A Coinductive Version of Milner's Proof System for Regular Expressions Modulo Bisimilarity

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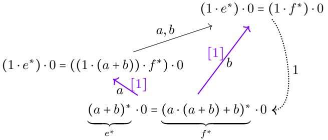
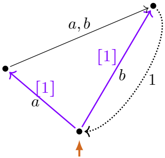


CALCO2021

September 2, 2021



Process semantics $[\![\cdot]\!]_P$ of regular (star) expr's *(Milner, 1984)*

$0 \stackrel{P}{\longmapsto}$ deadlock δ, no termination

$1 \stackrel{P}{\longmapsto}$ empty process ϵ, then terminate

$a \stackrel{P}{\longmapsto}$ atomic action a, then terminate

$e + f \stackrel{P}{\longmapsto}$ alternative composition of $[\![e]\!]_P$ and $[\![f]\!]_P$

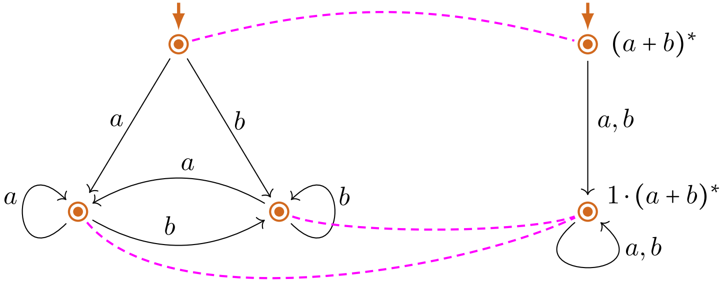
$e \cdot f \stackrel{P}{\longmapsto}$ sequential composition of $[\![e]\!]_P$ and $[\![f]\!]_P$

$e^* \stackrel{P}{\longmapsto}$ unbounded iteration of $[\![e]\!]_P$, option to terminate

$[\![e]\!]_P := [P(e)]_{\underline{\leftrightarrow}}$ (bisimilarity equivalence class of process $P(e)$)

$:= [\mathcal{C}(e)]_{\underline{\leftrightarrow}}$ (bisimilarity equivalence class of chart $\mathcal{C}(e)$)

Chart interpretation (example) (via TSS or *Antimirov's partial deriv's)*



$$\mathcal{C}(\,(a^* \cdot b^*)^*\,) \quad \underline{\leftrightarrow} \quad \mathcal{C}(\,(a+b)^*\,)$$

$$[\![(a^* \cdot b^*)^*]\!]_P \quad = \quad [\![(a+b)^*]\!]_P$$

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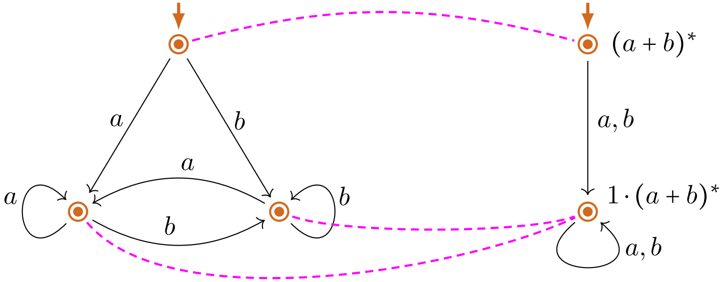


$$\mathcal{C}(\,(a^* \cdot b^*)^*\,) \quad \underline{\leftrightarrow} \quad \mathcal{C}(\,(a+b)^*\,)$$

$$(a^* \cdot b^*)^* \quad =_{[\![\cdot]\!]_P} \quad (a+b)^*$$

Milner's proof system Mil = Mil⁻ + RSP*

Axioms :

(assoc(+))	$(e + f) + g = e + (f + g)$	(id$_l$(·))	$1 \cdot e = e$
(neutr(+))	$e + 0 = e$	(id$_r$(·))	$e \cdot 1 = e$
(comm(+))	$e + f = f + e$	(deadlock)	$0 \cdot e = 0$
(idempot(+))	$e + e = e$	(rec(*))	$e^* = 1 + e \cdot e^*$
(assoc(·))	$(e \cdot f) \cdot g = e \cdot (f \cdot g)$	(trm-body(*))	$e^* = (1 + e)^*$
(r-distr(+, ·))	$(e + f) \cdot g = e \cdot g + f \cdot g$		

Inference rules : equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\ \text{RSP}^* \quad (\text{if } f \not\downarrow)$$

Milner's axiomatization question

Question *(Milner, 1984)*

Is Milner's system Mil = Mil^-+RSP* complete for bisimilarity of process interpretations of regular expressions?

$$\forall e, f \text{ reg. expr's } \left(\vdash_{\text{Mil}} e = f \quad \overset{\text{complete?}}{\underset{\text{sound}}{\Longleftarrow\!\!\Longrightarrow}} \quad e =_{[\![\cdot]\!]_P} f \right)?$$

"Yes" for restrictions to subclasses: *Zantema/Fokkink (1994), Fokkink (1996), Corradini, De Nicola, Labella (2002), G/Fokkink (2020)*.

Proposition *(G, CMCS 2006)*

The system Mil^-+USP, where:

USP : unique solvability of guarded, linear systems of equations,

is (sound and) complete.

Question (investigated here)

How can the derivational power be characterized that the fixed-point rule RSP* adds to the purely equational part Mil^- of Milner's system?

Answer developed

We use:

- the loop existence and elimination property (LEE) of charts
  - implies expressibility by a star expression
  - led to completeness result for 1-free star expressions (G/Fokkink, 2020)

We introduce:

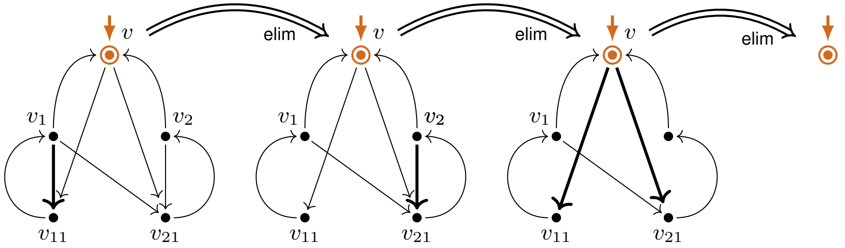
- a coinductive version cMil = (Mil^{-}+LCoProof) of Mil = (Mil^{-}+RSP*) based on LEE-witnessed coinductive proofs over Mil^{-}.

We construct / obtain:

- a proof transformation: Mil $\longmapsto$ cMil, (RSP* inst's $\mapsto$ LCoProof inst's),
- a proof transformation: Mil $\longleftarrow$ cMil, (bottom-up extraction procedure),
- theorem equivalence Mil ~ cMil :

$$\vdash_{\text{Mil}} e = f \quad \Longleftrightarrow \quad \vdash_{\text{cMil}} e = f \,.$$

LEE



Loop charts (interpretations of innermost iterations in 1-free expressions)

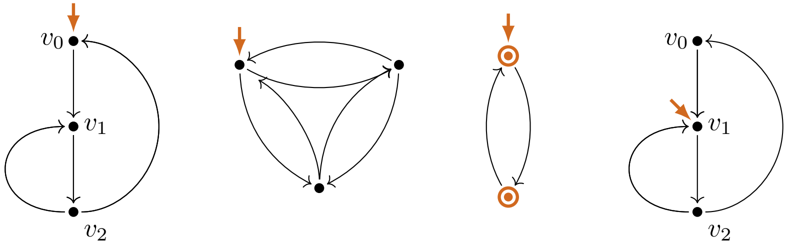
Definition

A chart is a loop chart if:

(L1) There is an infinite path from the start vertex.

