\tilde{R}$ is m-derivable in $\mathcal{S}_1$. Hence, due to our assumption (C.15), $\tilde{R}$ is also m-derivable in $\mathcal{S}_2$, which shows $\Delta = \mathrm{mset}(\sigma) \vdash^{(\mathrm{m})}_{\mathcal{S}_2} A$. Since Δ and A were arbitrary in this argument, we have shown (C.20). In this way we have now also justified the second implication in (C.19) and hence have demonstrated whole chain of implications (C.19). Now (C.18) and (C.19) imply

$$\mathrm{set}(\mathsf{prem}^{(R)}(\alpha)) \vdash^{(\mathrm{s})}_{\mathcal{S}_1} \mathsf{concl}_R(\alpha)\;. \qquad (\mathrm{C}.21)$$

Since we have considered an arbitrary application α of the rule R, we have shown (C.17). Therefore the rule R on $Fo_{\mathcal{S}_1}$, which is s-derivable in $\mathcal{S}_1$, is also s-derivable in $\mathcal{S}_2$.

Because above R was chosen to be an arbitrary s-derivable rule of $\mathcal{S}_1$ in this argument, we can conclude now that (C.16) holds. $\mathcal{S}_1$ and $\mathcal{S}_2$ having been arbitrary AHS's with the property (C.15), we have also shown (C.14), and thus the containment assertion $\preceq^{(\mathrm{m})}_{der} \subseteq \preceq^{(\mathrm{s})}_{der}$ of (C.8).

□

Proof of Theorem C.4. Item (i) of the theorem, the assertion (C.10), is equivalent to Lemma C.6, which is stated and proved immediately below this proof. For item (ii) of the theorem, we will only show (C.12) for the reason that (C.11) can be shown in a quite similar and to some extent formally easier way. Later we will prove item (iii) of the theorem by showing (C.13).

First we will prove (C.12) by demonstrating the three implications in the chain of implications $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2 \Rightarrow \mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2 \Rightarrow \mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2 \Rightarrow \mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$ in the below three items (A), (B) and (C) respectively.

(A) $[\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2]$: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's such that $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$ holds. We will show that then also $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2$ is the case. Since $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$ contains the statement $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$, it suffices to demonstrate that (a) all formulas in $Fo_{\mathcal{S}_1}$ that are m-derivable in $\mathcal{S}_1$ are also m-derivable in $\mathcal{S}_2$, and that (b) all rules on $Fo_{\mathcal{S}_1}$ that are m-derivable in $\mathcal{S}_1$ are also m-derivable in $\mathcal{S}_2$. We will only show (b), since (a) is an even easier consequence of $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$.

Let R be an arbitrary rule on $Fo_{\mathcal{S}_1}$ that is m-derivable in $\mathcal{S}_1$. For showing that R is m-derivable in $\mathcal{S}_2$, let α be an arbitrary application of R. Since R is m-derivable in $\mathcal{S}_1$, it holds that $\mathrm{mset}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{m})}_{\mathcal{S}_1} \mathsf{concl}(\alpha)$. But from this also $\mathrm{mset}(\mathsf{prem}(\alpha)) \vdash^{(\mathrm{m})}_{\mathcal{S}_2} \mathsf{concl}(\alpha)$ follows as a consequence of $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$. Since we have considered an arbitrary application α of R, we have shown (3.7) and therefore that R is also a m-derivable rule in $\mathcal{S}_2$. Because R was chosen as an arbitrary rule on $Fo_{\mathcal{S}_1}$ for this argument, we can conclude that all rules on $Fo_{\mathcal{S}_1}$ that are m-derivable in $\mathcal{S}_1$ are also m-derivable in $\mathcal{S}_2$.

(B) $[\mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2]$: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's with the property $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{der} \mathcal{S}_2$. Then we find that $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ holds, and furthermore that all formulas of $Fo_{\mathcal{S}_1}$ and rules on $Fo_{\mathcal{S}_1}$ that are m-derivable in $\mathcal{S}_1$ are also m-derivable in $\mathcal{S}_2$. From this it follows that all axioms and rules of $\mathcal{S}_1$ are m-derivable in $\mathcal{S}_2$ because all axioms and rules of $\mathcal{S}_1$ are clearly m-derivable in $\mathcal{S}_1$. Therefore $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2$ holds as well.

(C) $[\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2]$: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's for which we assume that $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2$ holds. Then the formula set of $\mathcal{S}_1$ is contained in the formula set of $\mathcal{S}_2$. For showing $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$, it therefore suffices to demonstrate

$$(\forall \Gamma \in \mathcal{M}_{\mathrm{f}}(Fo_{\mathcal{S}_1}))\,(\forall A \in Fo_{\mathcal{S}_1})\,\big[\,(\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_1} A) \implies (\Gamma \vdash^{(\mathrm{m})}_{\mathcal{S}_2} A)\,\big]\ . \tag{C.22}$$

By expanding the definitions of the two occurring relative derivability statements, it is easy to see that (C.22) is equivalent with the assertion

$$(\forall \mathcal{D} \in Der(\mathcal{S}_1))\,(\exists \mathcal{D}' \in Der(\mathcal{S}_2))\,\big[\,\mathsf{assm}(\mathcal{D}') = \mathsf{assm}(\mathcal{D})\ \&\ \&\ \mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D})\,\big]\ , \tag{C.23}$$

which we will prove by induction on the depth $|\mathcal{D}|$ of the derivation $\mathcal{D}$.

For the treatment of the base case of the induction, let $\mathcal{D}$ be a derivation of depth $|\mathcal{D}| = 0$ in $\mathcal{S}_1$. Then $\mathcal{D}$ consists of either an assumption or of an axiom of $\mathcal{S}_1$. In the first subcase, $\mathcal{D}$ is also a derivation in $\mathcal{S}_2$, and we have found the desired derivation $\mathcal{D}'$ in $\mathcal{D}$. In the second subcase, $\mathsf{assm}(\mathcal{D}) = \emptyset$ and $\mathsf{concl}(\mathcal{D}) = A$ holds for some $A \in Ax_{\mathcal{S}_1}$. Owing to $\mathcal{S}_1 \preceq_{r/der} \mathcal{S}_2$, it follows that A is m-derivable in $\mathcal{S}_2$, which means that $\vdash_{\mathcal{S}_2} A$ holds. Hence there exists a derivation $\mathcal{D}'$ in $\mathcal{S}_2$ with $\mathsf{assm}(\mathcal{D}') = \emptyset$ and $\mathsf{concl}(\mathcal{D}') = A$. By choosing an arbitrary such derivation $\mathcal{D}' \in \mathit{Der}(\mathcal{S}_2)$, we have demonstrated the base case of the induction in the here considered subcase.

For the induction step, we consider an arbitrary derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S}_1)$ of depth $|\mathcal{D}| > 0$. Let $A \in \mathit{Fo}_{\mathcal{S}_1}$ be the conclusion of $\mathcal{D}$, i.e. let $A = \mathsf{concl}(\mathcal{D})$. Then $\mathcal{D}$ contains at least one rule application. Let α be the bottommost rule application in $\mathcal{D}$ and let R be a rule of $\mathcal{S}_1$ to which it belongs. We distinguish two cases according to whether $\mathsf{arity}(\alpha)$ is zero, or a natural number.

If $\mathsf{arity}(\alpha) = 0$, then $\mathcal{D}$ must be of the form

$$\frac{}{A}$$

for some $A \in \mathit{Fo}_{\mathcal{S}_1}$ with

$$\mathsf{assm}(\mathcal{D}) = \mathsf{prem}(\alpha) = \emptyset \quad \text{and} \quad \mathsf{concl}(\mathcal{D}) = \mathsf{concl}(\alpha) = A \ .$$

Due to our assumption $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2$, R is derivable in $\mathcal{S}_2$, and hence there exists a derivation $\mathcal{D}_\alpha \in \mathit{Der}(\mathcal{S}_2)$ that mimics α with respect to $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}_2}}$, i.e. for which

$$\mathsf{assm}(\mathcal{D}_\alpha) = \mathrm{mset}(\mathsf{prem}(\alpha)) = \emptyset \quad \text{and} \quad \mathsf{concl}(\mathcal{D}_\alpha) = \mathsf{concl}(\alpha) = A$$

holds. Let $\mathcal{D}_\alpha$ be chosen as such a derivation. Since $\mathcal{D}_\alpha$ has the same assumptions (namely none) and the same conclusion as $\mathcal{D}$, we have found a desired derivation $\mathcal{D}' \in \mathit{Der}(\mathcal{S}_2)$ in $\mathcal{D}_\alpha$ and have thereby shown the induction step in this case.

Now we consider the case $\mathsf{arity}(\alpha) = n \in \omega \setminus \{0\}$. This means that $\mathcal{D}$ is of the form

$$\frac{\begin{array}{ccc}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \dots & A_n\end{array}}{A}$$

for some $A_1, \dots, A_n \in \mathit{Fo}_{\mathcal{S}_1}$ and $\mathcal{S}_1$-derivations $\mathcal{D}_1, \dots, \mathcal{D}_n$ such that it holds:

$$\mathsf{prem}(\alpha) = (A_1, \dots, A_n) \ , \quad \mathsf{concl}(\alpha) = A \ , \quad \mathsf{assm}(\mathcal{D}) = \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_i) \ .$$

As a consequence of our assumption $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{r/der} \mathcal{S}_2$, the rule R is m-derivable in $\mathcal{S}_2$, and hence its application α can be mimicked with respect to $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}_2}}$ by a derivation in $\mathcal{S}_2$. That is, there exists a derivation $\mathcal{D}_\alpha \in \mathit{Der}(\mathcal{S}_2)$ such that

$$\begin{gathered}\mathsf{assm}(\mathcal{D}_\alpha) = \mathrm{mset}(\mathsf{prem}(\alpha)) = \mathrm{mset}(\,(A_1, \dots, A_n)\,) \ , \\ \mathsf{concl}(\mathcal{D}_\alpha) = \mathsf{concl}(\alpha) = A \ .\end{gathered}$$

We choose such a derivation $\mathcal{D}_\alpha$ and denote it by the symbolic prooftree

$$\begin{array}{ccc} (A_1) & \ldots & (A_n) \\ & \mathcal{D}_\alpha & \\ & A & \end{array}$$

(where each of the expressions $(A_1), \ldots, (A_n)$ at the top designate single occurrences of the formulas $A_1, \ldots, A_n$ as *the* assumptions of $\mathcal{D}_\alpha$). From the induction hypotheses we conclude that there exist derivations $\mathcal{D}'_1, \ldots, \mathcal{D}'_n \in Der(\mathcal{S}_2)$ with

$$\begin{aligned} \mathsf{assm}(\mathcal{D}'_i) &= \mathsf{assm}(\mathcal{D}_i) && \text{(for all } i \in \{1, \ldots, n\}) , \\ \mathsf{concl}(\mathcal{D}'_i) &= \mathsf{concl}(\mathcal{D}_i) = A_i && \text{(for all } i \in \{1, \ldots, n\}) . \end{aligned}$$

Using these $\mathcal{S}_2$-derivations as well as the $\mathcal{S}_2$-derivation $\mathcal{D}_\alpha$, we find that the symbolic prooftree

$$\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ (A_1) & \ldots & (A_n) \\ & \mathcal{D}_\alpha & \\ & A & \end{array}$$

describes a derivation $\mathcal{D}'$ in $\mathcal{S}_2$ that arises from $\mathcal{D}_\alpha$ by substituting the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ respectively for its assumptions $(A_1), \ldots, (A_n)$. Then it holds for the derivation $\mathcal{D}'$:

$$\mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D}_\alpha) = A = \mathsf{concl}(\mathcal{D}) ,$$
$$\mathsf{assm}(\mathcal{D}') = \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}'_i) = \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_i) = \mathsf{assm}(\mathcal{D}) .$$

Thus we have found a derivation $\mathcal{D}'$ in $\mathcal{S}_2$ that has the same conclusion and the same assumptions as our given derivation $\mathcal{D}$ in $\mathcal{S}_1$. Therefore we have carried out the induction step in this case.

We have shown (C.23), and thereby also (C.22). As a consequence, we can now conclude that $\mathcal{S}_1 \preceq^{(\mathrm{m})}_{rth} \mathcal{S}_2$ holds.

We will now show item (iii) of the theorem. We have to show (C.13). By expanding the definitions, we see that this assertion is a consequence of the three statements

$$\mathcal{S}_1 \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}_2 \quad \Longleftarrow \quad \mathcal{S}_1 \preceq^{(\mathrm{s})}_{der} \mathcal{S}_2 \qquad \text{(for all } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}) , \tag{C.24}$$
$$\neg\, (\mathcal{S}_1 \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}_2 \quad \Longrightarrow \quad \mathcal{S}_1 \preceq^{(\mathrm{s})}_{rth} \mathcal{S}_2) \qquad \text{(for } \textit{some } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}) , \tag{C.25}$$
$$\mathcal{S}_1 \preceq^{(\mathrm{s})}_{der} \mathcal{S}_2 \quad \Longleftrightarrow \quad \mathcal{S}_1 \preceq^{(\mathrm{s})}_{rth} \mathcal{S}_2 \qquad \text{(for all } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}) . \tag{C.26}$$

Hereby (C.24) is obvious due to the fact that every axiom and every rule of an AHS $\mathcal{S}_1$ is clearly s-derivable in $\mathcal{S}_1$. (C.25) follows from what we know from Example C.2: For the AHS's $\mathcal{S}_2$ and $\mathcal{S}_3$ considered there, we noted in (C.3) and (C.6) that $\mathcal{S}_2 \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}_3$ and $\mathcal{S}_2 \not\preceq^{(\mathrm{s})}_{rth} \mathcal{S}_3$ holds.

(C.26), however, will be shown in the remaining part of this proof. For all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, the implication "$\Leftarrow$" in (C.26) follows easily by analyzing the definitions of $\mathcal{S}_1 \preceq^{(s)}_{der} \mathcal{S}_2$ and $\mathcal{S}_1 \preceq^{(s)}_{rth} \mathcal{S}_2$. For showing the implication "$\Rightarrow$" in (C.26), we let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ be given arbitrarily with the property $\mathcal{S}_1 \preceq^{(s)}_{der} \mathcal{S}_2$. We find $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$, which entails that for demonstrating $\mathcal{S}_1 \preceq^{(s)}_{rth} \mathcal{S}_2$, it remains to prove

$$(\forall\, \Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1}))\, (\forall A \in Fo_{\mathcal{S}_1}) \left[(\Sigma \vdash^{(s)}_{\mathcal{S}_1} A) \implies (\Sigma \vdash^{(s)}_{\mathcal{S}_2} A) \right] . \tag{C.27}$$

Let $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ and $A \in Fo_{\mathcal{S}_1}$ be arbitrary such that $\Sigma \vdash^{(s)}_{\mathcal{S}_1} A$ holds. Then there exists a derivation $\mathcal{D}$ in $\mathcal{S}_1$ with $\text{set}(\mathsf{assm}(\mathcal{D})) = \Sigma$ and $\mathsf{concl}(\mathcal{D}) = A$. We let $\sigma \in Seqs_{\text{f}}((Fo_{\mathcal{S}_1}))$ be such that $\text{mset}(\sigma) = \mathsf{assm}(\mathcal{D})$. Now we let R be the rule on $Fo_{\mathcal{S}_1}$ that possesses only the application of the form

$$\frac{\sigma}{A} \; .$$

More formally, we define $R = \langle \{\bullet\}, \mathsf{prem}, \mathsf{concl} \rangle$, where $\bullet$ denotes an arbitrary set and where $\mathsf{prem}(\bullet) = \sigma$ and $\mathsf{concl}(\bullet) = A$. Then R is s-derivable in $\mathcal{S}_1$, since for the single application $\bullet$ of R

$$\begin{aligned} \text{set}(\mathsf{prem}(\bullet)) &= \text{set}(\sigma) = \text{set}(\text{mset}(\sigma)) = \\ &= \text{set}(\mathsf{assm}(\mathcal{D})) = \text{set}(\Sigma) = \Sigma \vdash^{(s)}_{\mathcal{S}_1} A = \mathsf{concl}(\bullet) \end{aligned}$$

holds. By our assumption $\mathcal{S}_1 \preceq^{(s)}_{der} \mathcal{S}_2$, it follows that also R is s-derivable in $\mathcal{S}_2$ as well and that consequently

$$\Sigma = \ldots = \text{set}(\mathsf{prem}(\bullet)) \vdash^{(s)}_{\mathcal{S}_2} \mathsf{concl}(\bullet) = A$$

holds. This means that the relative derivability statement $\Sigma \vdash^{(s)}_{\mathcal{S}_2} A$ holds. Since $\Sigma \in \mathcal{P}(Fo_{\mathcal{S}_1})$ and $A \in Fo_{\mathcal{S}_1}$ have been arbitrary in this argument, we have shown (C.27) and thus that $\mathcal{S}_1 \preceq^{(s)}_{rth} \mathcal{S}_2$ holds.

□

Lemma C.6 *For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, the following implications hold:*

$$\begin{array}{ccc} \mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 & \Longrightarrow & \mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \\ & \searrow\!\!\!\!\searrow & \Downarrow \\ & & \mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \end{array} \tag{C.28}$$

And furthermore it holds: None of the implications in the above figure can be replaced by a logical equivalence in general.

Proof of Lemma C.6. The statement of the lemma that the implications in (C.28) hold for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$ is a consequence of the two assertions

$$\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \quad (\text{for } \mathit{all}\ \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}), \tag{C.29}$$
$$\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \implies \mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \quad (\text{for } \mathit{all}\ \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}); \tag{C.30}$$

the third implication in (C.28) obviously follows from (C.29) and (C.30). And the statement of the lemma that none of the implication arrows in (C.28) can be inverted in general, is a consequence of the following two assertions:

$$\mathcal{S}_1 \preceq_{th} \mathcal{S}_2 \quad \not\Rightarrow \quad \mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \;\; (\text{for } \textit{some } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}), \tag{C.31}$$

$$\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2 \quad \not\Rightarrow \quad \mathcal{S}_1 \preceq_{adm} \mathcal{S}_2 \quad (\text{for } \textit{some } \mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}). \tag{C.32}$$

That $\mathcal{S}_1 \preceq_{th} \mathcal{S}_2$ does not imply $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$ *in general* follows from (C.31) and (C.32). Hence it suffices to demonstrate (C.29)–(C.32) to prove the lemma.

We notice first that, for all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ is a consequence of $\mathcal{S}_1 \preceq_{adm} \mathcal{S}_2$ in view of the definitions of the relations $\preceq_{r/adm}$ and $\preceq_{adm}$. Hence (C.29) is indeed the case.