(L2) Every infinite path from the start vertex returns to it.

(L3) Immediate termination is only possible at the start vertex.



(L1),~~(L2)~~ (L1),(L2),~~(L3)~~

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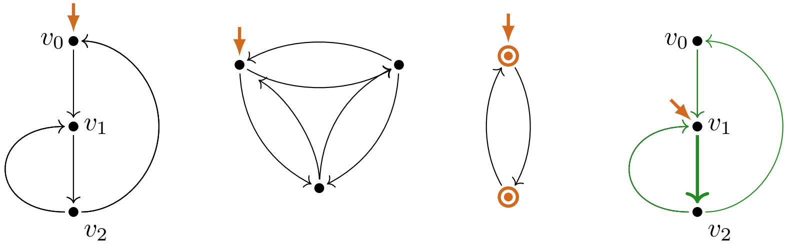

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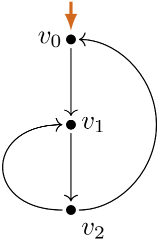

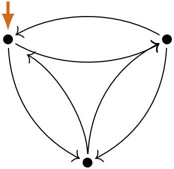

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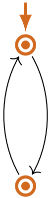

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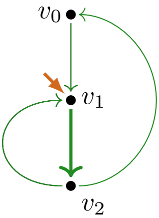

loop chart

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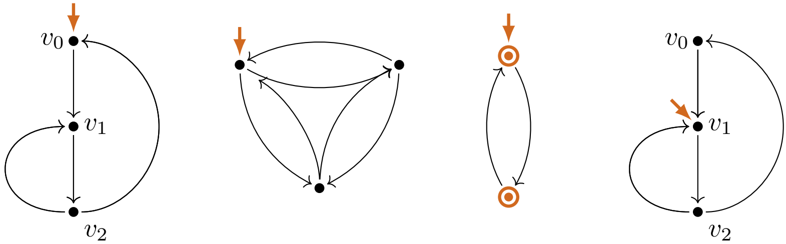

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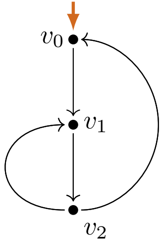

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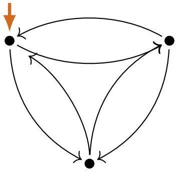

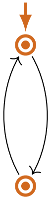

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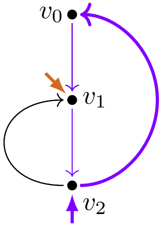
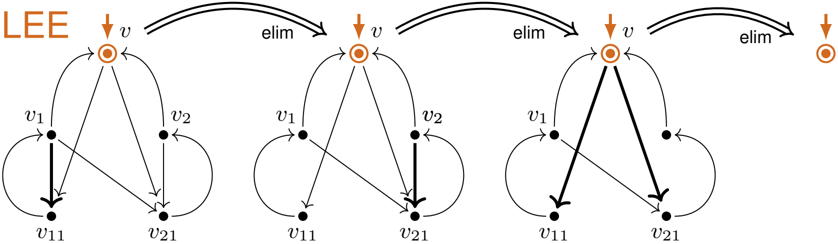
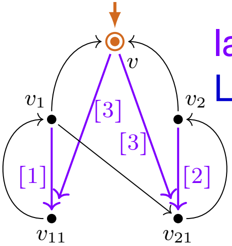


loop subchart

LEE, and LLEE-witness

LEE





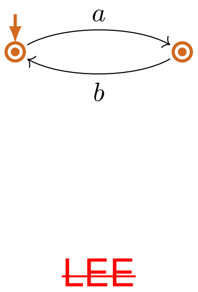
layered LEE-witness

LEE

Definition

A chart $\mathcal{C}$ satisfies LEE (*loop existence and elimination*) if:

$$\exists \mathcal{C}_0 \left(\mathcal{C} \Longrightarrow^{*}_{\text{elim}} \mathcal{C}_0 \not\Longrightarrow_{\text{elim}} \right.$$
$$\left. \wedge\ \mathcal{C}_0 \text{ permits no infinite path} \right).$$



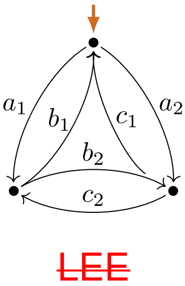
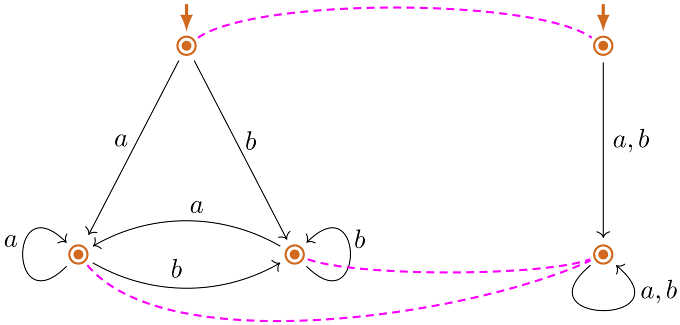


Chart interpretation (example)



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$$[\![(a^* \cdot b^*)^*]\!]_{\boldsymbol{P}} \quad = \quad [\![(a+b)^*]\!]_{\boldsymbol{P}}$$

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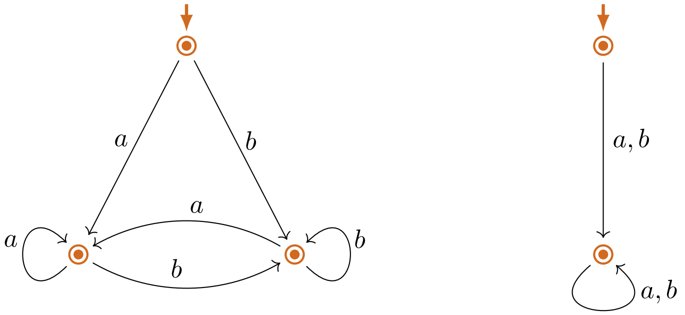

$\mathcal{C}(\,(a^* \cdot b^*)^*\,)$

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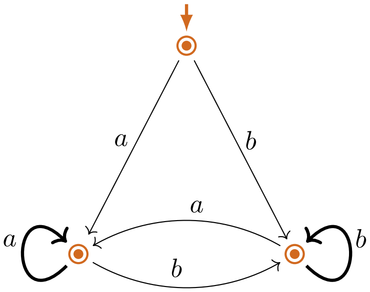

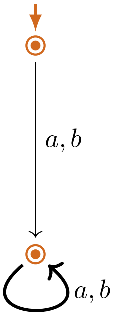

$\mathcal{C}(\,(a^* \cdot b^*)^*\,)$ $\mathcal{C}(\,(a+b)^*\,)$

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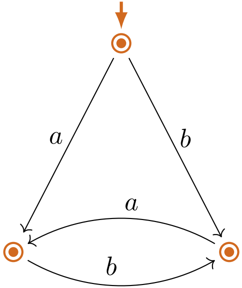

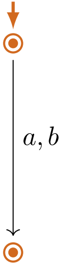

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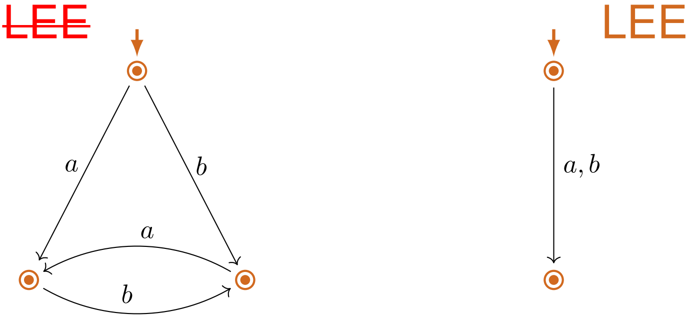