For proving (C.30), let $\mathcal{S}_1$ and $\mathcal{S}_2$ be arbitrary AHS's. We assume that $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ and show $\mathcal{S}_1 \preceq_{th} \mathcal{S}_2$. Since $Fo_{\mathcal{S}_1} \subseteq Fo_{\mathcal{S}_2}$ is part of the assertion $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$, it suffices to show that every theorem of $\mathcal{S}_1$ is also a theorem of $\mathcal{S}_2$. This, however, follows from

$$(\forall \mathcal{D} \in Der(\mathcal{S}_1)) \left[\mathsf{assm}(\mathcal{D}) = \emptyset \quad \Longrightarrow \quad \vdash_{\mathcal{S}_2} \mathsf{concl}(\mathcal{D}) \right], \tag{C.33}$$

which we will prove by induction on the rule application depth $|\mathcal{D}|$ of $\mathcal{D}$: If $|\mathcal{D}| = 0$ and $\mathsf{assm}(\mathcal{D}) = \emptyset$, then $\mathcal{D}$ consists of an axiom of $\mathcal{S}_1$, which due to $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ is also a theorem of $\mathcal{S}_2$. For the induction step, let $\mathcal{D}$ be a derivation in $\mathcal{S}_1$ with $|\mathcal{D}| \geq 1$ and with $\mathsf{assm}(\mathcal{D}) = \emptyset$. Then the conclusion of $\mathcal{D}$, which we let be the formula A, is the conclusion of an application α of a rule R of $\mathcal{S}_1$. If on the one hand α is a zero-premise application of R, then $\mathcal{D}$ consists only of this rule application; since $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ holds, α can be mimicked by a derivation $\mathcal{D}'$ in $\mathcal{S}_2$ without assumptions and with the same conclusion as $\mathcal{D}$, which shows $\vdash_{\mathcal{S}_2} A$ in this case. If on the other hand $\mathsf{arity}(\alpha) > 0$ is the case, then $\mathcal{D}$ is of the form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array}}{A}$$

with the application α of R as the bottommost application in $\mathcal{D}$, with derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\mathcal{S}_1$ that have no assumptions and that have the respective conclusions $A_1, \ldots, A_n$. By respectively applying the induction hypothesis to $\mathcal{D}_1, \ldots, \mathcal{D}_n$, it follows that all of $A_1, \ldots, A_n$ are theorems of $\mathcal{S}_2$. Since due to $\mathcal{S}_1 \preceq_{r/adm} \mathcal{S}_2$ the rule R of $\mathcal{S}_1$ is admissible in $\mathcal{S}_2$, we can then conclude that $\vdash_{\mathcal{S}_2} A$ holds, i.e. that A is a theorem of $\mathcal{S}_2$. Hence we have also carried out the induction step for (C.33) also in this case. In this way we have shown (C.33) and have finally proven (C.30).

For demonstrating (C.31) and (C.32), we employ two examples. Let $Fo = \{A, B, C\}$ be a three-element set and let $R_{A.B}$, $R_{A.C}$ and $R_{B.C}$ be the rules on Fo that have each only one application, namely

$$\frac{A}{B}\, R_{A.B} \qquad\qquad \frac{A}{C}\, R_{A.C} \qquad\qquad \frac{B}{C}\, R_{B.C}$$

(in these applications we have used $R_{A.B}$, $R_{A.C}$ and $R_{B.C}$ as "names" of the respective rules as if we considered these rule applications as derivations in n-AHS's). And we let $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ be the AHS's with axioms and rules as described by the following table:

AHS	set of axioms	set of rules
$\mathcal{S}_1$	$\emptyset$	$\{R_{A.C}\}$
$\mathcal{S}_2$	$\{A\}$	$\emptyset$
$\mathcal{S}_3$	$\{A\}$	$\{R_{A.B}\}$

Now we find that

$$\mathcal{S}_1 \preceq_{th} \mathcal{S}_3 \quad \& \quad \mathcal{S}_1 \not\preceq_{r/adm} \mathcal{S}_3 \tag{C.34}$$

holds: $\mathcal{S}_1 \preceq_{th} \mathcal{S}_3$ is a consequence of $Th(\mathcal{S}_1) = \emptyset \subseteq \{A, B\} = Th(\mathcal{S}_3)$, and furthermore $\mathcal{S}_1 \not\preceq_{r/adm} \mathcal{S}_3$ is the case since $R_{A.C}$ is not admissible in $\mathcal{S}_3$. The fact that (C.34) holds for the AHS's $\mathcal{S}_1$ and $\mathcal{S}_3$ as defined above now clearly shows (C.31).

Furthermore we find that

$$\mathcal{S}_2 \preceq_{r/adm} \mathcal{S}_3 \quad \& \quad \mathcal{S}_2 \not\preceq_{adm} \mathcal{S}_3 \tag{C.35}$$

holds: $\mathcal{S}_2 \preceq_{r/adm} \mathcal{S}_3$ is obvious since $\mathcal{S}_2$ has no rules and its single axiom is also an axiom of $\mathcal{S}_3$; and $\mathcal{S}_2 \not\preceq_{adm} \mathcal{S}_3$ is a consequence of the fact that the rule $R_{B.C}$ is admissible in $\mathcal{S}_2$ (trivially, since $Th(\mathcal{S}_2) = \{A\}$), but not in $\mathcal{S}_3$ (since the theory $Th(\mathcal{S}_3) = \{A, B\}$ of $\mathcal{S}_3$ is not closed under applications of $R_{B.C}$). Hereby we have demonstrated (C.35) for the AHS's $\mathcal{S}_2$ and $\mathcal{S}_3$ as defined above and hence we have shown (C.31).

We have shown (C.29), (C.30), (C.31) and (C.32) and hence we have proven the lemma. □

Proof of Theorem C.5. In view of Proposition C.3 and Theorem C.4, and in particular, of the visualization of these two statements by interrelation prisms as shown in Figure 3 (a), and of the fact that we already know $\preceq^{(s)}_{rth} \subsetneqq \preceq^{(s)}_{r/der}$ from Theorem C.4 (iii), we only have to show the following five assertions:

$$\preceq^{(m)}_{rth} \;\neq\; \preceq^{(s)}_{rth} \;, \tag{C.36}$$
$$\preceq^{(s)}_{r/der} \;\neq\; \preceq_{r/der} \;, \tag{C.37}$$
$$\preceq_{r/der} \;\neq\; \preceq_{r/adm} \;. \tag{C.38}$$
$$\preceq_{adm} \;\not\subseteq\; \preceq_{der} \;. \tag{C.39}$$
$$\preceq^{(m)}_{der} \;\not\subseteq\; \preceq_{adm} \;. \tag{C.40}$$

We start by showing (C.36). For a two-element set $Fo = \{A, B\}$, we let $\mathcal{S}_4$ and $\mathcal{S}_5$ be the AHS's with set Fo as their sets of formulas and with, respectively, only the rules R_{3a} and R_{3b} used in Example C.2. Then it is straightforward to check that

$$\mathcal{S}_4 \not\preceq^{(m)}_{rth} \mathcal{S}_5 \quad \& \quad \mathcal{S}_4 \preceq^{(s)}_{rth} \mathcal{S}_5$$

holds, which entails (C.36). Statement (C.37) is a consequence of the fact that for the two AHS's $\mathcal{S}_1$ and $\mathcal{S}_3$ from Example C.2 the following holds:

$$\mathcal{S}_1 \preceq_{r/der} \mathcal{S}_3 \quad \& \quad \mathcal{S}_1 \not\preceq^{(s)}_{r/der} \mathcal{S}_3 \ .$$

Hereby we have seen $\mathcal{S}_1 \not\preceq^{(s)}_{r/der} \mathcal{S}_3$ already in Example C.2. And $\mathcal{S}_1 \preceq_{r/der} \mathcal{S}_3$ can be verified easily by recognizing that the rules R_4 and R_{3a} of $\mathcal{S}_1$ are derivable in $\mathcal{S}_3$ (the fact that R_4 is not *s-derivable* in $\mathcal{S}_3$ was actually the reason for $\mathcal{S}_1 \not\preceq^{(s)}_{r/der} \mathcal{S}_3$).

Finally, we show (C.38). We let a three-element set $Fo = \{A, B, C\}$ be given, and we let $\mathcal{S}_6$ be the AHS with set Fo of formulas and with the two rules $R_{.A}$ and $R_{B.C}$ that each have only one application, namely

$$\frac{}{A}\, R_{.A} \qquad\qquad \frac{B}{C}\, R_{B.C}$$

where we allowed to use $R_{.A}$ and $R_{B.C}$ as respective "names" for these two rule applications as if we considered them in the context of an n-AHS. And we let $\mathcal{S}_7$ be the AHS with set Fo of formulas and with the single rule $R_{.A}$ as described above. Then it holds that $\mathcal{S}_6 \preceq_{r/adm} \mathcal{S}_7$ due to the fact that $R_{B.C}$ is admissible in $\mathcal{S}_7$ because $B \notin Th(\mathcal{S}_7)$. But $\mathcal{S}_6 \preceq_{r/der} \mathcal{S}_7$ does not hold because the rule $R_{B.C}$ is obviously not derivable in $\mathcal{S}_7$. We have shown

$$\mathcal{S}_6 \preceq_{r/adm} \mathcal{S}_7 \quad \& \quad \mathcal{S}_6 \not\preceq_{r/der} \mathcal{S}_7 \ ,$$

and therefore we can conclude that (C.38) holds.

For showing (C.39) and (C.40), we let three AHS's $\tilde{\mathcal{S}}_1$, $\tilde{\mathcal{S}}_2$ and $\tilde{\mathcal{S}}_3$ be given with set $Fo = \{A, B, C\}$ of formulas and with respective set of axioms and set of rules according to the following table:

AHS	set of axioms	set of rules
$\tilde{\mathcal{S}}_1$	$\{A\}$	$\emptyset$
$\tilde{\mathcal{S}}_2$	$\{A\}$	$\{R_{A.B}\}$
$\tilde{\mathcal{S}}_3$	$\{A\}$	$\{R_{B.C}\}$

Hereby the rules $R_{A.B}$ and $R_{B.C}$ have each only one application, namely

$$\frac{A}{B}\, R_{A.B} \qquad\qquad \frac{B}{C}\, R_{B.C}$$

(in these applications we have used $R_{A.B}$ and $R_{B.C}$ as "names" of the respective rules as if we considered these rule applications as derivations in n-AHS's). Now it is easy to see that

$$\tilde{\mathcal{S}}_3 \preceq_{adm} \tilde{\mathcal{S}}_1 \ \& \ \tilde{\mathcal{S}}_3 \not\preceq_{der} \tilde{\mathcal{S}}_1 \tag{C.41}$$

holds: $\tilde{\mathcal{S}}_3 \preceq_{adm} \tilde{\mathcal{S}}_1$ is a consequence of $Th(\tilde{\mathcal{S}}_1) = \{A\} = Th(\tilde{\mathcal{S}}_3)$, and $\tilde{\mathcal{S}}_3 \not\preceq_{der} \tilde{\mathcal{S}}_1$ is due to the fact that the rule $R_{B.C}$ of $\tilde{\mathcal{S}}_3$ is obviously not derivable in $\tilde{\mathcal{S}}_1$. This shows

(C.39). And we notice furthermore that

$$\tilde{\mathcal{S}}_1 \preceq^{(\mathrm{m})}_{der} \tilde{\mathcal{S}}_2 \;\;\&\;\; \tilde{\mathcal{S}}_1 \not\preceq_{adm} \tilde{\mathcal{S}}_2 \tag{C.42}$$

holds: $\tilde{\mathcal{S}}_1 \preceq^{(\mathrm{m})}_{der} \tilde{\mathcal{S}}_2$ is obvious (since $\tilde{\mathcal{S}}_1$ contains no rules and the single axiom of $\tilde{\mathcal{S}}_1$ is also an axiom of $\tilde{\mathcal{S}}_2$ and hence is m-derivable in $\tilde{\mathcal{S}}_2$), and $\tilde{\mathcal{S}}_1 \not\preceq_{adm} \tilde{\mathcal{S}}_2$ is due to the fact that the rule $R_{B.C}$ is clearly admissible in $\tilde{\mathcal{S}}_1$, but not in $\tilde{\mathcal{S}}_1$. And (C.42) now clearly shows (C.40).

□

Proof of Theorem 4.13. The statement of Theorem 4.13 consists of the assertions of Proposition C.7, and Theorem C.8, which are stated and then proved below. □

Corollary C.7 ('Vertical' and 'horizontal interrelations' between above defined mutual inclusion relations).

(i) *The following containment assertions hold between the mutual inclusion relations defined in Definitions 4.2, Definition 4.6 and Definition 4.10:*

$$\sim^{(m)}_{rth} \;\subseteq\; \sim^{(s)}_{rth} \;\subseteq\; \sim_{rth} \;\subseteq\; \sim_{th} \;, \tag{C.43}$$

$$\sim^{(m)}_{der} \;\subseteq\; \sim^{(s)}_{der} \;\subseteq\; \sim_{der} \;\subseteq\; \sim_{adm} \;, \tag{C.44}$$

$$\sim^{(m)}_{r/der} \;\subseteq\; \sim^{(s)}_{r/der} \;\subseteq\; \sim_{r/der} \;\subseteq\; \sim_{r/adm} \;. \tag{C.45}$$

These assertions justify, for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, the downwards-pointing, vertical implication arrows in an 'interrelation prism' as shown in Figure 3 (b).

(ii) *And the following relationships hold between these twelve mutual inclusion relations:*

$$\preceq_{r/adm} \;=\; \preceq_{adm} \;=\; \preceq_{th} \;, \tag{C.46}$$

$$\preceq_{r/der} \;=\; \preceq_{der} \;=\; \preceq_{rth} \;, \tag{C.47}$$

$$\preceq^{(s)}_{r/der} \;\supsetneqq\; \preceq^{(s)}_{der} \;=\; \preceq^{(s)}_{rth} \;, \tag{C.48}$$

$$\preceq^{(m)}_{r/der} \;=\; \preceq^{(m)}_{der} \;=\; \preceq^{(m)}_{rth} \;. \tag{C.49}$$

These assertions justify for all $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$, the horizontal logical equivalences in an 'interrelation prism' as shown in Figure 3 (b).

Theorem C.8 *The containment assertions* (C.43), (C.44) *and* (C.45) *in Corollary C.7 are proper inclusions, i.e. each inclusion symbol $\subseteq$ used there can be replaced by the symbol $\subsetneqq$.*

For showing (C.46) in Corollary C.7 above, we will need the following lemma.

Lemma C.9 *For all AHS's $\mathcal{S}_1$ and $\mathcal{S}_2$, the following equivalences hold:*

$$\mathcal{S}_1 \sim_{adm} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{r/adm} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{th} \mathcal{S}_2 \iff \mathcal{S}_1 \sim_{adm} \mathcal{S}_2 \tag{C.50}$$

Proof of Lemma C.6. The lemma is an obvious consequence of Lemma 4.8 (iii), Lemma C.9, and the definition of the mutual inclusion relations $\sim_{th}$, $\sim_{adm}$, $\sim_{r/adm}$ in respective terms of the inclusion relations $\preceq_{th}$, $\preceq_{adm}$, $\preceq_{r/adm}$. □

Proof of Corollary C.7. With the exception of its statements

$$\sim_{der} \; \subsetneq \; \sim_{adm} \; , \tag{C.51}$$

$$\sim^{(s)}_{r/der} \; \neq \; \sim^{(s)}_{rth} \; , \tag{C.52}$$

which we will show below, and of (C.46), which is equivalent to the statement of Lemma C.9 above, the corollary follows from the theorem using the following fact:

$$\preceq_{P_1,Q_1} \; \subseteq \; \preceq_{P_2,Q_2} \quad \Longrightarrow \quad \sim_{P_1,Q_1} \; \subseteq \; \sim_{P_2,Q_2} \tag{C.53}$$

holds for all inclusion relations $\preceq_{P_1,Q_1}$, $\preceq_{P_2,Q_2}$ and the respectively induced mutual inclusion relations $\sim_{P_1,Q_1}$, $\sim_{P_2,Q_2}$. This is, as we will show now, due to the way how mutual inclusion relations are defined in terms of inclusion relations.

Let two inclusions relations $\preceq_{P_1,Q_1}$, $\preceq_{P_2,Q_2}$ be given such that $\preceq_{P_1,Q_1} \subseteq \preceq_{P_2,Q_2}$ holds, and let $\sim_{P_1,Q_1}$ and $\sim_{P_2,Q_2}$ be the mutual inclusion relations that are induced by $\preceq_{P_1,Q_1}$ and $\preceq_{P_2,Q_2}$ respectively. For showing $\sim_{P_1,Q_1} \subseteq \sim_{P_2,Q_2}$, let $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{H}$ be two AHS's such that $\mathcal{S}_1 \sim_{P_1,Q_1} \mathcal{S}_2$. By the definition of $\sim_{P_1,Q_1}$, it follows that $\mathcal{S}_1 \preceq_{P_1,Q_1} \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_{P_1,Q_1} \mathcal{S}_1$. Due to $\preceq_{P_1,Q_1} \subseteq \preceq_{P_2,Q_2}$, it follows that $\mathcal{S}_1 \preceq_{P_2,Q_2} \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_{P_2,Q_2} \mathcal{S}_1$ hold as well. By using the definition of $\sim_{P_2,Q_2}$, these assertions show $\mathcal{S}_1 \sim_{P_2,Q_2} \mathcal{S}_2$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have been arbitrary AHS's with the property $\mathcal{S}_1 \sim_{P_1,Q_1} \mathcal{S}_2$, we have now proven $\sim_{P_1,Q_1} \subseteq \sim_{P_2,Q_2}$. Hence we have demonstrated (C.53) for all inclusion relations $\preceq_{P_1,Q_1}$, $\preceq_{P_2,Q_2}$ and mutual inclusion relations $\sim_{P_1,Q_1}$, $\sim_{P_2,Q_2}$ that are respectively induced by $\preceq_{P_1,Q_1}$ and $\preceq_{P_2,Q_2}$.

For showing (C.51), let $\mathcal{S}_1$ and $\mathcal{S}_2$ be AHS's such that $\mathcal{S}_1 \sim_{der} \mathcal{S}_2$ holds; we will show $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$. It is part of the assumption $\mathcal{S}_1 \sim_{der} \mathcal{S}_2$ that every derivable formula of $\mathcal{S}_1$ is also a derivable formula of $\mathcal{S}_2$, and vice versa. This implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems, i.e. that $\mathcal{S}_1 \sim_{th} \mathcal{S}_2$ holds. But this implies $\mathcal{S}_1 \sim_{adm} \mathcal{S}_2$ by Lemma C.9. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have been arbitrary AHS's with the property $\mathcal{S}_1 \sim_{der} \mathcal{S}_2$ for this argument, we have justified the containment assertion (C.51).

The assertion (C.52) is an immediate consequence of what we have found in Example C.2 about the two AHS's $\mathcal{S}_2$ and $\mathcal{S}_3$ considered there: In (C.3) and (C.6), we saw that $\mathcal{S}_2 \sim^{(s)}_{r/der} \mathcal{S}_3$ and $\mathcal{S}_2 \not\sim^{(s)}_{rth} \mathcal{S}_3$ hold.

□

Proof of Theorem C.8. The theorem is a consequence of the fact that all of the counterexamples given in the proof of Theorem C.5 for demonstrating (C.36), (C.37) and (C.38) can be used again here to show the analogous assertions

$$\sim^{(\mathrm{m})}_{rth} \;\neq\; \sim^{(\mathrm{s})}_{rth} \;, \tag{C.54}$$

$$\sim^{(\mathrm{s})}_{r/der} \;\neq\; \sim_{r/der} \;, \tag{C.55}$$

$$\sim_{r/der} \;\neq\; \sim_{r/adm} \;. \tag{C.56}$$

In view of the assertion $\sim^{(\mathrm{s})}_{rth} \subseteq \sim^{(\mathrm{s})}_{r/der}$ of Corollary C.7 and of the visualization of Corollary C.7 in an 'interrelation prism' as shown in Figure 3 (b), the assumptions (C.54), (C.55) and (C.56) are sufficient to demonstrate the statement of the theorem.

□

Proof of Corollary 4.15. Let $\mathcal{S}$ be an AHS and R a rule on $Fo_{\mathcal{S}}$. We saw earlier that (4.8) is just a reformulation of item (i) in Proposition 3.2. And the logical equivalence (4.9) follows from the theorem, or, more precisely, it follows easily from the assertion $\precsim_{r/der} = \precsim_{rth}$ of that part of Theorem 4.13, which is expressed by Corollary C.7:

R is derivable in $\mathcal{S}$	$\Longleftrightarrow$	$\mathcal{S}+R \precsim_{r/der} \mathcal{S}$	by the definition of $\precsim_{r/der}$,
	$\Longleftrightarrow$	$\mathcal{S}+R \precsim_{rth} \mathcal{S}$	due to the assertion $\precsim_{r/der} = \precsim_{rth}$ of Corollary C.7
	$\Longleftrightarrow$	$\mathcal{S}+R \sim_{rth} \mathcal{S}$	due to the definition of $\sim_{rth}$.

Analogously, (4.8) and (4.11) can be shown to be consequences of, respectively, the assertions in (i) and (ii) of the theorem; and similarly, (4.10) can be shown to follow from part (iii) of the theorem.

□

Example C.10 (Counterexample to "⇒" in (4.10)). That the implication "⇒" in (4.10) does not hold in general can be shown by a counterexample that again relies on the AHS's from Example C.2: For the system $\mathcal{S}_2$ considered there, we saw that $\mathcal{S}_2 \sim^{(\mathrm{s})}_{r/der} \mathcal{S}_3$ holds, and for this we have used that the rule R_{3a} of $\mathcal{S}_2$ is derivable in $\mathcal{S}_3$. However, $\mathcal{S}_3{+}R_{3a} \sim^{(\mathrm{s})}_{rth} \mathcal{S}_3$ does not hold, since $\{C_1, C_2\} \vdash^{(\mathrm{s})}_{\mathcal{S}_3+R_{3a}} B$ is clearly a relative derivability statement that holds in $\mathcal{S}_3{+}R_{3a}$ (the derivation in (C.4) is also a derivation in $\mathcal{S}_3{+}R_{3a}$), but on the other hand, $\mathcal{S}_3$ allows only holding relative derivability statements of the form $\{D\} \vdash^{(\mathrm{s})}_{\mathcal{S}_3} E$ for some formulas $D, E \in Fo$ due to the fact that $\mathcal{S}_3$ contains only one-premise rules. We have thereby shown

$$R_{3a} \text{ is s-derivable in } \mathcal{S}_3 \quad \& \quad \mathcal{S}_3{+}R_{3a} \not\sim^{(\mathrm{s})}_{rth} \mathcal{S}_3 \;,$$

a counterexample to the implication "⇒" in (4.10).

■

Appendix D: Auxiliary notions and proofs for statements in Section 5

This appendix consists of six parts: Sections D.1 and D.6 contain the proofs for Theorem 5.3 and Theorem 5.13, respectively. The biggest part, Section D.5, contains a proof for Lemma 5.10, i.e. for termination of ARS's of R-elimination by mimicking steps. In this proof the termination of an ARS of rule elimination by mimicking steps is reduced to the termination of an ARS of "multiset reduction" (which can be viewed to underly a multiset ordening) by means of a measure function. The necessary formal prerequisites for this proof are gathered in three earlier sections: In Section D.2 we review the concept of "abstract rewrite system" (ARS) and formally define basic notions for these systems such as sequences of composable steps, weak and strong normalization, and the transitive and reflexive-transitive closure of an ARS. In Section D.3 the concepts of "multiset reduction" and "multiset ordening" are introduced and the most important results for these notions are gathered. And in Section D.4 the method of reducing the termination of an ARS to the termination of another one by means of a "measure function" is described.

D.1 Proof of Theorem 5.3.

Proof of Theorem 5.3. We will only prove the theorem for the case of AHS's, since for in the case of n-AHS's it can be argued analogously. We let $\mathcal{S}$ be an AHS and R a rule of $\mathcal{S}$. (5.13) is a consequence of the following sequence of logical equivalences:

$$
\begin{aligned}
&R\text{-elimination holds in } \mathcal{S} \iff \\
&\quad\iff (\forall\,\mathcal{D}\in Der(\mathcal{S}))\big[\,\mathsf{assm}(\mathcal{D})=\emptyset \;\Rightarrow\; (\exists\,\mathcal{D}'\in Der(\mathcal{S}{-}R))\,[\,\mathcal{D}' \precsim \mathcal{D}\,]\,\big]\\
&\quad\iff \mathcal{S} \preceq_{th} \mathcal{S}{-}R\\
&\quad\iff \mathcal{S} \sim_{th} \mathcal{S}{-}R\\
&\quad\iff \mathcal{S} \sim_{r/adm} \mathcal{S}{-}R\\
&\quad\iff \mathcal{S} \preceq_{r/adm} \mathcal{S}{-}R\\
&\quad\iff R \text{ is admissible in } \mathcal{S}{-}R\,.
\end{aligned}
$$

The 6 logical equivalences occuring hereby are justified, in their order from the top one to the bottom one, by Definition 5.2 (i), by the definition of $\preceq_{th}$, by the definitions of $\sim_{th}$ and $\mathcal{S}{-}R$, by assertion (4.12) in Corollary 4.16, by the definitions of $\sim_{r/adm}$ and $\mathcal{S}{-}R$, and by the definition of $\preceq_{r/adm}$.

(5.15) is a consequence of the following sequence of logical implications and equivalences:

$$
\begin{aligned}
&R\text{-elimination holds in } Der(\mathcal{S}) \text{ w.r.t. } \simeq^{(\mathrm{s})} \iff\\
&\quad\iff (\forall\,\mathcal{D}\in Der(\mathcal{S}))\,(\exists\,\mathcal{D}'\in Der(\mathcal{S}{-}R))\,[\,\mathcal{D}' \simeq^{(\mathrm{s})} \mathcal{D}\,]\\
&\quad\iff \mathcal{S} \preceq^{(\mathrm{s})}_{rth} \mathcal{S}{-}R
\end{aligned}
$$

$$\iff \quad \mathcal{S} \sim^{(\mathrm{s})}_{rth} \mathcal{S}{-}R$$
$$\implies \quad \mathcal{S} \sim^{(\mathrm{s})}_{r/der} \mathcal{S}{-}R$$
$$\iff \quad \mathcal{S} \preceq^{(\mathrm{s})}_{r/der} \mathcal{S}{-}R$$
$$\iff \quad R \text{ is s-derivable in } \mathcal{S}{-}R\,.$$

The 5 logical equivalences and one implication above are justified, in their order from the top to the bottom one, by Definition 5.2 (i), by the definition of relative derivability statements with respect to $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}-R}}$, by the definition of $\preceq_{th}$, by the definitions of $\sim^{(\mathrm{s})}_{rth}$ and $\mathcal{S}{-}R$, by assertion (4.14) in Corollary 4.16, by the definitions of $\sim^{(\mathrm{s})}_{r/der}$ and $\mathcal{S}{-}R$, and by the definition of $\preceq^{(\mathrm{s})}_{r/der}$.

And (5.14) as well as (5.16) can be demonstrated analogously to our argumentation for (5.15) above.

□

D.2 Abstract Rewrite Systems

In this section we review the concept of "abstract rewrite system" in the way as it is defined in [9] on p. 317. We generally follow the definitions given there, but will also make some adjustments for our purposes.

We recall that an *abstract rewrite system* $\mathcal{A}$ is a quadruple $\langle A, \Phi, \mathsf{src}, \mathsf{tgt}\rangle$ consisting of a set A of *objects*, a set Φ of *steps*, and of *source* and *target functions* $\mathsf{src}, \mathsf{tgt} : \Phi \to A$. Various arrow-like symbols like $\to, \leadsto, \Rightarrow, \ldots$ are employed to range over ARS's, symbols $a, b, c, \ldots,$ will be used for objects, and $\phi, \psi, \chi, \ldots$ for steps. For a given abstract rewrite system $\to$, we write $\phi : a \to b$ to indicate that ϕ is a step with source a and target b; ϕ is a witness to the claim that some step from a to b exists.

Let $\langle A, \Phi, \mathsf{src}, \mathsf{tgt}\rangle$ be an ARS. Let ϕ_1 and ϕ_2 be steps. We say that ϕ_2 is composable with ϕ_1 if and only if $\mathsf{tgt}(\phi_1) = \mathsf{src}(\phi_2)$, i.e. iff the target of ϕ_1 is the source of ϕ_2. Let $I = \{0, \ldots, n\}$ for some $n \in \omega$, or $I = \omega$, and let $\{\phi_i\}_{i\in I}$ be a sequence of steps of $\mathcal{A}$, which is called *finite* for finite I, and *infinite* otherwise. We say that $\{\phi_i\}_{i\in I}$ is a *sequence of composable steps* iff it holds that

$$(\forall i \in \omega)\left[\, i+1 \in I \quad\Rightarrow\quad \mathsf{tgt}(\phi_i) = \mathsf{src}(\phi_{i+1})\,\right]$$

i.e. iff for each two successive steps ϕ_i and ϕ_{i+1} in the sequence ϕ_{i+1} is composable with ϕ_i. We call $\mathsf{src}(\phi_0)$ the *source* of the sequence $\{\phi_i\}_{i\in I}$, and if $I = \{0, \ldots, n\}$, we call $\mathsf{tgt}(\phi_n)$ the *target* of this sequence.

Let $\mathcal{A}$ again be an ARS and let ϕ, ψ be two of its steps. We say that the step ψ *mimics* the step ϕ, which assertion we abbreviate symbolically to $\psi \simeq \phi$, if and only if ϕ and ψ have respectively the same sources and the same targets, i.e. iff

$$\mathsf{src}(\psi) = \mathsf{src}(\phi) \quad \& \quad \mathsf{tgt}(\psi) = \mathsf{tgt}(\phi)$$

holds. Clearly $\simeq$ is an equivalence relation on the set of steps of $\mathcal{A}$.

An object a of an ARS $\mathcal{A}$ is a *normal form* of $\mathcal{A}$ if and only if a is not the source of any step of $\mathcal{A}$. We designate by

$$\mathcal{NF}(\mathcal{A}) = \{a \in A \mid a \text{ is normal form of } \mathcal{A}\}$$

the set of normal forms of $\mathcal{A}$.

We say that an ARS $\mathcal{A}$ is *weakly normalizing*, or just *normalizing*, if and only if for all $a \in A$ there exists a finite sequence $\{\phi_i\}_{i\in I}$ of composable steps of $\mathcal{A}$ with source a and a normal form of $\mathcal{A}$ as target. And we say that an ARS $\mathcal{A}$ is *strongly normalizing*, or *terminating*, if and only if there does not exist an infinite sequence of composable steps in $\mathcal{A}$.

In the following definition we deviate from a definition given in [9] in that we use the term "abstract rewrite system *with composition and identity*" for a notion that in [9] is just called "abstract rewrite system with composition". This is because we will also use the latter term for systems without explicit identity steps.

Definition D.1 (i) An abstract rewrite system *with composition* is a tuple $\langle \to, \cdot \rangle$, where $\to$ is an abstract rewrite system and $\cdot$ is a function from composable steps of $\to$ to steps of $\to$ such that:
- For every pair of composable steps $\phi : a \to b$ and $\psi : b \to c$ of $\to$, their *composition* $\phi \cdot \psi : a \to c$ exists.

(ii) An abstract rewrite system *with composition and identity* is a triple $\langle \to, 1, \cdot \rangle$, where $\to$ is an abstract rewrite system, 1 is a function from objects of $\to$ to steps of $\to$, and $\cdot$ is a function from composable steps of $\to$ to steps of $\to$ such that:
- For every object a of $\to$, its *trivial* step $1_a : a \to a$ exists.
- For every pair of composable steps $\phi : a \to b$ and $\psi : b \to c$ of $\to$, their *composition* $\phi \cdot \psi : a \to c$ exists.

Steps of the form 1_a and $(\phi \cdot \psi)$ are called *empty* and *composite* steps, respectively.

Now we define the "reflexive-transitive closure" of an ARS analogously as this is done in [9], and additionally, we introduce the notion of "transitive closure" of an ARS in an obvious and similar way.

Definition D.2 (i) The *transitive closure* of an abstract rewrite system of the form $\to = \langle A, \Phi, \mathsf{src}, \mathsf{tgt} \rangle$ is the abstract rewrite system with composition $\langle \to^+, \cdot \rangle$ defined by:
- A is the set of objects of $\to^+$.
- The steps of $\to^+$ together with their source and target objects are defined as the theorems of the following Hilbert-style proof system:

$$\frac{\phi : a \to b \in \Phi}{\phi : a \to^+ b} \qquad \frac{\phi : a \to^+ b \qquad \psi : b \to^+ c}{(\phi \cdot \psi) : a \to^+ c}$$

- The composition of steps $\phi : a \to^+ b$ and $\psi : b \to^+ c$ of $\to^+$ is $(\phi \cdot \psi) : a \to^+ c$.

(ii) The *reflexive-transitive closure* $\to^*$ of an ARS $\to = \langle A, \Phi, \mathsf{src}, \mathsf{tgt} \rangle$ is the abstract rewrite system with composition and identity $\langle ^*, 1, \cdot \rangle$ definied by:

– A is the set of objects of $\to^*$.
– The steps of $\to^*$ together with their source and target objects are defined as the theorems of the following Hilbert-style proof system:

$$\frac{a \in A}{1_a : a \to^* a} \qquad \frac{\phi : a \to b \in \Phi}{\phi : a \to^* b} \qquad \frac{\phi : a \to^* b \qquad \psi : b \to^* c}{(\phi \cdot \psi) : a \to^* c}$$

– The trivial step for an object a of $\to^*$ is $1_a : a \to^* a$.
– The composition of steps $\phi : a \to^* b$ and $\psi : b \to^* c$ of $\to^*$ is $(\phi \cdot \psi) : a \to^* c$.

D.3 Multiset Reduction ARS's and Multiset Ordening

In this section we review the definition of finite multisets over a given set, slightly extend the notation used for multisets in Section 2, introduce the notions of multiset reduction and multiset ordening, and report some of the most important results about these notions.

We recall the definition of finite multisets over some given set that was stated in Section 2, and we slightly extend the notation introduced there. For an arbitrary set X, there we let

$$\mathcal{M}_{\mathrm{f}}(X) = \{ M : X \to \omega \mid M(x) \neq 0 \text{ for only finitely many } x \in X \}$$

the set of *finite multisets over* X; we stipulated that $x \in X$ occurs in $M \in \mathcal{M}_{\mathrm{f}}(X)$ iff $M(x) \neq 0$, and that, for all $n \in \omega$, $x \in X$ *occurs* n *times in* M iff $M(x) = n$. The *union* $M_1 \uplus M_2$ and the *difference* $M_1 \setminus M_2$ of two finite multisets $M_1, M_2 \in \mathcal{M}_{\mathrm{f}}(X)$ over a set X are respectively defined by

$$\begin{aligned} M_1 \uplus M_2 :\ & X \to \omega \\ & x \mapsto (M_1 \uplus M_2)(x) = M_1(x) + M_2(x) , \\ M_1 \setminus M_2 :\ & X \to \omega \\ & x \mapsto (M_1 \setminus M_2)(x) = \max\{0, M_1(x) - M_2(x)\} . \end{aligned}$$

For arbitrary multisets $M, M_1, M_2 \in \mathcal{M}_{\mathrm{f}}(X)$ over a set X and arbitrary $x \in X$ we will furthermore use the following abbreviations:

$$\begin{aligned} x \in M \quad & \text{is short for} \quad M(x) > 0 , \\ M_1 \subseteq M_2 \quad & \text{is short for} \quad (\forall x \in X)[\, M_1(x) \leq M_2(x) \,] , \end{aligned}$$

and we will denote by $\emptyset$ the multiset in $\mathcal{M}_{\mathrm{f}}(X)$ defined by $\emptyset : X \to \omega,\ x \mapsto 0$. Furthermore we recall that we introduced in Section 2 an operation $\mathrm{mset}(\cdot)$, which converts finite sequences and finite sets over a set X into finite multisets over X. That is, we thereby defined, for all sets X, a function

$$\mathrm{mset} : \mathit{Seqs}_{\mathrm{f}}(X) \uplus \mathcal{P}_{\mathrm{f}}(X) \longrightarrow \mathcal{M}_{\mathrm{f}}(X)$$

that for finite sequences $\sigma \in Seqs_{\mathrm{f}}(X)$ and for all finite subsets B of X is defined by

$$\mathrm{mset}(\sigma)(x) = \text{number of occurrences of } x \text{ in } \sigma,$$
$$\mathrm{mset}(B)(x) = \begin{cases} 0 & \ldots\ x \notin B \\ 1 & \ldots\ x \in B \end{cases}$$

for all $x \in X$.