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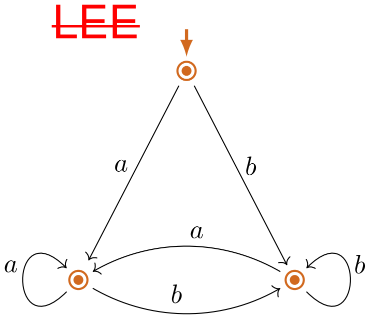
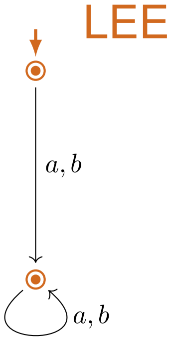
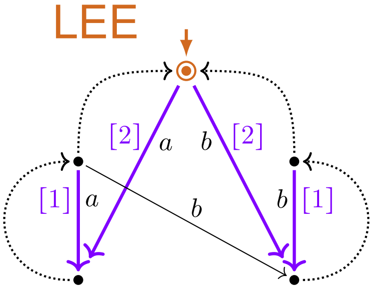
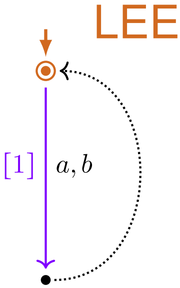




LLEE-1-chart interpretation (example) *(TERMGRAPH, 2020)*



$\underline{\mathcal{C}}(\,(a^* \cdot b^*)^*\,)$



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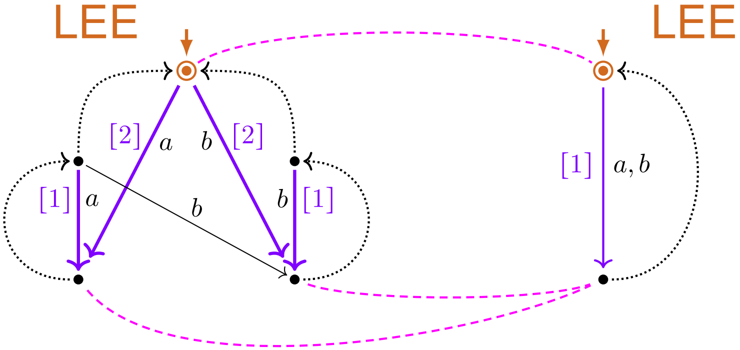

$$\underline{\mathcal{C}}(\,(a^* \cdot b^*)^*\,) \quad \underline{\leftrightarrow}^{(1)} \quad \underline{\mathcal{C}}(\,(a+b)^*\,)$$

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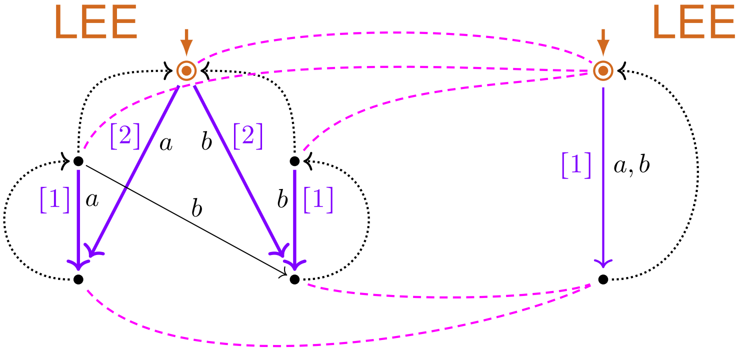
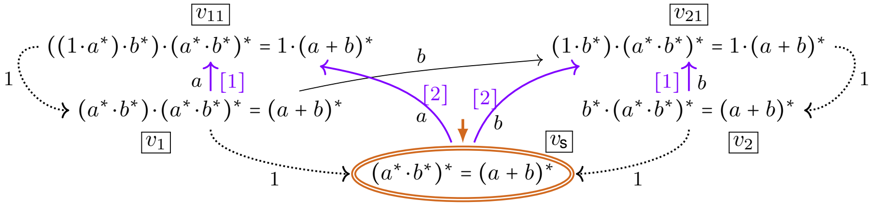


$\underline{\mathcal{C}}(\ (a^* \cdot b^*)^*\ )$ $\underline{\leftrightarrow}^{(1)}$ $\underline{\mathcal{C}}(\ (a+b)^*\ )$

LEE-witnessed coinductive proof over Mil⁻



Right- and left-hand sides are Mil⁻-provable solutions in every vertex.

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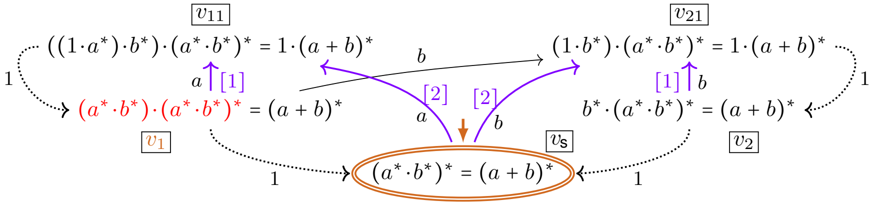
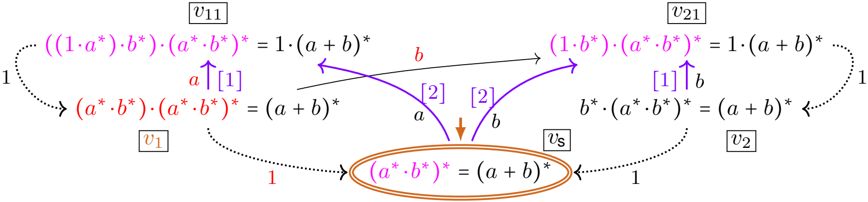


Right- and left-hand sides are Mil⁻-provable solutions in every vertex.

E.g. in v_1:

$(a^* \cdot b^*) \cdot (a^* \cdot b^*)^* =_{\text{Mil}^-}$

LEE-witnessed coinductive proof over Mil^{-}



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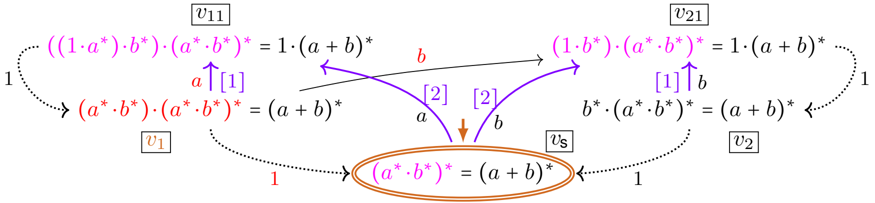

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$=_{\mathsf{Mil}^-} 1 \cdot (a^* \cdot b^*)^* + a \cdot (((1 \cdot a^*) \cdot b^*) \cdot (a^* \cdot b^*)^*) + b \cdot ((1 \cdot b^*) \cdot (a^* \cdot b^*)^*)$

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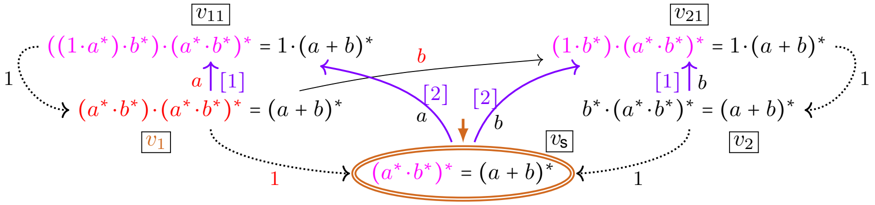