Let $\langle A, < \rangle$ be a strictly partially ordered set. The *ARS of multiset reduction on* $\mathcal{M}_f(A)$ is defined by

$$\rightarrow_{\mathrm{msr}} (A) = \langle \mathcal{M}_{\mathrm{f}}(A), \Phi_{\mathrm{msr}}(A), \mathsf{src}, \mathsf{tgt} \rangle$$

whose objects are finite multisets over A, whose steps are given by

$$\Phi_{\mathrm{msr}}(A) = \{ \langle M_1, a, X \rangle \mid M_1, X \in \mathcal{M}_{\mathrm{f}}(A),\ a \in M_1,\ (\forall x \in X)[\, x < a \,] \}$$

and where the source and target functions $\mathsf{src}, \mathsf{tgt} : \Phi_{\mathrm{msr}}(A) \rightarrow \mathcal{M}_{\mathrm{f}}(A)$ are, for all $\langle M_1, a, X \rangle \in \Phi_{\mathrm{msr}}(A)$, defined by

$$\mathsf{src}(\langle M_1, a, X \rangle) = M_1 \ , \quad \text{and} \quad \mathsf{tgt}(\langle M_1, a, X \rangle) = (M_1 \setminus \mathrm{mset}(\{a\})) \uplus X \ .$$

For all $a \in A$ and $X \in \mathcal{M}_{\mathrm{f}}(A)$, we will use the notation $\phi_{a,X}, \psi_{a,X}, \ldots$ for steps in $\Phi_{\mathrm{msr}}(A)$ that are of the form $\phi_{a,X} : M \rightarrow_{\mathrm{msr}} (M \setminus \{a\}) \uplus X$ (similar for $\psi_{a,X}$ and analogous notations) for some $M \in \mathcal{M}_{\mathrm{f}}(A)$; by the definition of $\Phi_{\mathrm{msr}}(A)$ it follows that a step of the form $\phi_{a,X}$ can only exist if $(\forall x \in X)\,[\, x < a \,]$ holds.

Let $\rightarrow^{+}_{\mathrm{msr}} (A)$ and $\rightarrow^{*}_{\mathrm{msr}} (A)$ denote the transitive closure of $\rightarrow_{\mathrm{msr}} (A)$ and the reflexive-transitive closure of $\rightarrow_{\mathrm{msr}} (A)$, respectively. Due to the fact that every composite step $(\phi_1 \cdot \phi_2) \cdot \phi_3$ in $\rightarrow^{+}_{\mathrm{msr}} (A)$ or $\rightarrow^{*}_{\mathrm{msr}} (A)$ can be mimicked by the step $\phi_1 \cdot (\phi_2 \cdot \phi_3)$, and vice versa, we will not distinguish composite steps in $\rightarrow^{+}_{\mathrm{msr}} (A)$ or $\rightarrow^{*}_{\mathrm{msr}} (A)$ according to in what order the composition is carried out. That is, we will drop the brackets in steps of $\rightarrow^{+}_{\mathrm{msr}} (A)$ and $\rightarrow^{*}_{\mathrm{msr}} (A)$ that consist of the composition of more than two steps.

Lemma D.3 *Let $\langle A, < \rangle$ be a strictly partially ordered set. In the transitive closure $\rightarrow^{+}_{msr} (A)$ of $\rightarrow_{msr} (A)$ there do not exist steps ϕ with $\mathsf{src}(\phi) = \mathsf{tgt}(\phi)$.*

A consequence of this lemma is the irreflexivity of the relation $<_{\mathrm{ms}}$ that will be introduced now, which shows the well-definedness of $<_{\mathrm{ms}}$ as a strict partial order.

Definition D.4 (Multiset ordening, strict multiset ordening). Let $\langle A, < \rangle$ be a set endowed with a strict partial order.

The *strict multiset ordening* $<_{\mathrm{ms}}$ on the set $\mathcal{M}_{\mathrm{f}}(A)$ of multisets over A is defined for all $M_1, M_2 \in \mathcal{M}_{\mathrm{f}}(A)$ by

$$M_1 <_{\mathrm{ms}} M_2 \iff (\exists\, \phi \in \Phi^{+}_{\mathrm{msr}}(A))\, \left[\, \phi : M_2 \rightarrow^{+}_{\mathrm{msr}} M_1 \,\right] \ . \tag{D.1}$$

And the *multiset ordering* $\leq_{\text{ms}}$ on the $\mathcal{M}_{\text{f}}(A)$ is defined by

$$M_1 \leq_{\text{ms}} M_2 \iff (\exists\, \phi \in \Phi^{*}_{\text{msr}}(A))\left[\, \phi : M_2 \rightarrow^{*}_{\text{msr}} M_1 \,\right] \tag{D.2}$$

for all $M_1, M_2 \in \mathcal{M}_{\text{f}}(A)$.

It is an obvious consequence of this definition that $<_{\text{ms}}$ is the strict part of $\leq_{\text{ms}}$, and that $\leq_{\text{ms}}$ is the partial order induced by $<_{\text{ms}}$: For all strictly partially ordered sets $\langle A, < \rangle$, the multiset ordering $\leq_{\text{ms}}$ and the strict multiset ordering $<_{\text{ms}}$ on $\mathcal{M}_{\text{f}}(A)$

$$M_1 \leq_{\text{ms}} M_2 \iff M_1 <_{\text{ms}} M_2 \;\vee\; M_1 = M_2$$

holds for all $M_1, M_2 \in \mathcal{M}_{\text{f}}(A)$. We gather reached knowledge about $<_{\text{ms}}$ and $\leq_{\text{ms}}$ in a lemma.

Lemma D.5 *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then $\langle \mathcal{M}_{f}(A), <_{ms} \rangle$ is a strictly partially ordered set, and $\langle \mathcal{M}_{f}(A), <_{ms} \rangle$ is a partially ordered set. $<_{ms}$ is the strict part of $\leq_{ms}$, and vice versa, $\leq_{ms}$ is the partial order induced by the strict partial order $<_{ms}$.*

The following lemma is an easy consequence of the definitions of $\leq_{\text{ms}}$ and $<_{\text{ms}}$, which we will need later.

Lemma D.6 *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then both the multiset ordering $\leq_{ms}$ and the strict multiset ordering $<_{ms}$ on $\mathcal{M}_{f}(A)$ are monotone. That is, for all $M_1, M_2, N \in \mathcal{M}_{f}(A)$ it holds that*

$$M_1 \leq_{ms} M_2 \implies M_1 \uplus N \leq_{ms} M_2 \uplus N \;, \tag{D.3}$$

$$M_1 <_{ms} M_2 \implies M_1 \uplus N <_{ms} M_2 \uplus N \;. \tag{D.4}$$

As a digression we want to mention and sketch a proof for the fact that ‘cancellation’ holds for the multiset ordenings $<_{\text{ms}}$ and $\leq_{\text{ms}}$. That is, also the implications “$\Leftarrow$” hold in (D.3) and (D.4). The fact that this is true for $\leq_{\text{ms}}$ follows obviously from the statement of the following lemma about its strict part $<_{\text{ms}}$.

Lemma D.7 (‘Cancellation’ for $<_{\text{ms}}$). *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then for all $M_1, M_2, N \in \mathcal{M}_{f}(A)$ it holds that*

$$M_1 <_{ms} M_2 \impliedby M_1 \uplus N <_{ms} M_2 \uplus N \;. \tag{D.5}$$

This lemma is an immediate consequence of Lemma D.8 and Lemma D.9, which are stated below. Hereby Lemma D.8 asserts essentially that, for all strictly partially ordered sets $\langle A, < \rangle$, every sequence $(\phi_{a_1,X_1}, \ldots, \phi_{a_n,X_n})$ of composable steps in the ARS $\rightarrow_{\text{msr}}(A)$ (where $a_1, \ldots, a_n \in A$ and $X_1, \ldots, X_n \in \mathcal{M}_{\text{f}}(A)$) can be replaced by a sequence $(\psi_{b_1,Y_1}, \ldots, \psi_{b_m,Y_m})$ of composable steps of $\rightarrow_{\text{msr}}(A)$ (for some $b_1, \ldots, b_m \in A$ and $Y_1, \ldots, Y_m \in \mathcal{M}_{\text{f}}(A)$) where the sequence $(b_1, \ldots, b_m)$ in A is *non-increasing*, i.e. is such that

$$\neg(\exists\, i, j \in \{1, \ldots, m\})[\, i < j \;\&\; b_i < b_j \,]$$

holds. And Lemma D.9 states that, for all multisets M_1, M_2, N over a strictly partially ordered set $\langle A, < \rangle$, every sequence of composable steps of $\rightarrow_{\text{msr}}(A)$ between $M_2 \uplus N$ and $M_1 \uplus N$ can be replaced by a similar sequence of composable steps of $\rightarrow_{\text{msr}}(A)$ between M_2 and M_1.

Lemma D.8 *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then it holds for all multisets $M_1, M_2 \in \mathcal{M}_f(A)$ that*

$$\begin{aligned} M_1 <_{ms} M_2 \iff & (\exists\, n \in \omega \backslash \{0\})\, (\exists\, a_1, \ldots, a_n \in A)\, (\exists\, X_1, \ldots, X_n \in \mathcal{M}_f(A)) \\ & (\exists\, \phi_{a_1,X_1}, \ldots, \phi_{a_n,X_n} \in (\Phi_{msr}(A))^+) \\ & \big[\, (a_1, \ldots, a_n) \text{ is non-increasing } \& \\ & \& \;\; \phi_{a_1,X_1} \cdot \phi_{a_2,X_2} \cdot \ldots \cdot \phi_{a_n,X_n} : M_2 \rightarrow^+_{msr} M_1 \,\big] \;. \end{aligned}$$

Lemma D.9 *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then it holds for all $M_1, M_2, N \in \mathcal{M}_f(A)$, $n \in \omega$, elements $a_1, \ldots, a_n$ of A and multisets M_1, M_2 in $\mathcal{M}_f(A)$ that*

$$\begin{aligned} (a_1, \ldots, a_n) & \text{ is non-increasing } \implies \\ & (\forall \phi_{a_1,X_1}, \ldots \phi_{a_n,X_n} \in \Phi_{msr}(A)) \\ & \big[\, \phi_{a_1,X_1} \cdot \ldots \cdot \phi_{a_n,X_n} : M_2 \uplus N \rightarrow^+_{msr} M_1 \uplus N \implies \\ & \implies (\exists\, \psi_{a_1,X_1}, \ldots \psi_{a_n,X_n} \in \Phi_{msr}(A)) \\ & \big[\, \psi_{a_1,X_1} \cdot \ldots \cdot \psi_{a_n,X_n} : M_2 \rightarrow^+_{msr} M_1 \,\big] \big] \;. \end{aligned}$$

Lemma D.9 can be shown by a rather easy induction on n. And Lemma D.8 follows easily from the next lemma, which expresses that for all $a, a' \in A$ and $X, X' \in \mathcal{M}_{\text{f}}(A)$ such that (a, a') is non-increasing, every step $\phi_{a',X'} \in \Phi_{\text{msr}}(A)$ that is composable with a step of the form $\phi_{a,X} \in \Phi_{\text{msr}}(A)$ *can be permuted* over $\phi_{a,X}$, that is to say, the composite step $\phi_{a,X} \cdot \phi_{a',X'}$ of the transitive closure $\rightarrow^+_{\text{msr}}(A)$ of $\rightarrow_{\text{msr}}(A)$ can be mimicked by a composite step $\psi_{a',X'} \cdot \psi_{a,X}$ of $\rightarrow^+_{\text{msr}}(A)$ for some $\psi_{a,X}, \psi_{a',X'} \in \Phi_{\text{msr}}(A)$.

Lemma D.10 (Permutation of certain composable steps of $\rightarrow_{\text{msr}}$). *Let $\langle A, < \rangle$ be a strictly partially ordered set and $\rightarrow_{msr}(A) = \langle \mathcal{M}_f(A), \Phi_{msr}(A), \mathsf{src}, \mathsf{tgt} \rangle$ the ARS of multiset reduction on $\mathcal{M}_f(A)$. Then for all $a, a' \in \mathcal{M}_f(A)$ it holds:*

$$\begin{aligned} (\forall \phi_{a,X}, \phi_{a',X'} \in \Phi_{msr}(A))\, \big[\, a < a' \,\vee\, (\neg(a \leq a')\ \&\ \neg(a' \leq a)) \implies \\ \implies (\exists\, \psi_{a,X}, \psi_{a',X'} \in \Phi_{msr}(A))\, [\, \phi_{a,X} \cdot \phi_{a',X'} \simeq \psi_{a',X'} \cdot \psi_{a,X} \,] \big] \;. \end{aligned}$$

This lemma can be proved by an easy case analysis. We have hereby concluded our digression about the proof for Lemma D.7 and continue with reviewing properties of the multiset orderings $<_{\text{ms}}$ and $\leq_{\text{ms}}$.

The next lemma contains a characterization of the strict multiset ordening on $\mathcal{M}_{\text{f}}(A)$, for some strictly partially ordered set $\langle A, < \rangle$, which by some authors (for example by [2]) is taken as *the* definition of the strict multiset ordening $<_{\text{ms}}$.

Lemma D.11 (Alternative definition of $<_{\mathrm{ms}}$). *Let $\langle A, < \rangle$ be a strictly partially ordered set and let $<_{ms}$ be the strict multiset ordering on $\mathcal{M}_f(A)$. Then it holds:*

$$M_1 <_{ms} M_2 \iff (\exists X, Y \in \mathcal{M}_f(A)) \left[\emptyset \neq X \subseteq M_1 \;\&\; M_1 = (M_2 \setminus X) \uplus Y \;\& \right.$$
$$\left. \& \; (\forall y \in Y)(\exists x \in X)[\, y < x \,] \right] . \quad \text{(D.6)}$$

Hint of Proof. "$\Leftarrow$" in (D.6) can be proved by a straightforward induction on the length of a sequence $(\phi_1, \ldots, \phi_n)$ of composable steps in $\rightarrow_{\mathrm{msr}}(A)$ between two multisets M_1 and M_2, i.e. such that in $\rightarrow^{+}_{\mathrm{msr}}$

$$\mathsf{src}(\phi_1 \cdot \ldots \cdot \phi_n) = M_2 \quad \text{and} \quad \mathsf{tgt}(\phi_1 \cdot \ldots \cdot \phi_n) = M_1$$

holds. The direction "$\Rightarrow$" in (D.6) can be proved by induction on the number of elements of the set X, i.e. by induction on $|\{x \in X \mid X(x) > 0\}|$.

The important nontrivial property of multiset reduction is termination.[10]

Lemma D.12 (Termination of multiset reduction). *Let $\langle A, < \rangle$ be a strictly partially ordered set. Then the ARS $\rightarrow_{msr}(A)$ of multiset reduction on $\mathcal{M}_f(A)$ is strongly normalizing.*

A proof for this theorem can be carried out in an analogous way to a proof given for the multiset ordering $<_{\mathrm{ms}}$ in [2] on p. 23 for Theorem 2.5.5 (Baader and Nipkow give a proof for termination of the strict multiset ordering, here denoted by $<_{\mathrm{ms}}$, that they define according to the equivalence (D.6); this proof can easily be adapted for a proof of Lemma D.11). A nice alternative, proof-theoretic proof for Lemma D.11 can be found on the note [6]; that proof is due to W. Buchholz.

The following theorem about the well-foundedness of the multiset ordering is an immediate consequence of Lemma D.12 in view of the definitions in (D.1) and (D.2) of the orderings $<_{\mathrm{ms}}$ and $\leq_{\mathrm{ms}}$ on $\mathcal{M}_{\mathrm{f}}(A)$ for some given strictly partially ordered set $\langle A, < \rangle$. We recall the usual definition of well-foundedness for partially ordered sets: A strictly partially ordered set $\langle A, < \rangle$ is *well-founded* if and only if there does not exist an infinite chain of the form $\ldots < a_n < a_{n-1} < \ldots < a_2 < a_1 < a_0$ in $\langle A, < \rangle$. And a partially ordered set $\langle B, \leq \rangle$ is *well-founded* if and only if its strict part $\langle B, < \rangle$ is well-founded, where $<$ is defined by

$$a < b \iff a \leq b \;\&\; a \neq b$$

for all $a, b \in B$.

Theorem D.13 (Well-foundedness of the multiset ordening). *Let $\langle A, < \rangle$ be a strictly partially ordered set, that is well-founded.*

Then also the strict multiset ordening $<_{ms}$ and the multset ordening $\leq_{ms}$ on the set $\mathcal{M}_f(A)$ are well-founded.

[10] In [2] this phrase is used in connection with the multiset *ordering*.

D.4 Using measure functions to prove termination for ARS's

In the context of their treatment of *abstract reduction systems*, which are of the form $\langle \tilde{A}, \tilde{R} \rangle$ with $\tilde{R}$ a binary relation on $\tilde{A}$, [2] argue that the most basic method for proving termination of an abstract reduction system $\langle A, \rightarrow \rangle$ is to embed it into another abstract reduction system $\langle B, > \rangle$ that is known to terminate. This requires a *measure function* between $\langle A, \rightarrow \rangle$ and $\langle B, > \rangle$, by which a monotone function $m : A \rightarrow B$ is meant, i.e. for which

$$a \rightarrow b \quad \Longrightarrow \quad m(a) > m(b) \qquad (\text{for all } a, b \in A)$$

holds. If such a measure function m exists between $\langle A, \rightarrow \rangle$ and $\langle B, > \rangle$ and if $>$ is terminating on B, then also $\langle A, \rightarrow \rangle$ must be terminating: Every infinite reduction sequence $a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \ldots$ in $\langle A, \rightarrow \rangle$ would namely give rise to an infinite reduction sequence $m(a_0) > m(a_1) > m(a_2) > \ldots$ in $\langle B, > \rangle$.

Using the easy fact that an abstract reduction system $\langle A, \rightarrow \rangle$ is terminating if and only if $\langle A, \rightarrow^+ \rangle$ is terminating, where $\rightarrow^+$ denotes the transitive closure of $\rightarrow$, the described method can be slightly generalized: If for abstract reduction systems $\langle A, \rightarrow \rangle$ and $\langle B, > \rangle$ a function $m : A \rightarrow B$ with the property

$$a \rightarrow b \quad \Longrightarrow \quad m(a) >^+ m(b) \qquad (\text{for all } a, b \in A)$$

exists (where $>^+$ denotes the transitive closure of $>$ on B), then termination of $\langle B, > \rangle$ also implies termination of $\langle A, \rightarrow \rangle$.

We will now transfer this method for proving termination in a straightforward way to abstract rewrite systems. For this we consider two ARS's of the forms $\rightarrow_1 = \langle A_1, \Phi_1, \mathsf{src}_1, \mathsf{tgt}_1 \rangle$ and $\rightarrow_2 = \langle A_2, \Phi_2, \mathsf{src}_2, \mathsf{tgt}_2 \rangle$, and let $\rightarrow_2^+$ the transitive closure ARS of $\rightarrow_2$. We call a function $m : A_1 \rightarrow A_2$ a *measure function* between $\rightarrow_1$ and $\rightarrow_2$ if and only if it holds that

$$(\forall \phi \in \Phi_1)\, (\forall a, b \in A_1) \left[\, \Phi : a \rightarrow_1 b \;\; \Rightarrow \;\; (\exists \phi' \in \Phi_2^+)\, [\, \phi' : m(a) \rightarrow_2^+ m(b)\,] \right] .$$

Based on this definition, the following lemma holds.

Lemma D.14 *Let $\rightarrow_1$ and $\rightarrow_2$ be ARS's, and let $m : A_1 \rightarrow A_2$ a measure function between $\rightarrow_1$ and $\rightarrow_2$. Then it holds that*

$$\rightarrow_2 \textit{ is strongly normalizing} \quad \Longrightarrow \quad \rightarrow_1 \textit{ is strongly normalizing}\,.$$

Sketch of Proof. If, under the assumption of the lemma, $\{\phi_i\}_{i \in \omega}$ is an infinite sequence of composable steps in $\rightarrow_1$, then there exists, due to the existence of a measure function between $\rightarrow_1$ and $\rightarrow_2$, an infinite sequence $\{\phi_i'\}_{i \in \omega}$ of composable steps in $\rightarrow_2^+$, which also entails the existence of an infinite sequence $\{\phi_i''\}_{i \in \omega}$ of composable steps in $\rightarrow_2$. Hence $\rightarrow_2$ cannot be strongly normalizing if $\rightarrow_1$ isn't. □

D.5 Proof of Lemma 5.10

In this section we will carry out the proof for Lemma 5.10, i.e. we will demonstrate that ARS's of rule elimination by mimicking steps are strongly normalizing. As mentioned before, we will reduce the termination problem for such ARS's to the termination problem for ARS's of multiset reduction on $\mathcal{M}_{\mathrm{f}}(\omega)$ over $\langle \omega, > \rangle$, which ARS's are terminating due to Lemma D.12. For the purpose of defining an appropriate measure function between ARS's of these two kinds, we will need the following notion of R-depth of derivations in an n-AHS $\mathcal{S}$ with $R \in Fo_{\mathcal{S}}$: Whereas the (rule application) depth $|\mathcal{D}|$ of a derivation $\mathcal{D}$ in $\mathcal{S}$ stands for the maximal number of rule applications in a thread of $\mathcal{D}$ (from a leaf at the top down to the conclusion of $\mathcal{D}$), R-depth $|\mathcal{D}|_R$ of a derivation $\mathcal{D}$ will be defined so as to denote the maximal number of applications of R in a thread of $\mathcal{D}$.

Definition D.15 (R-depth of derivations). Let $\mathcal{S} = \langle Fo, Ax, \mathcal{R}, Na, \mathsf{name} \rangle$ be an n-AHS and let R be a rule of $\mathcal{S}$.

We define the *R-depth* $|\mathcal{D}|_R$ of arbitrary derivations $\mathcal{D} \in Der(\mathcal{S})$ according to the function

$$|\cdot|_R : Der(\mathcal{S}) \longrightarrow \omega\,, \quad \mathcal{D} \longmapsto |\mathcal{D}|_R$$

on the set $Der(\mathcal{S})$ of derivations of $\mathcal{D}$, which in its turn is defined with induction on the (rule application) depth $|\mathcal{D}|$ of derivations $\mathcal{D}$ in $\mathcal{S}$ by the following clauses: [11]

(i) If $|\mathcal{D}| = 0$, then $\mathcal{D}$ consists of an axiom or of an assumption and does not contain rules. In this case we let

$$|\mathcal{D}|_R = 0 \;\;.$$

(ii) If $|\mathcal{D}| > 0$, then we distinguish two cases concerning the arity n of the bottommost application of a rule in $\mathcal{D}$.

Case 1. $n = 0$: Then $\mathcal{D}$ is of the form

$$\frac{}{A}\,\mathsf{name}(R')$$

for some $A \in Fo$ and a rule $R' \in \mathcal{R}$. Here we stipulate

$$|\mathcal{D}|_R = \begin{cases} 1 & \ldots \; R' = R \\ 0 & \ldots \; R' \neq R \;. \end{cases}$$

Case 2. $n > 0$: Here $\mathcal{D}$ is of the form

$$\frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array}}{A}\,\mathsf{name}(R')$$

[11] The rule application depth $|\mathcal{D}|$ of a derivation $\mathcal{D}$ in an n-AHS has been defined in Definition 2.4.

for a rule $R' \in \mathcal{R}$, formulas $A_1, \ldots, A_n \in Fo$ and derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n \in Der(\mathcal{S})$. Then we set

$$|\mathcal{D}|_R = \begin{cases} 1 + \max\{\,|\mathcal{D}_i|_R \mid 1 \le i \le n\} & \ldots\ R' = R \\ \max\{\,|\mathcal{D}_i|_R \mid 1 \le i \le n\} & \ldots\ R' \neq R\ . \end{cases}$$

Two easy properties of the notion of R-depth of derivations in an n-AHS with R as a rule are given in the following lemma. These properties will be needed in the proof of Lemma 5.10.

Lemma D.16 *Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$.*

(i) *Let $\mathcal{D}$ be a derivation and $\mathcal{D}_0$ be a subderivation of $\mathcal{S}$. Then $|\mathcal{D}_0|_R \le |\mathcal{D}|_R$ holds.*

(ii) *For all derivations $\mathcal{D}_{00}$, $\mathcal{D}'_{00}$ and $\begin{matrix}(A)\\ \mathcal{D}_0\end{matrix}$ in $\mathcal{S}$, where a particular assumption of A in $\mathcal{D}_0$ is symbolically singled out, it holds that*

$$|\mathcal{D}'_{00}|_R \le |\mathcal{D}_{00}|_R \quad\Longrightarrow\quad \left|\begin{matrix}\mathcal{D}'_{00}\\ (A)\\ \mathcal{D}_0\end{matrix}\right|_R \le \left|\begin{matrix}\mathcal{D}_{00}\\ (A)\\ \mathcal{D}_0\end{matrix}\right|_R .$$

We want to mention that the assertions of Lemma D.16 hold also, if R-depth $|\cdot|_R$ is replaced everywhere by usual (rule application) depth $|\cdot|$.

Proof of Lemma 5.10. Let $\mathcal{S}$ be an n-AHS and R be a rule of $\mathcal{S}$ that is derivable in $\mathcal{S}{-}R$. We have to show that the ARS $\to^{(R)}_{\text{mim}}(\mathcal{S}) = \langle Der(\mathcal{S}), \Phi^{(R)}_{\text{mim}}(\mathcal{S}), \mathsf{src}, \mathsf{tgt}\rangle$ of R-elimination by mimicking steps as introduced on page 31 is strongly normalizing.

To prove this, it is sufficient due to Lemma D.14 to find a measure function between $\to^{(R)}_{\text{mim}}(\mathcal{S})$ and a strongly normalizing ARS. We will now define a function m between $\to^{(R)}_{\text{mim}}(\mathcal{S})$ and the ARS $\to_{\text{msr}}(\omega)$ of multiset reduction on $\mathcal{M}_{\text{f}}(\omega)$, which is strongly normalizing as a consequence of Lemma D.12. We define m as the function

$$\begin{aligned} m : Der(\mathcal{S}) &\longrightarrow \mathcal{M}_{\text{f}}(\omega) \\ \mathcal{D} &\longmapsto m(\mathcal{D}) : \omega \to \omega \\ &\qquad l \mapsto \#\big\{\,\tilde{\mathcal{D}}_0 \,\big|\, \tilde{\mathcal{D}}_0 \text{ is a subderivation of } \mathcal{D} \text{ with } \big|\tilde{\mathcal{D}}_0\big|_R = l \\ &\qquad\qquad \text{that is ending with an application of } R\big\}\ , \end{aligned}$$

where we used the symbol # as a notation for an operation that gives the cardinality of a finite set: For every finite set A, we wrote and will do so again below $\#A$ for the cardinality of A, i.e. the finite number of elements of A. We will show that m is indeed a measure function between $\to^{(R)}_{\text{mim}}(\mathcal{S})$ and $\to_{\text{msr}}(\omega)$ in the remaining part of this proof.

To show that this is the case, we have to demonstrate

$$(\forall \phi \in \Phi^{(R)}_{\text{mim}}(\mathcal{S}))\,(\forall \mathcal{D}, \mathcal{D}' \in \mathit{Der}(\mathcal{S}))\,\big[\,\Phi : \mathcal{D} \to^{(R)}_{\text{mim}} \mathcal{D}' \implies \implies (\exists \phi' \in \Phi^{+}_{\text{msr}})\,[\,\phi' : m(\mathcal{D}) \to^{+}_{\text{msr}} m(\mathcal{D}')\,]\,\big]\;. \tag{D.7}$$

For proving this, we let $\mathcal{D}, \mathcal{D}' \in \mathit{Der}(\mathcal{S})$ and a step $\phi \in \Phi^{(R)}_{\text{mim}}(\mathcal{S})$ be arbitrary such that $\phi : \mathcal{D} \to^{(R)}_{\text{mim}} \mathcal{D}'$. We have to show

$$m(\mathcal{D}') \;<^{+}_{\text{ms}}\; m(\mathcal{D})\;. \tag{D.8}$$

By the definition of $\to^{(R)}_{\text{mim}}(\mathcal{S})$, the step ϕ is of the form (5.17) or (5.18). We will only consider the case that ϕ is of the more complicated form (5.18), since the argument to show (D.8) in the case of a step of the form (5.17) is basically just a special situation of the following demonstration.

We will now consider the case that ϕ is a mimicking step $\phi : \mathcal{D} \to \mathcal{D}'$ (for some $\mathcal{D}, \mathcal{D}' \in \mathit{Der}(\mathcal{S})$) of the particular form

$$\phi \;:\; \begin{array}{c} \begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n \end{array} \\ \hline (A) \\ \mathcal{D}_0 \end{array}\,\mathsf{name}(R) \quad \to^{(R)}_{\text{mim}} \quad \begin{array}{c} \begin{array}{ccc} \mathcal{D}_{i_1} & & \mathcal{D}_{i_k} \\ (A_{i_1}) & \ldots & (A_{i_k}) \end{array} \\ \mathcal{D}_\alpha \\ (A) \\ \mathcal{D}_0 \end{array} \tag{D.9}$$

whereby

- $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n \in \mathit{Der}(\mathcal{S})$, $A_1, \ldots, A_n, A \in \mathit{Fo}_{\mathcal{S}}$, $n \in \omega \backslash \{0\}$ and $k \in \omega$;
- $\mathcal{D}$ and $\mathcal{D}'$ are the derivations in $\mathcal{S}$ on the left and right sides of the step shown in (D.9), respectively;
- $\mathcal{D}_\alpha$ is a mimicking derivation in $\mathcal{S}{-}R$ for the R-application

  $$\frac{A_1 \quad \ldots \quad A_n}{A}\,\mathsf{name}(R)\quad,$$

  or equivalently, for this derivation $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ corresponding to α in $\mathcal{S}$;
- for the indices $i_1, \ldots, i_k \in \omega$ it holds that $1 \le i_1, i_2, \ldots, i_k \le n$, the expressions $(A_{i_1}), \ldots, (A_{i_k})$ at the top of $\mathcal{D}_\alpha$ in $\mathcal{D}'$ stand for single occurrences of the assumptions $A_{i_1}, \ldots, A_{i_k}$ in $\mathcal{D}_\alpha$, which together make up all assumptions of $\mathcal{D}_\alpha$, i.e. such that it holds $\mathsf{assm}(\mathcal{D}_\alpha) = \text{mset}((A_{i_1}, A_{i_2}, \ldots, A_{i_k}))$.

We first notice that $m(\mathcal{D})$ is of the form

$$m(\mathcal{D}) = M_1 \uplus \ldots \uplus M_n \uplus \text{mset}(\{|\mathcal{D}_{00}|_R\}) \uplus M_0 \uplus M \tag{D.10}$$

with $M_1, \ldots, M_n \in \mathcal{M}_{\text{f}}(\omega)$ that are defined by

$$M_i(l) = \#\big\{\, \tilde{\mathcal{D}}_0 \;\big|\; \tilde{\mathcal{D}}_0 \text{ is a subderivation of } \mathcal{D}_i \text{ that is ending with an application of } R \text{ and fulfills } \big|\tilde{\mathcal{D}}_0\big|_R = l \,\big\}\;,$$

for all $i \in \{1,\ldots,n\}$ and $l \in \omega$, and $M_0, M \in \mathcal{M}_{\mathrm{f}}(\omega)$ by

$$M_0(l) = \#\{\, \tilde{\mathcal{D}}_0 \mid \tilde{\mathcal{D}}_0 \text{ is a subderivation of } \mathcal{D} \text{ that ends in an application of } R \text{ in the part } \mathcal{D}_0 \text{ of } \mathcal{D} \textit{ below} \text{ the subderivation } \mathcal{D}_{00} \text{ and fulfills } \big|\tilde{\mathcal{D}}_0\big|_R = l \,\}\ ,$$

$$M(l) = \#\{\, \tilde{\mathcal{D}}_0 \mid \tilde{\mathcal{D}}_0 \text{ is a subderivation of } \mathcal{D} \text{ with } \big|\tilde{\mathcal{D}}_0\big|_R = l \text{ that ends in an } R\text{-application in the part } \mathcal{D}_0 \text{ of } \mathcal{D} \text{ and that does not have } \mathcal{D}_{00} \text{ as a subderivation} \,\}$$

for all $l \in \omega$, respectively. And furthermore, we observe that $m(\mathcal{D})$ is of a similar form as $m(\mathcal{D}')$ in (D.10), namely

$$m(\mathcal{D}') = M_{i_1} \uplus \ldots \uplus M_{i_k} \uplus M_0' \uplus M\ , \tag{D.11}$$

where $M_0' \in \mathcal{M}_{\mathrm{f}}(\omega)$ is defined by

$$M_0'(l) = \#\{\, \tilde{\mathcal{D}}_0 \mid \tilde{\mathcal{D}}_0 \text{ is a subderivation of } \mathcal{D}' \text{ that ends in an application of } R \text{ in the part } \mathcal{D}_0' \text{ of } \mathcal{D} \textit{ below} \text{ the subderivation } \mathcal{D}_{00}' \text{ and fulfills } \big|\tilde{\mathcal{D}}_0\big|_R = l \,\}$$

for all $l \in \omega$. For eventually showing (D.8), we are going to use the assertions

$$M_{i_1} \uplus \ldots \uplus M_{i_k} <_{\mathrm{ms}} M_1 \uplus \ldots \uplus M_n \uplus \mathrm{mset}(\{|\mathcal{D}_{00}|_R\}) \tag{D.12}$$

$$M_0' \leq_{\mathrm{ms}} M_0 \tag{D.13}$$

which we will now demonstrate subsequently.

(D.12) is a consequence of

$$(\forall j \in \{1,\ldots,n\})\,(\forall\, m \in M_j)\,\big[\, m < |\mathcal{D}_{00}|_R \,\big]\ , \tag{D.14}$$

since this implies

$$M_1 \uplus \ldots \uplus M_n \uplus \mathrm{mset}(\{|\mathcal{D}_{00}|_R\}) \ \to_{\mathrm{msr}}\ M_{i_1} \uplus \ldots \uplus M_{i_k}$$

(recall that $i_1,\ldots,i_k \in \{1,\ldots,n\}$). Hence we proceed to show (D.14). For this we let $j \in \{1,\ldots,n\}$ and $m \in M_j$ be arbitrary. Then $m = \big|\tilde{\mathcal{D}}_0\big|_R$ for some subderivation $\tilde{\mathcal{D}}_0$ of $\mathcal{D}_j$ that ends in an application of R. We note that

$$|\mathcal{D}_{00}|_R = 1 + \max\{\, |\mathcal{D}_i|_R \mid 1 \leq i \leq n \} \tag{D.15}$$

follows from the definition of $|\cdot|_R$. By Lemma D.16 (i) and (D.15), we can now conclude:

$$m = \big|\tilde{\mathcal{D}}_0\big|_R \leq |\mathcal{D}_j|_R < 1 + \max\{\, |\mathcal{D}_i|_R \mid 1 \leq i \leq n \} = |\mathcal{D}_{00}|_R\ .$$

Since $j \in \{1,\ldots,n\}$ and $m \in M_j$ were arbitrary, we have shown (D.14), and as a consequence, (D.12).

Next we show (D.13). First we notice that due to (D.15) and the fact that $\mathcal{D}_\alpha$ does not contain applications of R it holds

$$|\mathcal{D}'_{00}|_R = \max\left\{|\mathcal{D}_{i_l}|_R \mid 1 \le l \le k\right\} \le \max\{|\mathcal{D}_i|_R \mid 1 \le i \le n\} < |\mathcal{D}_{00}|_R\ . \qquad \text{(D.16)}$$

(the equality here can be shown by induction on the (rule application) depth $|\mathcal{D}_\alpha|$ of $\mathcal{D}_\alpha$). Due to the definition of M_0 and M'_0 there is a bijective correspondence between these two multisets that relates elements of the form

$$\text{elements of the form} \quad \left|\begin{array}{c}\mathcal{D}_{00}\\ (A)\\ \tilde{\mathcal{D}}_0\end{array}\right|_R \quad \text{of } M_0 \text{ with elements} \quad \left|\begin{array}{c}\mathcal{D}'_{00}\\ (A)\\ \tilde{\mathcal{D}}_0\end{array}\right|_R \quad \text{of } M'_0,$$

where $\tilde{\mathcal{D}}_0$ is a subderivation of $\mathcal{D}_0$ that ends in an application of R. Since for all such corresponding elements of M_0 and M'_0 respectively

$$\left|\begin{array}{c}\mathcal{D}'_{00}\\ (A)\\ \tilde{\mathcal{D}}_0\end{array}\right|_R \le \left|\begin{array}{c}\mathcal{D}_{00}\\ (A)\\ \tilde{\mathcal{D}}_0\end{array}\right|_R$$

holds as a consequence of (D.16) and Lemma D.16 (ii), (D.13) follows: By a finite (and possibly zero-length) chain of $\to_{\text{msr}}$-steps, every element of M_0 can be replaced by its corresponding element in M'_0, if this is in fact smaller (otherwise no replacement is necessary). We have shown (D.13).