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$$
\begin{aligned}
(a^* \cdot b^*) \cdot (a^* \cdot b^*)^* &=_{\mathsf{Mil}^-} ((1 + a \cdot a^*) \cdot (1 + b \cdot b^*)) \cdot (a^* \cdot b^*)^* \\
&=_{\mathsf{Mil}^-} (1 \cdot 1 + a \cdot a^* \cdot 1 + 1 \cdot b \cdot b^* + a \cdot a^* \cdot b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
&=_{\mathsf{Mil}^-} (1 + a \cdot a^* + a \cdot a^* \cdot b \cdot b^* + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
&=_{\mathsf{Mil}^-} (1 + a \cdot a^* \cdot (1 + b \cdot b^*) + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
&=_{\mathsf{Mil}^-} (1 + a \cdot a^* \cdot b^* + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
&=_{\mathsf{Mil}^-} 1 \cdot (a^* \cdot b^*)^* + a \cdot (((1 \cdot a^*) \cdot b^*) \cdot (a^* \cdot b^*)^*) + b \cdot ((1 \cdot b^*) \cdot (a^* \cdot b^*)^*)
\end{aligned}
$$

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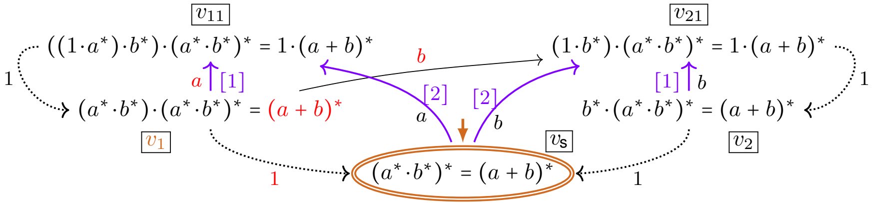

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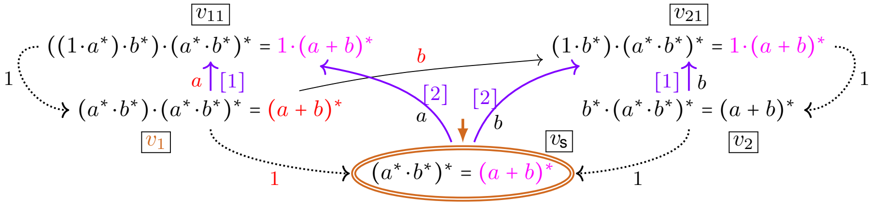

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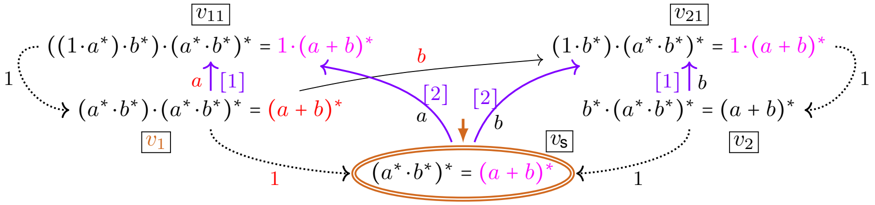

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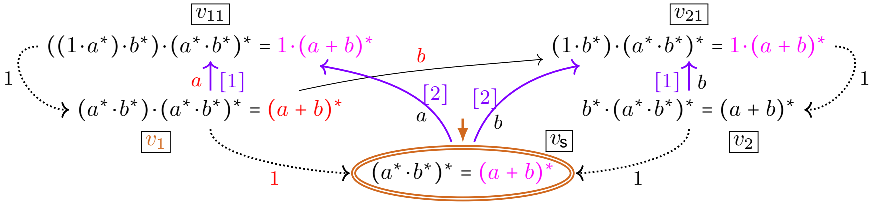

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E.g. in v_1:

$$
\begin{aligned}
(a+b)^* &=_{\mathsf{Mil}^-} (a+b)^* + (a+b)^* =_{\mathsf{Mil}^-} 1 + (a+b) \cdot (a+b)^* + 1 + (a+b) \cdot (a+b)^* \\
&=_{\mathsf{Mil}^-} 1 + 1 + (a+b) \cdot (a+b)^* + a \cdot (a+b)^* + b \cdot (a+b)^* \\
&=_{\mathsf{Mil}^-} 1 + (a+b) \cdot (a+b)^* + a \cdot (1 \cdot (a+b)^*) + b \cdot (1 \cdot (a+b)^*) \\
&=_{\mathsf{Mil}^-} 1 \cdot (a+b)^* + a \cdot (1 \cdot (a+b)^*) + b \cdot (1 \cdot (a+b)^*)
\end{aligned}
$$

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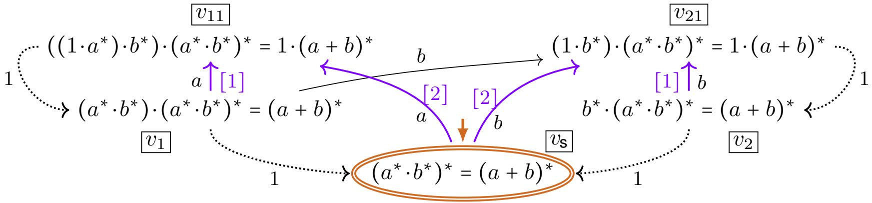

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Coinductive proof systems

Rule scheme for combining LEE-witnessed coinductive proofs:

$$\frac{g_1 = h_1 \quad \ldots \quad g_n = h_n \qquad \mathcal{LCP}(e = f)}{e = f}\ \mathsf{LCoProof}_n$$

- $\mathcal{LCP}(e = f)$ is a LEE-witnessed coinductive proof of $e = f$ over $\mathsf{Mil}^{-}+\{g_1 = h_1, \ldots, g_n = h_n\}$.

We define the proof systems:

CLC := rules $\{\mathsf{LCoProof}_n\}_{n\in\mathbb{N}}$

cMil := $\mathsf{Mil}^{-} + \{\mathsf{LCoProof}_n\}_{n\in\mathbb{N}}$

Lemma

$\mathsf{CLC} \sim \mathsf{cMil}$

Coinductive proof systems

Rule scheme for combining coinductive proofs:

$$\frac{g_1 = h_1 \quad \ldots \quad g_n = h_n \qquad \mathcal{CP}(e = f)}{e = f}\ \text{CoProof}_n$$

- $\mathcal{CP}(e = f)$ is a coinductive proof of $e = f$ over $\text{Mil}^{-}+\{g_1 = h_1, \ldots, g_n = h_n\}$.

We define the proof systems:

CLC := rules $\{\text{LCoProof}_n\}_{n \in \mathbb{N}}$

cMil := Mil^{-} + $\{\text{LCoProof}_n\}_{n \in \mathbb{N}}$

CC := rules $\{\text{CoProof}_n\}_{n \in \mathbb{N}}$

$\overline{\text{cMil}}$:= Mil^{-} + $\{\text{CoProof}_n\}_{n \in \mathbb{N}}$

Lemma

CLC ∼ cMil

Lemma

(i) CC ∼ $\overline{\text{cMil}}$.

(ii) $\overline{\text{cMil}}$ ∼ Mil^{-}+USP,

(iii) CC and $\overline{\text{cMil}}$ are complete for $=_{[\![\cdot]\!]_P}$.

Coinductive proof systems

Rule scheme for combining coinductive proofs:

$$\frac{g_1 = h_1 \quad \dots \quad g_n = h_n \quad \mathcal{CP}(e = f)}{e = f}\ \text{CoProof}_n$$

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CLC := rules $\{\text{LCoProof}_n\}_{n\in\mathbb{N}}$ CC := rules $\{\text{CoProof}_n\}_{n\in\mathbb{N}}$

cMil := Mil^{-} + $\{\text{LCoProof}_n\}_{n\in\mathbb{N}}$ $\overline{\text{cMil}}$:= Mil^{-} + $\{\text{CoProof}_n\}_{n\in\mathbb{N}}$

Lemma

CLC ∼ cMil

Consequence

cMil ∼ CLC ≲ CC ∼ $\overline{\text{cMil}}$.