Using (D.10), (D.11), (D.12), (D.13), and Lemma D.6, we now find

$$\begin{aligned} m(\mathcal{D}') &= M_{i_1} \uplus \ldots \uplus M_{i_k} \uplus M'_0 \uplus M \\ &\le_{\text{ms}} M_{i_1} \uplus \ldots \uplus M_{i_k} \uplus M_0 \uplus M \\ &<_{\text{ms}} M_1 \uplus \ldots \uplus M_n \uplus \text{mset}(\{|\mathcal{D}_{00}|_R\}) \uplus M_0 \uplus M \\ &= m(\mathcal{D})\ . \end{aligned}$$

By transitivity of $\le_{\text{ms}}$ and $<_{\text{ms}}$, we have shown that $m(\mathcal{D}') <_{\text{ms}} m(\mathcal{D}')$ holds and thus, by the definition of $<_{\text{ms}}$, that $m(\mathcal{D}) \to^{+}_{\text{msr}} m(\mathcal{D}')$ is the case, i.e. (D.8). Since we have considered an arbitrary mimicking step $\phi : \mathcal{D} \to \mathcal{D}'$ of the form (5.18) and have shown (D.8), and since this can be shown—as remarked above—in an easier way for the case of mimicking steps of the form (5.17), we have proven (D.7) and hence that m is a measure function between $\to^{(R)}_{\text{mim}}(\mathcal{S})$ and $\to_{\text{msr}}(\omega)$. As argued above, this implies that the ARS $\to^{(R)}_{\text{mim}}(\mathcal{S})$ of R-elimination in $\mathcal{S}$ by mimicking steps is strongly normalizing.

□

D.6 Proofs for other statements in Section 5

Proof of Lemma 5.7. Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$.

The assertion in Lemma 5.7 that the result $\mathcal{D}'$ of applying a sequence of mimicking (s-mimicking, m-mimicking) steps to a derivation $\mathcal{D} \in \mathit{Der}(\mathcal{S})$ has the property that it mimics (s-mimics, m-mimics) $\mathcal{D}$ follows by induction on the length of the respective sequence of applied (s-, m-) mimicking steps from the following assertion: For every mimicking (s-mimicking, m-mimicking) step $\phi : \mathcal{D} \to \mathcal{D}'$ of $\to^{(R)}_{\text{mim}}(\mathcal{S})$, the target $\mathcal{D}'$ of ϕ mimics (s-mimics, m-mimics) the source $\mathcal{D}$ of ϕ. More precisely, the assertion that, for all $\mathcal{D}, \mathcal{D}' \in \mathit{Der}(\mathcal{S})$, (5.26), (5.27) and (5.28) hold is a consequence of the assertion that, for all $\mathcal{D}, \mathcal{D}' \in \mathit{Der}(\mathcal{S})$,

$$(\exists\,\phi \in \Phi^{(R)}_{\text{mim}}(\mathcal{S}))\,\big[\,\phi : \mathcal{D} \to^{(R)}_{\text{mim}} \mathcal{D}'\,\big] \quad\Longrightarrow\quad \mathcal{D}' \precsim \mathcal{D}\ , \tag{D.17}$$

$$(\exists\,\phi \in \Phi^{(R)}_{\text{m-mim}}(\mathcal{S}))\,\big[\,\phi : \mathcal{D} \to^{(R)}_{\text{m-mim}} \mathcal{D}'\,\big] \quad\Longrightarrow\quad \mathcal{D}' \simeq^{(\text{m})} \mathcal{D}\ . \tag{D.18}$$

hold ((5.27) follows by an inductive proof based on (D.17) in which it is used that every s-mimicking step is also a mimicking step). It therefore suffices to prove (D.17) and (D.18).

To show (D.17), let us consider an arbitrary mimicking step $\phi : \mathcal{D} \to^{(R)}_{\text{mim}} \mathcal{D}'$ of the ARS $\to^{(R)}_{\text{mim}}(\mathcal{S})$. If ϕ is of the form (5.17) with the denotations explained there, then due to $\mathsf{assm}(\mathcal{D}_\alpha) = \emptyset$ (this is the case because $\mathcal{D}_\alpha$ is a mimicking derivation for a zero-premise application of R) we find

$$\mathsf{assm}(\mathcal{D}') = (\mathsf{assm}(\mathcal{D}_0) \setminus \{A\}) \uplus \mathsf{assm}(\mathcal{D}_\alpha) = \mathsf{assm}(\mathcal{D}_0) \setminus \{A\} = \mathsf{assm}(\mathcal{D})\ ,$$
$$\text{and}\quad \mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D}_\alpha) = A = \mathsf{concl}(\mathcal{D})\ .$$

Therefore we can conclude that

$$\mathcal{D}' \precsim \mathcal{D} \tag{D.19}$$

holds, i.e. that the target derivation $\mathcal{D}'$ of ϕ mimics the source derivation $\mathcal{D}$ of ϕ in this case.

If ϕ is of the form (5.18) with the denotations explained there, in particular

$$\mathsf{assm}(\mathcal{D}_\alpha) = \text{mset}((A_{i_1}, \ldots, A_{i_k}))$$

holds. Due to this we find

$$\begin{aligned}
\mathsf{assm}(\mathcal{D}') &= (\mathsf{assm}(\mathcal{D}_0) \setminus \{A\}) \uplus (\mathsf{assm}(\mathcal{D}_\alpha) \setminus \text{mset}((A_{i_1}, \ldots, A_{i_k}))) \uplus \\
&\qquad \uplus \biguplus_{j=1}^{k} \mathsf{assm}(\mathcal{D}_{i_j}) \\
&= (\mathsf{assm}(\mathcal{D}_0) \setminus \{A\}) \uplus \biguplus_{j=1}^{k} \mathsf{assm}(\mathcal{D}_{i_j}) \\
&= (\mathsf{assm}(\mathcal{D}_0) \setminus \{A\}) \uplus \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_{i}) \\
&= \mathcal{D}\ .
\end{aligned}$$

Since $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, it follows that

$$\mathrm{set}(\mathsf{assm}(\mathcal{D}')) \subseteq \mathrm{set}(\mathsf{assm}(\mathcal{D})) ,$$

and hence, since also

$$\mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D}_0) = \mathsf{concl}(\mathcal{D})$$

is the case, that is, $\mathcal{D}'$ mimics $\mathcal{D}$. Thus (D.19) holds.

Because we have now shown (D.19) for an arbitrary mimicking step $\phi : \mathcal{D} \to^{(R)}_{\mathrm{mim}} \mathcal{D}'$ of $\to^{(R)}_{\mathrm{mim}}(\mathcal{S})$, we have proven (D.17).

To prove (D.18), let $\phi : \mathcal{D} \to^{(R)}_{\text{m-mim}} \mathcal{D}'$ be an arbitrary m-mimicking step of the ARS $\to^{(R)}_{\text{m-mim}}(\mathcal{S})$. If ϕ is an m-mimicking step of the form (5.17) (in which a zero-premise application of R gets replaced by an m-mimicking derivation), then it can be argued analogously as in the proof of (D.17) above that

$$\mathcal{D}' \simeq^{(\mathrm{m})} \mathcal{D} \tag{D.20}$$

holds, i.e. that $\mathcal{D}'$ m-mimics $\mathcal{D}$.

Otherwise ϕ is of the form (5.18), and since ϕ is a m-mimicking step here, it is in particular of the form

$$\phi \;:\; \dfrac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n \\ A_1 & \ldots & A_n\end{matrix}}{\begin{matrix}(A) \\ \mathcal{D}_0\end{matrix}}\,\mathsf{name}(R) \quad \to^{(R)}_{\text{m-mim}} \quad \begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n \\ (A_1) & \ldots & (A_n) \\ & \mathcal{D}_\alpha & \\ & (A) & \\ & \mathcal{D}_0 & \end{matrix} \tag{D.21}$$

for some derivation $\mathcal{D}_\alpha \in \mathit{Der}(\mathcal{S}{-}R)$ with

$$\mathcal{D}_\alpha \quad \simeq^{(\mathrm{m})} \quad \frac{A_1 \,\ldots\, A_n}{A}\,\mathsf{name}(R)$$

(i.e. $\mathcal{D}_\alpha$ m-mimics the application α of R on the right-hand side above) and with

$$\mathsf{assm}(\mathcal{D}_\alpha) = \mathrm{mset}((A_1, \ldots, A_n)) \,. \tag{D.22}$$

In the symbolic prooftree for $\mathcal{D}'$ on the right-hand side of (D.21) the expressions $(A_1), \ldots, (A_n)$ represent the occurrences of single formulas $A_1, \ldots, A_n$ as assumptions at the top of the prooftree $\mathcal{D}_\alpha$ (into which the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are respectively substituted). Using (D.22), we find here that

$$\begin{aligned}\mathsf{assm}(\mathcal{D}') \;=\; & (\mathsf{assm}(\mathcal{D}_\alpha) \setminus \{A\}) \uplus (\mathsf{assm}(\mathcal{D}_\alpha) \setminus \mathrm{mset}((A_1, \ldots, A_n))) \uplus \\ & \uplus \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_i)\end{aligned}$$

$$= (\mathsf{assm}(\mathcal{D}_\alpha) \setminus \{A\}) \uplus \biguplus_{i=1}^{n} \mathsf{assm}(\mathcal{D}_i)$$
$$= \mathsf{assm}(\mathcal{D})$$

and
$$\mathsf{concl}(\mathcal{D}') = \mathsf{concl}(\mathcal{D}_0) = \mathsf{concl}(\mathcal{D})$$
holds. Thus $\mathcal{D}'$ m-mimics $\mathcal{D}$ and hence (D.20) is the case.

We have shown (D.20) for arbitrary steps $\phi : \mathcal{D} \to^{(R)}_{\text{m-mim}} \mathcal{D}'$ in $\to^{(R)}_{\text{m-mim}}(\mathcal{S})$ and we have therefore demonstrated (D.18).

□

Proof of Lemma 5.8. We let Fo be the four-element set $\{A, B, C_1, C_2\}$. And we let $\mathcal{S}$ be the n-AHS that has Fo as its set of formulas, that possesses no axioms and that contains precisely four rules, each of which has only one application, namely:

$$\frac{C_1}{A}\,R_1 \qquad \frac{C_2}{A}\,R_2 \qquad \frac{A \quad A}{B}\,R_{3a} \qquad \frac{A}{B}\,R_{3b}$$

(we have previously encountered these rules, and a similar argumentation as will be given below, in Example C.2). Due to the fact that

$$\frac{A \quad A}{B}\,R_{3a} \quad \simeq^{(\mathrm{s})} \quad \frac{A}{B}\,R_{3b}$$

holds, we find that

$$\phi \;:\; \frac{\dfrac{C_1}{A}\,R_1 \quad \dfrac{C_2}{A}\,R_2}{B}\,R_{3a} \quad \to^{(R_{3a})}_{\text{s-mim}} \quad \frac{\dfrac{C_1}{A}\,R_1}{B}\,R_{3b}$$

is an s-mimicking step of the ARS $\to^{(R)}_{\text{s-mim}}(\mathcal{S})$, the source and target of which we respectively denote by $\mathcal{D}$ and $\mathcal{D}'$. Since clearly

$$\mathsf{assm}(\mathcal{D}') = \mathrm{mset}(\{C_2\}) \neq \mathrm{mset}(\{C_1, C_2\}) = \mathsf{assm}(\mathcal{D})$$

is the case, we find that
$$\mathcal{D}' \not\simeq^{(\mathrm{s})} \mathcal{D}\,.$$
As a consequence, (5.27) does not hold for all $\mathcal{D}, \mathcal{D}' \in Der(\mathcal{S})$ and hence R_{3a}-elimination in $Der(\mathcal{S})$ by s-mimicking steps is *not correct* with respect to $\simeq^{(\mathrm{s})}$.

□

Proof of Theorem 5.13. Let $\mathcal{S}$ be an n-AHS and let R be a rule of $\mathcal{S}$. We will only show (5.33) since (5.32) and (5.34) can be shown analogously.

For showing "$\Leftarrow$" in (5.33), let R be s-derivable in $\mathcal{S}$. Then it follows by Lemma 5.11 (i) that the ARS $\to^{(R)}_{\text{s-mim}}(\mathcal{S})$ is strongly normalizing. And by Lemma 5.5 it follows that $\mathcal{NF}(\to^{(R)}_{\text{s-mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R)$. Hence in view of Definition 5.12, we can conclude that R-elimination by s-mimicking steps holds in $Der(\mathcal{S})$.

For demonstrating "⇒" in (5.33), we assume that strong R-elimination by s-mimicking steps holds in $Der(\mathcal{S})$. This entails

$$\mathcal{NF}(\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{S})) = Der(\mathcal{S}{-}R)\ . \tag{D.23}$$

To show that R is s-derivable in $\mathcal{S}$, let α be an arbitrary application of R in $\mathcal{S}$. We have to establish that α can be s-mimicked by a derivation in $\mathcal{S}{-}R$, or what is the same in the light of (5.6), that a derivation $\mathcal{D}_\alpha \in Der(\mathcal{S}{-}R)$ exists with the property

$$\mathcal{D}_\alpha \simeq^{(\text{s})} \mathcal{D}_{(\alpha,R,\mathcal{S})}\ , \tag{D.24}$$

where $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ is the derivation that corresponds to application α of R in $\mathcal{S}$. Due to (D.23) and $\mathcal{D}_{(\alpha,R,\mathcal{S})} \notin Der(\mathcal{S}{-}R)$ (this is because $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ *contains* an application of R), the derivation $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ cannot be a normal form of $\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{S})$. Hence there must exist an s-mimicking step in $\rightarrow^{(R)}_{\text{s-mim}}(\mathcal{S})$ with $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ as its source. Let $\phi \in \Phi^{(R)}_{\text{s-mim}}(\mathcal{S})$ be arbitrary such that $\mathsf{src}(\phi) = \mathcal{D}_{(\alpha,R,\mathcal{S})}$ and let $\mathcal{D}_\alpha = \mathsf{tgt}(\phi)$, i.e. let $\mathcal{D}_\alpha$ be the result of performing the s-mimicking step ϕ to $\mathcal{D}_{(\alpha,R,\mathcal{S})}$. Since $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ contains only a single rule application, namely one of R, the derivation $\mathcal{D}_\alpha$ must be an s-mimicking derivation for $\mathcal{D}_{(\alpha,R,\mathcal{S})}$ in $\mathcal{S}$. We have thereby shown the existence of a derivation $\mathcal{D}_\alpha \in Der(\mathcal{S}{-}R)$ with (D.24), and via the already mentioned statement (5.6), that the application α of R is s-mimicked by a derivation $\mathcal{D}_\alpha$ in $\mathcal{S}{-}R$. Since α was an arbitrary application of R in this argument, we can conclude now that every application of R can be s-mimicked by a respective derivation in $\mathcal{S}{-}R$. Hence R is s-derivable in $\mathcal{S}{-}R$.

□

Appendix E: Relationship with Hilbert systems for consequence à la Avron

In this appendix we establish a connection between our notion of abstract Hilbert system and the notion of "Hilbert-type system for consequence" introduced by Avron in [1]. We show the existence of natural correspondences between, on the one hand, AHS's that are endowed with consequence relations of the sort introduced in Definition 2.8, and on the other hand, certain systems from the sequent-style formalization of Hilbert systems by Avron. In particular, we prove that for every AHS $\mathcal{S}$ there exists a "pure" and single-conclusioned sequent-style "Hilbert system for consequence" $\mathcal{H}$ such that the consequence relations $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}, \vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$ and $\vdash_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$ are respectively "axiomatized" by $\mathcal{H}$, by the extension of $\mathcal{H}$ with the contraction rule, and by the extension of $\mathcal{H}$ with contraction and weakening rules. And we will also formulate a similar 'reverse statement'.

As a byproduct of these correspondences, it becomes apparent that, apart from the consequence relations of Definition 2.8, another consequence relation that is similarly definable on AHS's does naturally fit into our correspondence results: One that is axiomatized by a "Hilbert system for consequence" containing weakening, but not contraction rules. We will formally introduce this additional consequence relation $\vdash^{\mathbf{(mw)}}$ and show its corresondence with a respective sequent-style system. Furthermore we will prove that, for all sets *Fo*, a relation $\Vdash$ between *sets* on *Fo* and *Fo*, or between *multisets* on *Fo* and *Fo* is "naturally axiomatizable" by a HSC if and only if it is one of the consequence relations $\vdash_{\mathcal{S}}$, $\vdash_{\mathcal{S}}^{\mathbf{(s)}}$, $\vdash_{\mathcal{S}}^{\mathbf{(mw)}}$ and $\vdash_{\mathcal{S}}^{\mathbf{(m)}}$ on an AHS $\mathcal{S}$ with formula set *Fo*.

In the following definition, a *language* L is understood (in the sense of [1] recalled at the start of Section 2) to consist of several syntactic categories, among which there is the category of 'well-formed formulae' (wff). The class of wff of a language L will be denoted by wff(L). Furthermore, for every language L, by a *sequent* in L we will mean an expression of the form $\Gamma \Rightarrow \Delta$ for some $\Gamma, \Delta \in \mathcal{M}_{\mathrm{f}}(\mathrm{wff}(L))$, i.e. finite multisets over L. In a sequent $\Gamma \Rightarrow \Delta$ we call Γ the *antecedens* and Δ the *succedens*. In writing sequents we will furthermore conform to some standard abbreviations from proof theory. For example, a sequent $A \Rightarrow A$ over L with $A \in \mathrm{wff}(L)$ stands short for the sequent $\mathrm{mset}(\{A\}) \Rightarrow \mathrm{mset}(\{A\})$, and a sequent $A, \Gamma_1\Gamma_2 \Rightarrow B, C, \Delta$ with $A, B, C \in \mathrm{wff}(L)$ and $\Gamma_1, \Gamma_2, \Delta \in \mathcal{M}_{\mathrm{f}}(\mathrm{wff}(L))$ stands short for the sequent

$$\mathrm{mset}(\{A\}) \uplus \Gamma_1 \uplus \Gamma_2 \;\Rightarrow\; \mathrm{mset}(\{B\}) \uplus \mathrm{mset}(\{C\}) \uplus \Delta \;\;.$$

For multisets $\Gamma_1, \ldots, \Gamma_n$ of wff, we will abbreviate their union $\biguplus_{i=1}^{n} \Gamma_i$ by $\Gamma_1 \ldots \Gamma_n$.

The definition of "Hilbert systems for consequence" given below follows the definition of what Avron in [1] calls "Hilbert-type systems for consequence". [12]

Definition E.1 (Hilbert systems for consequence). A *Hilbert system for consequence* (a HSC) $\mathcal{H}$ in the language L is an axiomatic system such that:

(i) The *formulas* of $\mathcal{H}$ are sequents in L.

(ii) The *axioms* of $\mathcal{H}$ include $A \Rightarrow A$ for all A. All other axioms of $\mathcal{H}$ are of the form $\Rightarrow A$.

(iii) Every *rule* R of $\mathcal{H}$ is an *n-premise rule* for some $n \in \omega$; that is, for every rule R of $\mathcal{H}$ there exists $n \in \omega$ such that all applications of R have arity n.

(iv) With the exception of the (optionally present) *structural rules* weakening and contraction

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,\mathrm{Weak}_l \qquad\qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\,\mathrm{Weak}_r$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\,\mathrm{Contr}_l \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\,\mathrm{Contr}_r$$

(Weak_l, Weak_r and Contr_l, Contr_r will symbolically be gathered under the respective 'names' Weak and Contr) and of the (optionally present) *cut rule*

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma\Gamma' \Rightarrow \Delta\Delta'}\,\mathrm{Cut} \quad ,$$

[12] In order to avoid an unwanted allusion to type theory here, we prefer not to speak of 'Hilbert-type systems'.

all rules of $\mathcal{H}$ fulfill the *left-hand side property*: The *set* of formulas which appear on the left-hand side of the conclusion of a rule is the union of the *sets* of formulas which appear on the left-hand side of the premises.

The language of a HSC $\mathcal{H}$ will be denoted by $L(\mathcal{H})$. For a HSC $\mathcal{H}$ and a sequent $\Gamma \Rightarrow \Delta$ in $L(\mathcal{H})$, the formal expression $\vdash_{\mathcal{H}} \Gamma \Rightarrow \Delta$ stands for the assertion that there exists a derivation in $\mathcal{H}$ with conclusion (end-formula) $\Gamma \Rightarrow \Delta$ and without unproven assumptions.

We call a HSC $\mathcal{H}$ *single-conclusioned* if and only if, for every sequent $\Gamma \Rightarrow \Delta$ that occurs as an axiom of $\mathcal{H}$ or that occurs in an application of a rule of $\mathcal{H}$, the succedens Δ consists only of a single wff. Furthermore, we denote by $\mathfrak{HC}$ the class of all Hilbert systems for consequence, and by $\mathfrak{HC}_1$ the class of all single-conclusioned HSC's.

For the sake of clarity, we want to describe the left-hand side property for rules in an HSC more formally. This is done in the following remark, which also contains an elegant characterization by Avron of the above defined systems.

Remark E.2 Let $\mathcal{H}$ be a Hilbert system for consequence with language L.

(a) Condition (iii) on HSC's in Definition E.1 is not an explicit part of the stipulations in [1]. However, it seems to be assumed there, and because it constitutes a formal difference with rules in AHS's (where a rule may possess applications of different arities), it was taken up into the definition of an HSC here. This minor conceptual difference with AHS's has to be taken into account for defining correspondences between AHS's and HSC's.

(b) A zero-premise rule R of $\mathcal{H}$ fulfills the left-hand side property if and only if all applications of R are of the form

$$\frac{}{\Rightarrow \Delta}$$

for some $\Delta \in \mathcal{M}_{\mathrm{f}}(\mathrm{wff}(L))$. An n-premise rule R (where $n \in \omega \backslash \{0\}$) in a Hilbert system for consequence has the left-hand side property if and only if for all applications

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \dots \qquad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta} \qquad \text{the assertion} \quad \mathrm{set}(\Gamma) = \bigcup_{i=1}^{n} \mathrm{set}(\Gamma_i)$$

holds.

(c) The presence of the right-weakening and right-contraction rules Weak_r and Contr_r in all HSC's is perhaps debatable [I am not certain about how to interpret [1] on this point correctly, C.G.]. However, Weak_r and Contr_r will play no role below since we will only be interested in single-conclusioned HSC's here.

(d) Avron sums up his characterization of Hilbert systems in sequent-style format in the following succinct way: "If we take axioms as rules with 0 premises then Hilbert representations can be characterized as those systems which *have besides the basic reflexivity and transitivity rules only structural rules and/or rules with the left-hand side property*" ([1, p. 26], emphasis in the original).

In [1] the subclass of "pure" sequent-style Hilbert systems is introduced in essentially the following way (a minor difference will be explained below).

Definition E.3 (Pure rules and pure Hilbert systems for consequence).

(i) Let $\mathcal{H}$ be a Hilbert system for consequence with language L.

A rule R of $\mathcal{H}$ is called *pure* if and only if the following property holds with respect to its applications: Whenever, for some $n \in \omega \backslash \{0\}$ and multisets $\Gamma, \Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_n \in \mathcal{M}_{\mathrm{f}}(\mathrm{wff}(L))$,

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \ldots \qquad \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta} \tag{E.1}$$

is an application of R, then

$$\Gamma = \Gamma_1 \ldots \Gamma_n \tag{E.2}$$

holds, and for all $\Gamma'_1, \ldots, \Gamma'_n \in \mathcal{M}_{\mathrm{f}}(\mathrm{wff}(L))$, also

$$\frac{\Gamma'_1 \Rightarrow \Delta_1 \qquad \ldots \qquad \Gamma'_n \Rightarrow \Delta_n}{\Gamma'_1 \ldots \Gamma'_n \Rightarrow \Delta} \tag{E.3}$$

is an application of R (hence zero-premise rules are pure trivially).

(ii) A HSC $\mathcal{H}$ is called *pure* if and only if all rules of $\mathcal{H}$ are pure.

(iii) We denote by $\mathfrak{HC}^{\mathrm{pure}}$ the class of pure HSC's, and by $\mathfrak{HC}_1^{\mathrm{pure}}$ the class of pure, single-conclusioned HSC's.

For an inference rule R of a HSC to be "pure", in [1] it is only demanded that for every application of R of the form (E.1) also all applications of the form (E.3) are applications of R; there the condition (E.2) does not have to be fulfilled necessarily. Because we do not want to admit pure rules with inbuilt contraction or weakening (in the interest of results given below that distinguish between HSC's as to whether weakening and/or contraction rules are present or not) we use a stricter definition of "pure" inference rules.

Due to their special property, pure rules can be represented by schemes consisting only of the formulas in the antecedents of the premises and the conclusion. For example, a rule with applications of the form (E.1) can be communicated in the form

$$\frac{\Delta_1 \qquad \ldots \qquad \Delta_n}{\Delta} \quad .$$

The following proposition, which is not mentioned in [1], is not very difficult to prove. However, we do not include its proof here.

Proposition E.4 (Cut-elimination in $\mathfrak{HC}_1^{\mathrm{pure}}$). *Cut-elimination holds in every pure, single-conclusioned Hilbert system for consequence. That is, for every HSC $\mathcal{H} \in \mathfrak{HC}_1^{pure}$ with language L, it holds for all $A \in wff(L)$ and $\Gamma \in \mathcal{M}_f(wff(L))$:*[13]

$$\vdash_{\mathcal{H}} \Gamma \Rightarrow A \quad \Longleftrightarrow \quad \vdash_{\mathcal{H}-\mathrm{Cut}} \Gamma \Rightarrow A \ .$$

[13] If $\mathcal{H}$ contains the cut rule, then by $\mathcal{H}-\mathrm{Cut}$ we mean the result of dropping the cut rule from $\mathcal{H}$; otherwise $\mathcal{H}-\mathrm{Cut}$ is just $\mathcal{H}$.

Figure 4 Definition of the Hilbert system for consequence $\mathcal{H}(\mathcal{S})$ for every abstract Hilbert system $\mathcal{S} = \langle Fo, Ax, \mathcal{R}\rangle$.

The *axioms* of $\mathcal{H}(\mathcal{S})$:

$$\text{(Assm)}\;\; A \Rightarrow A \quad (\text{for all } A \in Fo) \qquad\qquad \text{(Ax)}\;\; \Rightarrow A \quad (\text{for all } A \in Ax)$$

The *rules* of $\mathcal{H}(\mathcal{S})$:

For all rules $R = \langle Apps, \mathsf{prem}, \mathsf{concl}\rangle$ of $\mathcal{S}$, if R has zero-premise applications then rules $R^{(0)}$, and for all $n \in \omega\backslash\{0\}$ such that R has at least an application of arity n, rules $R^{(n)}$. These rules have the respective applications

$$\frac{}{\Rightarrow \mathsf{concl}(\alpha)}\, R^{(0)} \qquad\qquad \frac{\Gamma_1 \Rightarrow \mathsf{prem}_1(\beta) \qquad \dots \qquad \Gamma_n \Rightarrow \mathsf{prem}_n(\beta)}{\Gamma_1 \dots \Gamma_n \Rightarrow \mathsf{concl}(\beta)}\, R^{(n)}$$

for all $\alpha, \beta \in Apps_R$ with $\mathsf{arity}(\alpha) = 0$ and $\mathsf{arity}(\beta) = n$, and for all $\Gamma_1, \dots, \Gamma_n$ multisets of formulas over Fo.

And moreover, every derivation $\mathcal{D}$ in an HSC $\mathcal{H} \in \mathfrak{H}\mathfrak{C}_1^{pure}$ can effectively be transformed into a derivation $\mathcal{D}'$ in $\mathcal{H}$ that has the same conclusion as $\mathcal{D}$ and that does not contain applications of Cut.

We want to mention that this result can easily be generalized to the assertion that cut-elimination also holds for all pure HSC's (i.e. also for 'multiple-conclusioned' systems in $\mathfrak{H}\mathfrak{C}^{\text{pure}}$) that do not contain right-weakening or right-contraction rules.

We will now define a pure and single-conclusioned HSC $\mathcal{H}(\mathcal{S})$ for every AHS $\mathcal{S}$. The principal idea consists in modelling relative derivability statements in $\mathcal{S}$ with respect to the consequence relation $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$ by derivable sequents, i.e. theorems, of $\mathcal{H}(\mathcal{S})$. In this way, every axiom A of $\mathcal{S}$ gives rise to an axiom $\Rightarrow A$ in $\mathcal{H}(\mathcal{S})$, and every application

$$\frac{A_1 \qquad \dots \qquad A_n}{A}$$

of a rule R in $\mathcal{S}$ gives rise to applications of the form

$$\frac{\Gamma_1 \Rightarrow A_1 \qquad \dots \qquad \Gamma_n \Rightarrow A_n}{\Gamma_1 \dots \Gamma_n \Rightarrow A}$$

(for all $\Gamma_1, \dots, \Gamma_n$ multisets of formulas) of a rule in $\mathcal{H}(\mathcal{S})$.

Definition E.5 (Mapping $\mathcal{H}(\cdot)$ from AHS's to HSC's). We define a function

$$\mathcal{H}(\cdot) : \mathfrak{H} \longrightarrow \mathfrak{H}\mathfrak{C}_1^{\text{pure}}, \;\; \mathcal{S} \longmapsto \mathcal{H}(\mathcal{S})$$

between the class $\mathfrak{H}$ of Hilbert systems for consequence and the class $\mathfrak{H}\mathfrak{C}_1^{\text{pure}}$ of pure, single-conclusioned Hilbert systems for consequence: For every AHS $\mathcal{S} = \langle Fo, Ax, \mathcal{R}\rangle$, the HSC $\mathcal{H}(\mathcal{S})$ has a language L with $\text{wff}(L) = Fo$, and its axioms and rules are defined in Figure 4; we say that $\mathcal{H}(\mathcal{S})$ is the HSC that is *induced by* the AHS $\mathcal{S}$.

The following proposition sums up obvious properties of the mapping $\mathcal{H}(\cdot)$ and justifies that the range of $\mathcal{H}(\cdot)$ is indeed contained in the class $\mathfrak{H}\mathfrak{C}_1^{\mathrm{pure}}$.

Proposition E.6 *Let $\mathcal{S}$ be an AHS. Then the image $\mathcal{H}(\mathcal{S})$ of $\mathcal{S}$ under the mapping $\mathcal{H}(\cdot)$ as defined in Definition E.5 is a pure, single-conclusioned Hilbert system for consequence, which neither contains structural rules nor* Cut.

As a consequence of this proposition, the HSC $\mathcal{H}(\mathcal{S})$ defined for a given AHS $\mathcal{S}$ according to the mapping $\mathcal{H}(\cdot)$ possesses only rules that fulfill the left-hand side property.

In the following definition we will define an AHS $\mathcal{S}(\mathcal{H})$ for every pure, single-conclusioned HSC $\mathcal{H}$. For its motivation, we observe the following: Due to the definition of "pure rule" in an HSC $\mathcal{H}$, it is sufficient (as we have argued earlier) to retain, for a pure n-premise rule R of $\mathcal{H}$ (where $n > 0$), from every application

$$\frac{\Gamma_1 \Rightarrow A_1 \qquad \dots \qquad \Gamma_n \Rightarrow A_n}{\Gamma \Rightarrow A} \tag{E.4}$$

of R only the part

$$\frac{A_1 \qquad \dots \qquad A_n}{A} \tag{E.5}$$

consisting of the inference between the respective formulas in the succedents of the premises and the conclusion of (E.4). Since $\Gamma = \Gamma_1 \dots \Gamma_n$ must hold in (E.4) due to purity of R, the original application (E.4) can always be found among all those HSC-applications that result from building applications of a pure HSC-rule from the 'AHS-rule application' (E.5). In this way we have gained an idea of how to define an AHS-rule $\check{R}$ for every rule R in an HSC *in a natural and faithful way*. This is worked out precisely in the following definition of an AHS $\mathcal{S}(\mathcal{H})$ for every pure, single-conclusioned HSC $\mathcal{H}$.

Definition E.7 (Mapping $\mathcal{S}(\cdot)$ from HSC's to AHS's). We define a function

$$\mathcal{S}(\cdot) : \mathfrak{H}\mathfrak{C}_1^{\mathrm{pure}} \longrightarrow \mathfrak{H}, \;\; \mathcal{H} \longmapsto \mathcal{S}(\mathcal{H})$$

between the class $\mathfrak{H}\mathfrak{C}_1^{\mathrm{pure}}$ of pure, single-conclusioned Hilbert systems for consequence and the class $\mathfrak{H}$ of abstract Hilbert systems: For every HSC $\mathcal{H}$ with language L, the AHS $\mathcal{S}(\mathcal{H})$ is defined in Figure 5; we will call $\mathcal{S}(\mathcal{H})$ the AHS that is *induced by* by the HSC $\mathcal{H}$.

As an obvious property of the mapping $\mathcal{S}(\cdot)$, we find the following proposition.

Proposition E.8 *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{H}\mathfrak{C}_1^{pure}$ be such that $\mathcal{H}_1$ and $\mathcal{H}_2$ differ only by the presence or absence of (one or all of) the structural rules or of* Cut. *Then*

$$\mathcal{S}(\mathcal{H}_1) = \mathcal{S}(\mathcal{H}_2)$$

holds, i.e. $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same image under $\mathcal{S}(\cdot)$.

As a further basic observation, the next lemma formulates and generalizes the observation that, for HSC's in $\mathfrak{H}\mathfrak{C}_1^{\mathrm{pure}}$ without structural rules and without cut, the mapping $\mathcal{H}(\cdot)$ is the inverse of the mapping $\mathcal{S}(\cdot)$.

Figure 5 Definition of an abstract Hilbert system $\mathcal{S}(\mathcal{H})$ for every pure, single-conclusioned Hilbert system for consequence $\mathcal{H}$ with language L.

$\mathcal{S}(\mathcal{H})$ is defined as the AHS $\langle Fo, Ax, \mathcal{R}\rangle$ with

$$\begin{aligned} Fo &= \text{wff}(L)\ , \\ Ax &= \{A \in Fo \mid\ \Rightarrow A \text{ is an axiom of } \mathcal{H}\}\ , \\ \mathcal{R} &= \left\{\check{R} \mid R \text{ is a rule of } \mathcal{H}\right\}\ , \end{aligned}$$

whereby for all rules R of $\mathcal{H}$ the rule $\check{R}$ of $\mathcal{S}(\mathcal{H})$ is defined as follows: If R is an n-premise rule for some $n \in \omega$, then $\check{R}$ has only n-premise applications and it holds for all $A, A_1, \ldots, A_n \in Fo$ that (depending on whether $n = 0$ or $n \geq 1$)

$$\frac{}{A} \qquad \text{or respectively} \qquad \frac{A_1 \qquad \ldots \qquad A_n}{A}$$

is an application of $\check{R}$ if and only if there exists an application

$$\frac{}{\Rightarrow A} \qquad \text{or respectively} \qquad \frac{\Gamma_1 \Rightarrow A_1 \qquad \ldots \qquad \Gamma_n \Rightarrow A_n}{\Gamma_1 \ldots \Gamma_n \Rightarrow A}$$

of R for some multisets $\Gamma_1, \ldots, \Gamma_n$ of formulas over Fo.

Lemma E.9 *Let $\mathcal{H}$ be a pure, single-conclusioned HSC that does not contain the cut rule.*

(i) *If $\mathcal{H}$ contains neither weakening nor contraction, then $\mathcal{H}(\mathcal{S}(\mathcal{H})) = \mathcal{H}$ holds.*

(ii) *If $\mathcal{H}$ contains weakening, but not contraction, then $\mathcal{H}(\mathcal{S}(\mathcal{H}))+Weak = \mathcal{H}$ holds.*

(iii) *If $\mathcal{H}$ contains contraction, but not weakening, then $\mathcal{H}(\mathcal{S}(\mathcal{H}))+Contr = \mathcal{H}$ holds.*

(iv) *If $\mathcal{H}$ contains weakening and contraction, then $\mathcal{H}(\mathcal{S}(\mathcal{H}))+Weak+Contr = \mathcal{H}$ holds.*

Sketch of Proof. Assertion (i) of the lemma can be checked easily. Assertions (ii), (iii) and (iv) are immediate consequences of the fact that

$$\mathcal{H}(\mathcal{S}(\mathcal{H})) = \mathcal{H} - \{\text{Weak, Contr, Cut}\} \qquad (\text{for all } \mathcal{H} \in \mathfrak{H}\mathfrak{C}_1^{\text{pure}})$$

holds, which follows by (i) from the consequence

$$\mathcal{S}(\mathcal{H} - \{\text{Weak, Contr, Cut}\}) = \mathcal{S}(\mathcal{H}) \qquad (\text{for all } \mathcal{H} \in \mathfrak{H}\mathfrak{C}_1^{\text{pure}}).$$

of Proposition E.8. □

It turns out that a fourth kind of consequence relation on AHS's (next to the consequence relations of Definition 2.8) is able to fill a 'gap' that would otherwise arise in correspondence assertions given below. For being able to state our results without this 'gap', we first define this consequence relation $\vdash^{(\mathbf{mw})}$, which is not treated otherwise in this report. For an arbitrary AHS or n-AHS $\mathcal{S}$, $\vdash^{(\mathbf{mw})}$ relates to $\vdash^{\mathbf{(m)}}_{\boldsymbol{\mathcal{S}}}$ in an analogous way as $\vdash_{\boldsymbol{\mathcal{S}}}$ does to $\vdash^{\mathbf{(s)}}_{\boldsymbol{\mathcal{S}}}$.