Lemma

(i) CC ∼ $\overline{\text{cMil}}$.

(ii) $\overline{\text{cMil}}$ ∼ Mil^{-}+USP,

(iii) CC and $\overline{\text{cMil}}$ are complete for $=_{[\![\cdot]\!]_P}$.

Proof transformation Mil $\longmapsto$ cMil

$$\frac{e = f \cdot e + g}{e = f^{*} \cdot g}\ \mathsf{RSP}^{*}$$

$$\Longmapsto \quad \frac{e = f \cdot e + g \qquad \mathcal{LCP}_{\mathsf{Mil}^{-}+\{e=f\cdot e+g\}}(e = f^{*} \cdot g)}{e = f^{*} \cdot g}\ \mathsf{LCoProof}_1$$

Proof transformation Mil $\longmapsto$ cMil

$$\frac{\overbrace{(a+b)^*}^{e} = \overbrace{((a\cdot a^* + b)\cdot b^*)}^{f} \cdot \overbrace{(a+b)^*}^{e} + \overbrace{1}^{g}}{(a+b)^* = \underbrace{((a\cdot a^* + b)\cdot b^*)^*}_{f^*} \cdot \underbrace{1}_{g}} \ \text{RSP}^*$$

From fixed-point rule instances to coinductive proofs

$$\frac{\overbrace{(a+b)^*}^{e} = \overbrace{((a \cdot a^* + b) \cdot b^*)}^{f} \cdot \overbrace{(a+b)^*}^{e} + \overbrace{1}^{g}}{(a+b)^* = \underbrace{((a \cdot a^* + b) \cdot b^*)^*}_{f^*} \cdot \underbrace{1}_{g}} \; \mathsf{RSP}^*$$

From fixed-point rule instances to coinductive proofs

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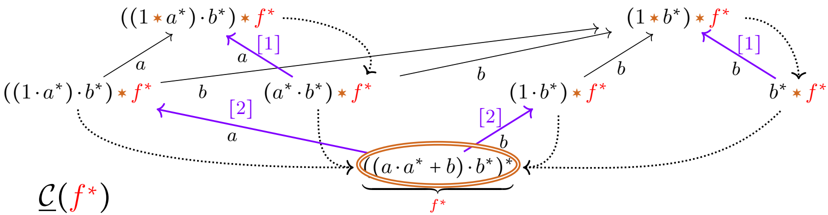
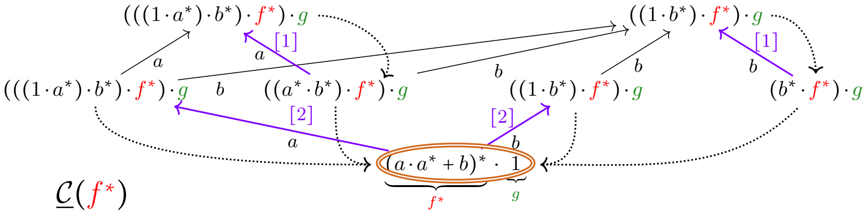


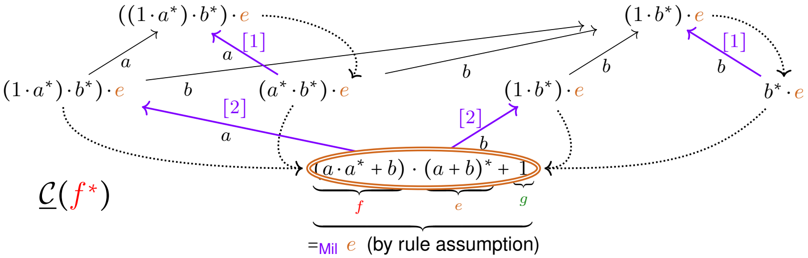
From fixed-point rule instances to coinductive proofs

$$\frac{\overbrace{(a+b)^*}^{e} = \overbrace{((a \cdot a^* + b) \cdot b^*)}^{f} \cdot \overbrace{(a+b)^*}^{e} + \overbrace{1}^{g}}{(a+b)^* = \underbrace{((a \cdot a^* + b) \cdot b^*)^*}_{f^*} \cdot \underbrace{1}_{g}} \ \text{RSP*}$$



From fixed-point rule instances to coinductive proofs

$$\frac{\overbrace{(a+b)^*}^{e} = \overbrace{((a\cdot a^* + b)\cdot b^*)}^{f}\cdot\overbrace{(a+b)^*}^{e} + \overbrace{1}^{g}}{(a+b)^* = \underbrace{((a\cdot a^* + b)\cdot b^*)^*}_{f^*}\cdot\underbrace{1}_{g}}\ \text{RSP*}$$



From fixed-point rule instances to coinductive proofs

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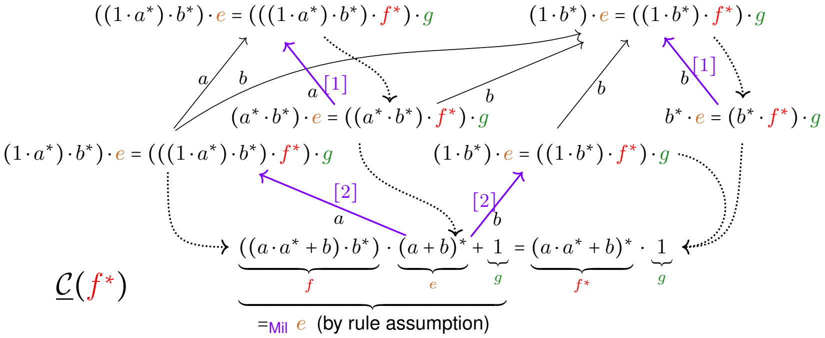


LEE-witnessed coinductive proof over $\textsf{Mil}^- + \{e = f\cdot e + g\}$

Proof transformation Mil $\longmapsto$ cMil

.

$$\frac{e = f \cdot e + g}{e = f^* \cdot g}\,\text{RSP}^*$$

$$\Longmapsto \quad \frac{e = f \cdot e + g \qquad \mathcal{LCP}_{\mathsf{Mil}^- + \{e = f \cdot e + g\}}(e = f^* \cdot g)}{e = f^* \cdot g}\,\text{LCoProof}_1$$

Proof transformation Mil $\longmapsto$ cMil

Theorem

Mil $\precsim$ cMil, because:

every derivation in Mil with conclusion $e = f$ can be transformed effectively into a derivation in cMil with conclusion $e = f$.

Proof idea.

$$\frac{e = f \cdot e + g}{e = f^{*} \cdot g}\ \text{RSP}^{*}$$

$$\Longmapsto \quad \frac{e = f \cdot e + g \qquad \mathcal{LCP}_{\mathsf{Mil}^{-}+\{e=f\cdot e+g\}}(e = f^{*} \cdot g)}{e = f^{*} \cdot g}\ \text{LCoProof}_1$$

Corollary

Mil $\sim$ cMil $\sim$ CLC.

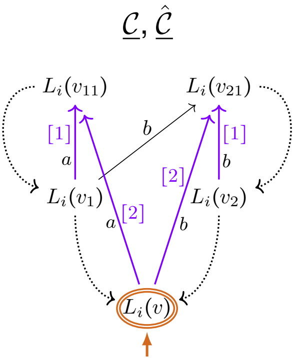
Proof transformation cMil $\longmapsto$ Mil

Lemma

For all star expression e, f, and equations $\Gamma \subseteq =_{\text{Mil}}$:

$$e \overset{\text{LEE}}{=\!=}_{\text{Mil}^- + \Gamma} f \quad \Longrightarrow \quad e =_{\text{Mil}} f$$

Extraction of Mil-derivation from LEE-witn. coind. proof



$$L_i(v_{21}) =^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 \cdot L_i(v_2) =_{\mathsf{Mil}^-} L_i(v_2)$$

($=^{(\mathrm{sol})}_{\mathsf{Mil}^-}$ means use of ‘is Mil⁻-provable solution’)

$$L_i(v_2) =^{(\mathrm{sol})}_{\mathsf{Mil}^-} b \cdot L_i(v_{21}) + 1 \cdot L_i(v) =_{\mathsf{Mil}^-} b \cdot L_i(v_2) + L_i(v)$$

⇓ applying RSP*

$$L_i(v_2) =_{\mathsf{Mil}} b^* \cdot L_i(v)$$

$$L_i(v_{11}) =^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 \cdot L_i(v_1) =_{\mathsf{Mil}^-} L_i(v_1)$$

$$L_i(v_1) =_{\mathsf{Mil}^-} a \cdot L_i(v_{11}) + b \cdot L_i(v_{21}) + 1 \cdot L_i(v)$$
$$=_{\mathsf{Mil}} a \cdot L_i(v_1) + (b \cdot b^* + 1) \cdot L_i(v)$$
$$=_{\mathsf{Mil}^-} a \cdot L_i(v_1) + b^* \cdot L_i(v)$$

⇓ applying RSP*

$$L_i(v_1) =_{\mathsf{Mil}} a^* \cdot (b^* \cdot L_i(v)) =_{\mathsf{Mil}^-} (a^* \cdot b^*) \cdot L_i(v)$$

$$L_i(v) =^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 + a \cdot L_i(v_{11}) + b \cdot L_i(v_{21}) =_{\mathsf{Mil}^-} 1 + a \cdot L_i(v_1) + b \cdot L_i(v_2)$$
$$=_{\mathsf{Mil}} (a \cdot (a^* \cdot b^*) + b \cdot b^*) \cdot L_i(v) + 1$$

⇓ applying RSP*

$$L_i(v) =_{\mathsf{Mil}} (a \cdot (a^* \cdot b^*) + b \cdot b^*)^* \cdot 1 =_{\mathsf{Mil}^-} (a \cdot (a^* \cdot b^*) + b \cdot b^*)^* = s_{\underline{\hat{\mathcal{C}}}}(v)$$

Proof transformation cMil $\longmapsto$ Mil

Lemma (extraction and unique solvability)

Let $\underline{\mathcal{C}}$ be a LEE-1-chart.

- From $\underline{\mathcal{C}}$ a Mil^{-}-(hence Mil-)provable solution can be extracted.
- Any two Mil-provable solutions of $\underline{\mathcal{C}}$ are Mil-provably equal.

Lemma

For all star expression e, f, and equations $\Gamma \subseteq =_{\text{Mil}}$:

$$e \overset{\text{LEE}}{=\!=}_{\text{Mil}^{-}+\Gamma} f \quad \Longrightarrow \quad e =_{\text{Mil}} f$$

Theorem

cMil $\lesssim$ Mil, because:

every derivation in cMil with conclusion $e = f$ can be transformed effectively into a derivation in Mil with conclusion $e = f$.

Summary

We define:

- (LEE-witnessed) coinductive proofs over Mil^- :
  - 1-charts $\underline{\mathcal{C}}$ (with LEE) whose vertices are labeled by equations between the values of two provable solutions of $\underline{\mathcal{C}}$
- proof systems
  - systems cMil / CLC with LEE-witnessed coind. proofs over Mil^-
  - systems $\overline{\mathsf{cMil}}$ / CC with coinductive proofs over Mil^-

Results:

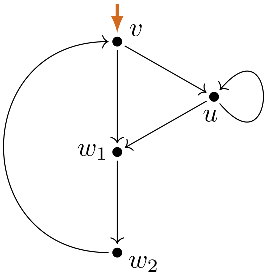
- $\mathsf{Mil} = (\mathsf{Mil}^- + \mathsf{RSP}^*) \sim (\mathsf{Mil}^- + \mathsf{LCoProof}) = \mathsf{cMil} \sim \mathsf{CLC}$
- $\mathsf{Mil} \lesssim (\mathsf{Mil}^- + \mathsf{USP}) \sim (\mathsf{Mil}^- + \mathsf{CoProof}) = \overline{\mathsf{cMil}} \sim \mathsf{CC}$ ((clearly) complete).

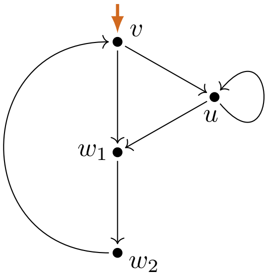
Desired application: proof strategy for completeness proof of Mil

- $\vdash_{\mathsf{Mil}} e = f \Longleftarrow \vdash_{\mathsf{cMil}} e = f \Longleftarrow e =_{[\![\cdot]\!]_P} f$

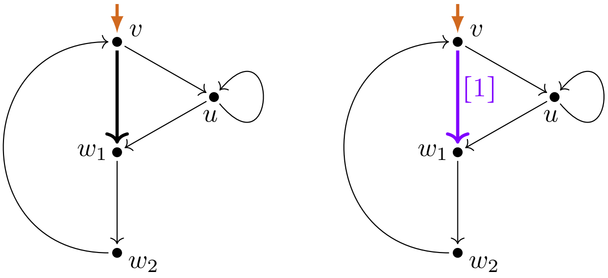
- Technical report: arXiv:2108.13104

Layered LEE-witness

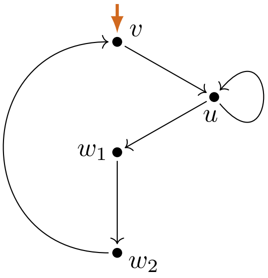


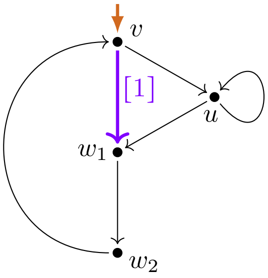


Layered LEE-witness

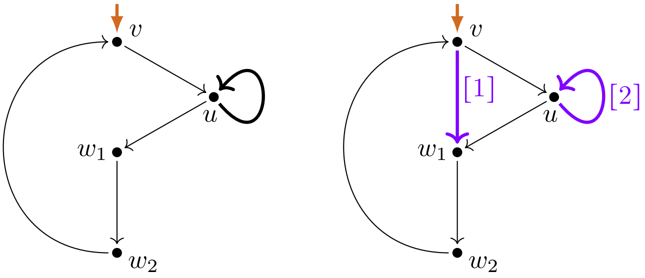


Layered LEE-witness





Layered LEE-witness



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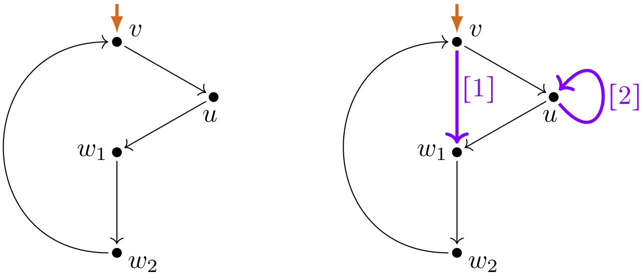