Definition E.10 (The consequence relation $\vdash^{\mathbf{(mw)}}$ for AHS's and n-AHS's and relative derivability statements w.r.t. $\vdash^{\mathbf{(mw)}}$).
Let $\mathcal{S}$ be an AHS or n-AHS with formula set Fo. We define the consequence relation $\vdash^{\mathbf{(mw)}}_{\boldsymbol{\mathcal{S}}}$, where $\vdash^{\mathbf{(mw)}}_{\boldsymbol{\mathcal{S}}} \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$ by stipulating for all $A \in Fo$ and *multisets* $\Gamma \in \mathcal{M}_{\mathrm{f}}(Fo)$:

$$\langle \Gamma, A \rangle \in \vdash^{\mathbf{(mw)}}_{\boldsymbol{\mathcal{S}}} \iff (\exists \mathcal{D} \in Der(\mathcal{S}))\big[\,\mathsf{assm}(\mathcal{D}) \subseteq \Gamma \;\;\&\;\; \mathsf{concl}(\mathcal{D}) = A\,\big]\,.$$

And we define *relative derivability statements* with respect to this consequence relation: For all $\Gamma \in \mathcal{M}_{\mathrm{f}}(Fo)$ and $A \in Fo$, the statement $\Gamma \vdash^{(\mathrm{mw})}_{\mathcal{S}} A$ holds if and only if $\langle \Gamma, A \rangle \in \vdash^{\mathbf{(mw)}}_{\boldsymbol{\mathcal{S}}}$.

We will now give our main theorem about the correspondence $\mathcal{H}(\cdot)$ between AHS's and pure, single-conclusioned HSC's. It gives a characterization of the consequence relations $\vdash^{\mathbf{(m)}}$, $\vdash^{\mathbf{(mw)}}$, $\vdash^{\mathbf{(s)}}$ and $\vdash$ on an AHS $\mathcal{S}$ through respective variant systems of the HSC $\mathcal{H}(\mathcal{S})$ that is induced by $\mathcal{S}$.

Theorem E.11 *Let $\mathcal{S}$ be an AHS, and let $\mathcal{H}(\mathcal{S})$ be the Hilbert system for consequence that is induced by $\mathcal{S}$. Then the following four logical assertions hold for all $A \in Fo_{\mathcal{S}}$ and $\Gamma \in \mathcal{M}_f(Fo_{\mathcal{S}})$:*

$$\Gamma \vdash^{(m)}_{\mathcal{S}} A \iff \vdash_{\mathcal{H}(\mathcal{S})} \Gamma \Rightarrow A\;, \tag{E.6}$$
$$\Gamma \vdash^{(mw)}_{\mathcal{S}} A \iff \vdash_{\mathcal{H}(\mathcal{S})+Weak} \Gamma \Rightarrow A\;, \tag{E.7}$$
$$\mathrm{set}(\Gamma) \vdash^{(s)}_{\mathcal{S}} A \Longleftarrow \vdash_{\mathcal{H}(\mathcal{S})+Contr} \Gamma \Rightarrow A\;, \tag{E.8}$$
$$\mathrm{set}(\Gamma) \vdash_{\mathcal{S}} A \iff \vdash_{\mathcal{H}(\mathcal{S})+Weak+Contr} \Gamma \Rightarrow A\;. \tag{E.9}$$

And furthermore, for all $A \in Fo_{\mathcal{S}}$ and $\Sigma \in \mathcal{P}_f(Fo_{\mathcal{S}})$ it holds:

$$\Sigma \vdash^{(s)}_{\mathcal{S}} A \iff \vdash_{\mathcal{H}(\mathcal{S})+Contr} \mathrm{mset}(\Sigma) \Rightarrow A\;, \tag{E.10}$$
$$\Sigma \vdash_{\mathcal{S}} A \iff \vdash_{\mathcal{H}(\mathcal{S})+Weak+Contr} \mathrm{mset}(\Sigma) \Rightarrow A\;. \tag{E.11}$$

Idea of the Proof. All implications in the assertions (E.6)–(E.11) of the theorem can be shown by straightforward inductions on the depth of derivations in $\mathcal{S}$, $\mathcal{H}(\mathcal{S})$, $\mathcal{H}(\mathcal{S})$+Weak, $\mathcal{H}(\mathcal{S})$+Contr, and $\mathcal{H}(\mathcal{S})$+Weak+Contr, respectively. The implications "$\Leftarrow$" in (E.10) and (E.11) are special cases of the respective implications "$\Leftarrow$" in (E.8) and (E.9).

The implication "$\Rightarrow$" in (E.8) does actually not hold in general as can be seen in this way: Let, for a non-empty set Fo, $\mathcal{S} = \langle Fo, \emptyset, \emptyset \rangle$ be an AHS with no axioms and no rules. Then $\mathcal{H}(\mathcal{S})$ contains only the axioms $A \Rightarrow A$ (for all $A \in Fo$), but no rules. As a consequence both $\mathcal{H}(\mathcal{S})$ and $\mathcal{H}(\mathcal{S})$+Contr have only sequents of the form $A \Rightarrow A$ (for $A \in Fo$) as theorems. Hence, for all $A \in Fo$, $\vdash_{\mathcal{H}(\mathcal{S})+\mathrm{Contr}} \Gamma \Rightarrow A$

only holds for $\Gamma = \{A\}$; but $\mathrm{set}(\Gamma) \vdash_{\mathcal{S}}^{(\mathrm{s})} A$ clearly holds for other multisets Γ over *Fo* as well (for instance, for $\Gamma = \mathrm{mset}((A, A))$).

We continue by giving our main theorem about the correspondence $\mathcal{S}(\cdot)$ between pure, single-conclusioned HSC's and AHS's. This theorem gives a characterization of the theorems, i.e. the derivable sequents, of a pure, single-conclusioned HSC $\mathcal{H}$ using a respective one of the consequence relation of $\vdash^{\mathbf{(m)}}$, $\vdash^{\mathbf{(mw)}}$, $\vdash^{\mathbf{(s)}}$ and $\vdash$ on the AHS $\mathcal{S}(\mathcal{H})$ that is induced by $\mathcal{H}$.

Theorem E.12 *Let $\mathcal{H}$ be a pure, single-conclusioned Hilbert system for consequence with language L, and let $\mathcal{S}(\mathcal{H})$ be the AHS that is induced by $\mathcal{H}$. Then the following four assertions hold for all $A \in \mathit{wff}(L)$, $\Gamma \in \mathcal{M}_f(\mathit{wff}(L))$ and $\Sigma \in \mathcal{P}_f(\mathit{wff}(L))$:*

(i) If $\mathcal{H}$ does neither contain weakening nor contraction, then

$$\Gamma \vdash_{\mathcal{S}(\mathcal{H})}^{(m)} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{H}} \Gamma \Rightarrow A \ .$$

(ii) If $\mathcal{H}$ contains weakening, but not contraction, then

$$\Gamma \vdash_{\mathcal{S}(\mathcal{H})}^{(mw)} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{H}} \Gamma \Rightarrow A \ .$$

(iii) If $\mathcal{H}$ contains contraction, but not weakening, then

$$\begin{aligned} \mathrm{set}(\Gamma) \vdash_{\mathcal{S}(\mathcal{H})}^{(s)} A \quad &\Longleftarrow \quad \vdash_{\mathcal{H}} \Gamma \Rightarrow A \ , \\ \Sigma \vdash_{\mathcal{S}(\mathcal{H})}^{(s)} A \quad &\Longleftrightarrow \quad \vdash_{\mathcal{H}} \mathrm{mset}(\Sigma) \Rightarrow A \ . \end{aligned} \tag{E.12}$$

(iv) If $\mathcal{H}$ contains both weakening and contraction, then

$$\begin{aligned} \mathrm{set}(\Gamma) \vdash_{\mathcal{S}(\mathcal{H})} A \quad &\Longleftrightarrow \quad \vdash_{\mathcal{H}} \ \Gamma \Rightarrow A \ , \\ \Sigma \vdash_{\mathcal{S}(\mathcal{H})} A \quad &\Longleftrightarrow \quad \vdash_{\mathcal{H}} \mathrm{mset}(\Sigma) \Rightarrow A \end{aligned}$$

Proof. All assertions of the theorem follow in analogous ways from respective assertions of Theorem E.11 by using appropriate statements from Proposition E.9. For example, to show (E.12) let $\mathcal{H} \in \mathfrak{H}\mathfrak{C}_1^{\mathrm{pure}}$ be such that it contains contraction, but not weakening, and let $\mathcal{H}_0 = \mathcal{H} - \mathrm{Cut}$. Then we find that for all $A \in \mathrm{wff}(L)$ and $\Sigma \in \mathcal{P}_{\mathrm{f}}(\mathrm{wff}(L))$

$$\begin{aligned} \Sigma \vdash_{\mathcal{S}(\mathcal{H})}^{(\mathrm{s})} A \quad &\Longleftrightarrow \quad \Sigma \vdash_{\mathcal{S}(\mathcal{H}_0)}^{(\mathrm{s})} A \\ &\Longleftrightarrow \quad \vdash_{\mathcal{H}(\mathcal{S}(\mathcal{H}_0))+\mathrm{Contr}} \ \mathrm{mset}(\Sigma) \Rightarrow A \\ &\Longleftrightarrow \quad \vdash_{\mathcal{H}_0} \ \mathrm{mset}(\Sigma) \Rightarrow A \\ &\Longleftrightarrow \quad \vdash_{\mathcal{H}} \ \mathrm{mset}(\Sigma) \Rightarrow A \end{aligned}$$

holds, due to the consequence $S(\mathcal{H}) = \mathcal{S}(\mathcal{H}_0)$ of Proposition E.8, the equivalence (E.10) of Theorem E.11, item (iii) of Proposition E.9, and the cut-elimination theorem Proposition E.4. In this way we have shown (E.12).

□

In the remaining part of this appendix, we give applications of the two theorems above to the question of when a relation between sets, or multisets, of formulas and formulas can actually be "axiomatized" by a pure, single conclusioned HSC. First, we stipulate what we mean by saying that a HSC "axiomatizes" such a relation.

Definition E.13 Let Fo be a set, and let $\mathcal{H}$ be a Hilbert system for consequence with language L such that $\text{wff}(L) = Fo$ holds.

(i) Let $\Vdash^{(\text{mset})} \subseteq \mathcal{M}_{\text{f}}(Fo) \times Fo$ be a relation between finite *multisets* of formulas of Fo and formulas of Fo. We say that $\mathcal{H}$ *axiomatizes* the relation $\Vdash^{(\text{mset})}$ if and only if for all $A \in Fo$ and all $\Gamma \in \mathcal{M}_{\text{f}}(Fo)$ it is the case that:

$$\Gamma \Vdash^{(\text{mset})} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{H}} \Gamma \Rightarrow A \ .$$

(ii) Let $\Vdash^{(\text{set})} \subseteq \mathcal{P}_{\text{f}}(Fo) \times Fo$ be a relation between finite *sets* of formulas of Fo and formulas of Fo. We say that $\mathcal{H}$ *axiomatizes* the relation $\Vdash^{(\text{set})}$ if and only if it holds for all $A \in Fo$ and all $\Sigma \in \mathcal{P}_{\text{f}}(Fo)$:

$$\Sigma \Vdash^{(\text{set})} A \quad \Longleftrightarrow \quad \vdash_{\mathcal{H}} \text{mset}(\Sigma) \Rightarrow A \ .$$

By restating the assertions of Theorem E.11 using these definitions, we find the following corollary.

Corollary E.14 *Let $\mathcal{S}$ be an AHS, and let $\mathcal{H}(\mathcal{S})$ be the HSC that is induced by $\mathcal{S}$. Then the following four assertions hold:*

(i) *$\mathcal{H}(\mathcal{S})$ axiomatizes the consequence relation $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$.*

(ii) *$\mathcal{H}(\mathcal{S})$+Weak axiomatizes the consequence relation $\vdash^{(\mathbf{mw})}_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$.*

(iii) *$\mathcal{H}(\mathcal{S})$+Contr axiomatizes the consequence relation $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$.*

(iv) *$\mathcal{H}(\mathcal{S})$+Weak+Contr axiomatizes the consequence relation $\vdash_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$.*

And similarly, by restating the assertions of Theorem E.12 in the light of the terminology from Definition E.13, we also find the following corollary.

Corollary E.15 *Let $\mathcal{H}$ be a pure, single-conclusioned HSC, and let $\mathcal{S}(\mathcal{H})$ be the AHS that is induced by $\mathcal{H}$. Then it holds:*

(i) *If $\mathcal{H}$ does neither contain weakening nor contraction, then it axiomatizes the consequence relation $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}(\mathcal{H})}}$ on $\mathcal{S}(\mathcal{H})$.*

(ii) *If $\mathcal{H}$ contains weakening, but not contraction, then it axiomatizes the consequence relation $\vdash^{(\mathbf{mw})}_{\boldsymbol{\mathcal{S}(\mathcal{H})}}$ on $\mathcal{S}(\mathcal{H})$.*

(iii) *If $\mathcal{H}$ contains contraction, but not weakening, then it axiomatizes the consequence relation $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}(\mathcal{H})}}$ on $\mathcal{S}(\mathcal{H})$.*

(iv) *If $\mathcal{H}$ contains contraction as well as weakening, then it axiomatizes the consequence relation $\vdash_{\boldsymbol{\mathcal{S}(\mathcal{H})}}$ on $\mathcal{S}(\mathcal{H})$.*

Corollary E.14 asserts that, for every AHS $\mathcal{S}$, each of the consequence relations $\vdash_{\boldsymbol{\mathcal{S}}}$, $\vdash^{(\mathbf{s})}_{\boldsymbol{\mathcal{S}}}$, $\vdash^{(\mathbf{mw})}_{\boldsymbol{\mathcal{S}}}$ and $\vdash^{(\mathbf{m})}_{\boldsymbol{\mathcal{S}}}$ on $\mathcal{S}$ is "axiomatizable" by a HSC. As our last result in this appendix, we will now give an inverse assertion to this fact: Under a reasonable restriction of the term "axiomatizable", we find that every relation that is axiomatizable by a HSC is actually equal to one of the consequence relations $\vdash_{\mathcal{S}}$, $\vdash^{(\mathbf{s})}_{\mathcal{S}}$, $\vdash^{(\mathbf{mw})}_{\mathcal{S}}$ and $\vdash^{(\mathbf{m})}_{\mathcal{S}}$ for some AHS $\mathcal{S}$. The restriction mentioned hereby is that we exclude, on the one hand, relations $\Vdash \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$ (for some set Fo) from being axiomatized by HSC's *with contraction*, and on the other hand, relations $\Vdash \subseteq \mathcal{P}_{\mathrm{f}}(Fo) \times Fo$ (for some set Fo) from being axiomatized by HSC's *without contraction*. The reason is that consequence relations between sets of formulas and formulas lean themselves naturally to being axiomatized by HSC's with contraction, whereas consequence relations between multisets of formulas and formulas are certainly more appropriately axiomatized by systems without contraction.

As a formalization of this curtailed notion of "axiomatizable" relation, we introduce the notion of "naturally axiomatizable" relation.

Definition E.16 Let Fo be a set, and let $\Vdash$ be a relation with either $\Vdash \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$ or with $\Vdash \subseteq \mathcal{P}_{\mathrm{f}}(Fo) \times Fo$. Let furthermore $\mathcal{H}$ be a HSC with language L such that $\mathrm{wff}(L) = Fo$ holds.

We say that $\Vdash$ is *naturally axiomatizable* by a HSC if and only if there exists a HSC $\mathcal{H}$ such that:

(i) $\mathcal{H}$ has a language L such that $\mathrm{wff}(L) = Fo$ holds.

(ii) $\mathcal{H}$ axiomatizes $\Vdash$.

(iii) If $\Vdash \subseteq \mathcal{M}_{\mathrm{f}}(Fo) \times Fo$ is the case, then $\mathcal{H}$ does not contains the contraction rule.

(iv) If $\Vdash \subseteq \mathcal{P}_{\mathrm{f}}(Fo) \times Fo$ is the case, then $\mathcal{H}$ contains the contraction rule.

With this notion, the following corollary is an easy consequence of Theorem E.11 and Theorem E.12.

Corollary E.17 *Let Fo be a set, and let $\Vdash$ be a relation with either $\Vdash \subseteq \mathcal{M}_f(Fo) \times Fo$ or with $\Vdash \subseteq \mathcal{P}_f(Fo) \times Fo$. Then the following logical equivalence holds:*

$$\begin{aligned} &\Vdash \textit{ is naturally axiomatizable by a pure, single-conclusioned HSC} \iff \\ &\iff (\exists\, \mathcal{S} \textit{ AHS with } Fo_{\mathcal{S}} = Fo) \\ &\qquad \big[\Vdash \textit{ is equal to one of the consequence relations} \\ &\qquad\qquad \vdash_{\mathcal{S}},\ \vdash^{(\mathbf{s})}_{\mathcal{S}},\ \vdash^{(\mathbf{mw})}_{\mathcal{S}} \textit{ and } \vdash^{(\mathbf{m})}_{\mathcal{S}} \textit{ on } \mathcal{S}\big]\,. \end{aligned}$$

Due to this result, it can be said that in this report three (namely $\vdash$, $\vdash^{(\mathbf{s})}$ and $\vdash^{(\mathbf{m})}$) out of those four consequence relations on AHS's (the mentioned three plus $\vdash^{(\mathbf{mw})}$) are studied that are naturally axiomatizable by pure, single conclusioned Hilbert systems for consequence.