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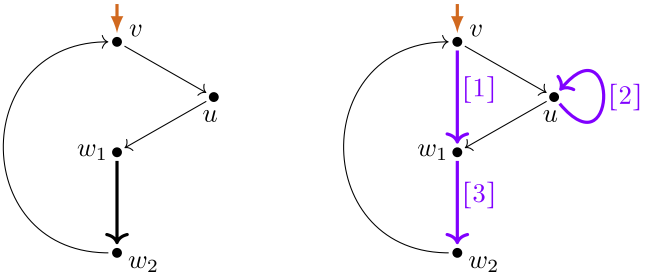
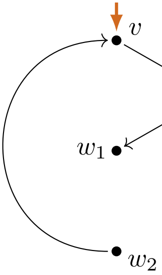
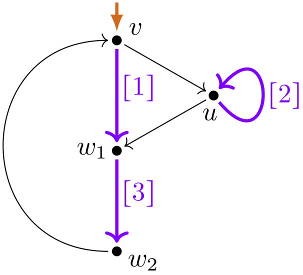


Layered LEE-witness





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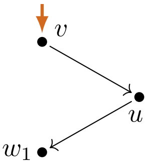

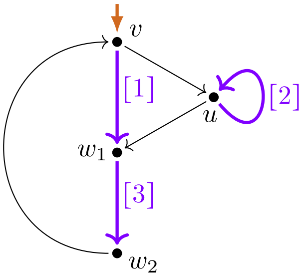

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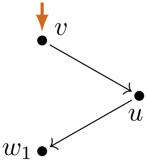
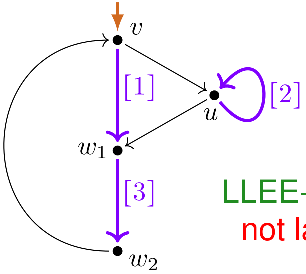
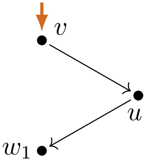
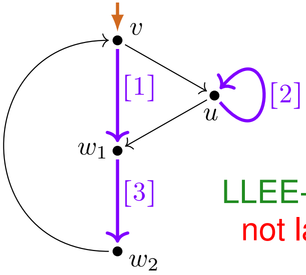




LLEE-witness
not layered

Layered LEE-witness





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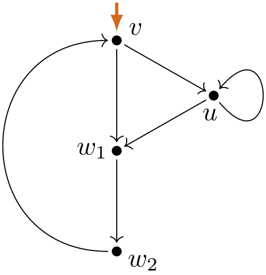
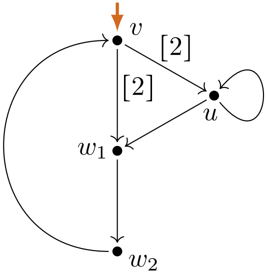
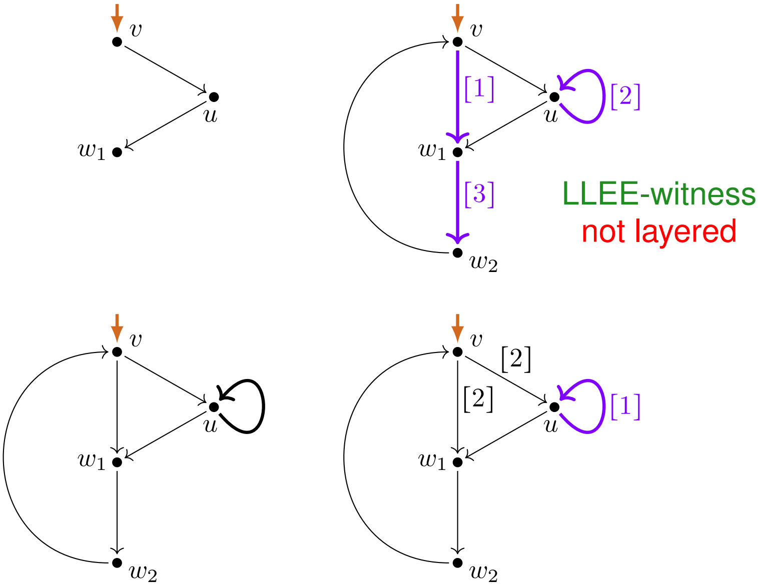
not layered

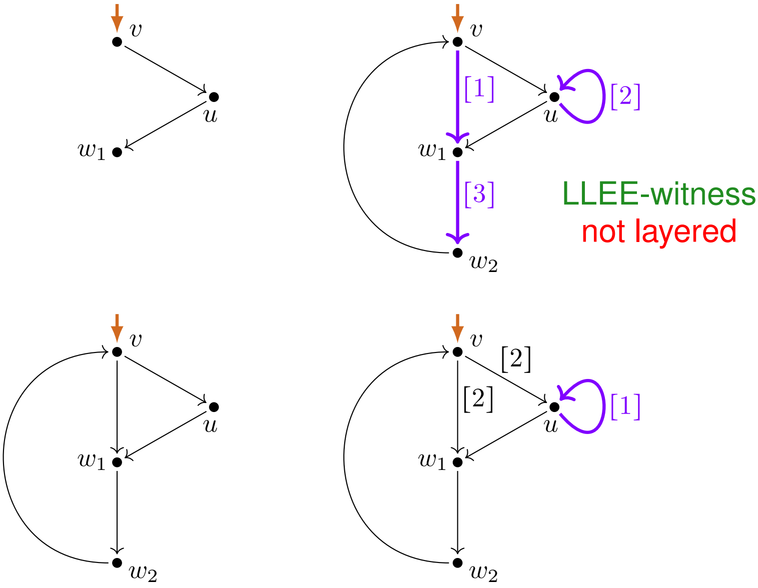




Layered LEE-witness



Layered LEE-witness



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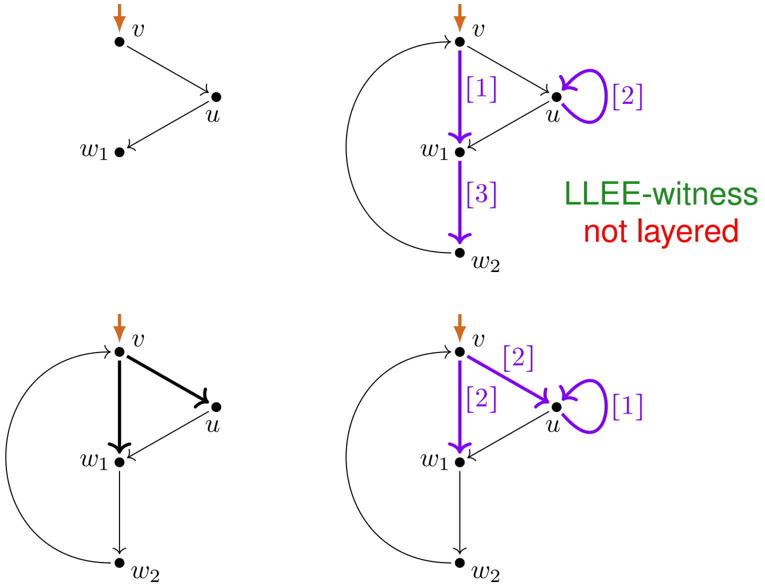

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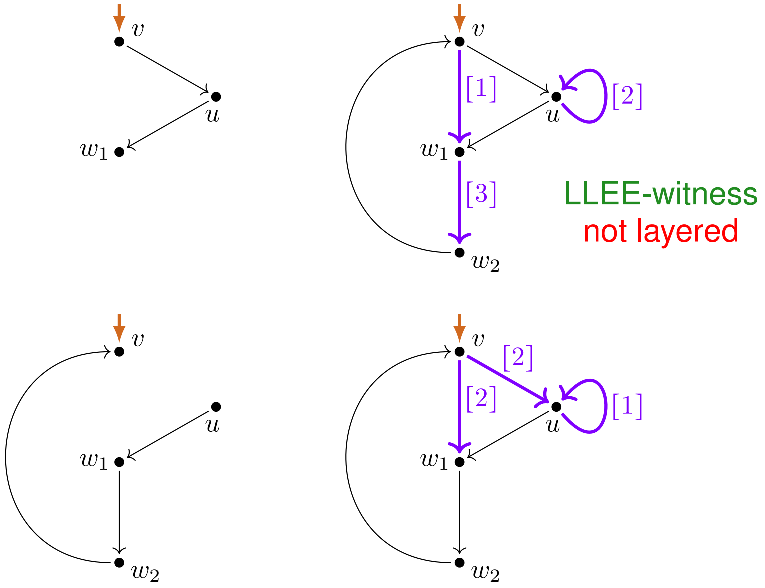

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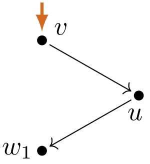
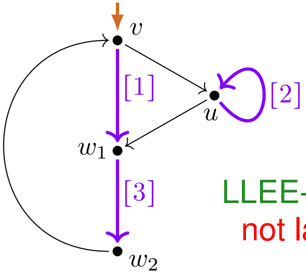
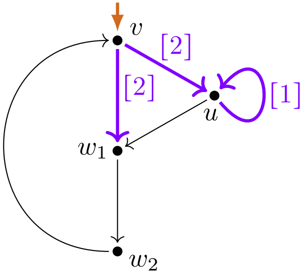
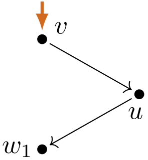


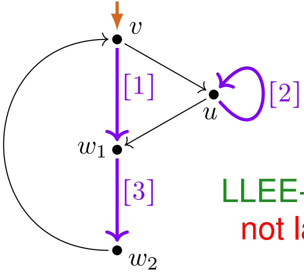


LLEE-witness
not layered



Layered LEE-witness





LLEE-witness
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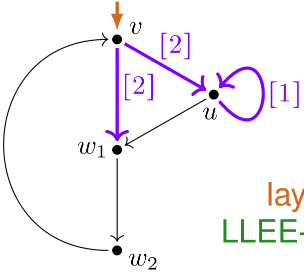
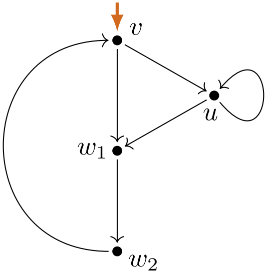
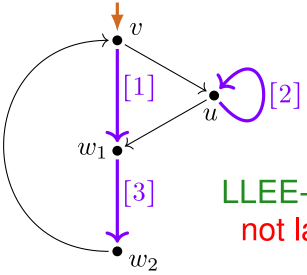


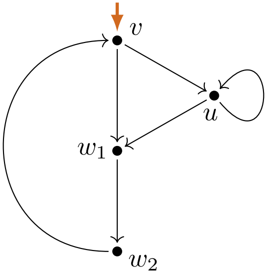
layered
LLEE-witness

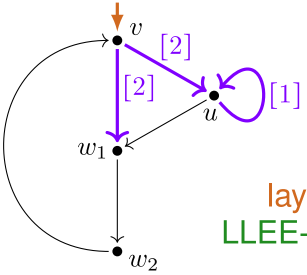
Layered LEE-witness





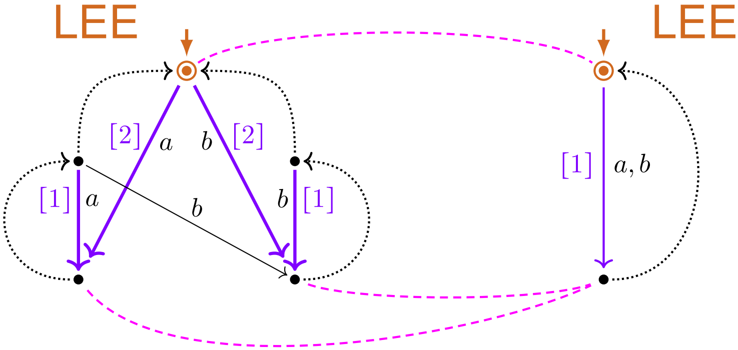
LLEE-witness
not layered





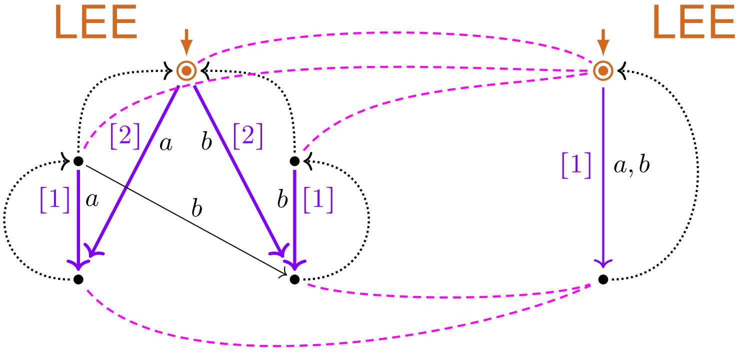
layered
LLEE-witness

LLEE-1-chart interpretation (example) *(TERMGRAPH, 2020)*



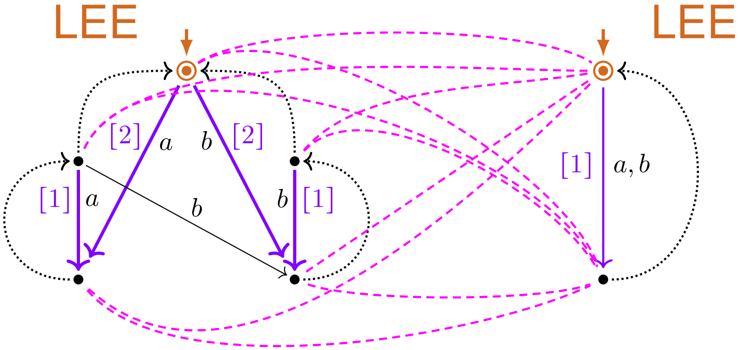
$\underline{\mathcal{C}}(\ (a^* \cdot b^*)^*\ )$ $\underline{\leftrightarrow}^{(1)}$ $\underline{\mathcal{C}}(\ (a + b)^*\ )$

LLEE-1-chart interpretation (example) *(TERMGRAPH, 2020)*



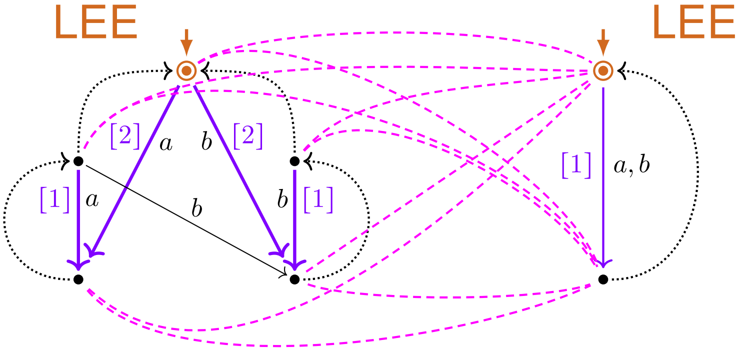
$\underline{\mathcal{C}}(\,(a^* \cdot b^*)^*\,)$ $\underline{\leftrightarrow}^{(1)}$ $\underline{\mathcal{C}}(\,(a+b)^*\,)$

LLEE-1-chart interpretation (example) *(TERMGRAPH, 2020)*



$\underline{\mathcal{C}}(\ (a^* \cdot b^*)^*\ )$ $\underline{\leftrightarrow}^{(1)}$ $\underline{\mathcal{C}}(\ (a+b)^*\ )$

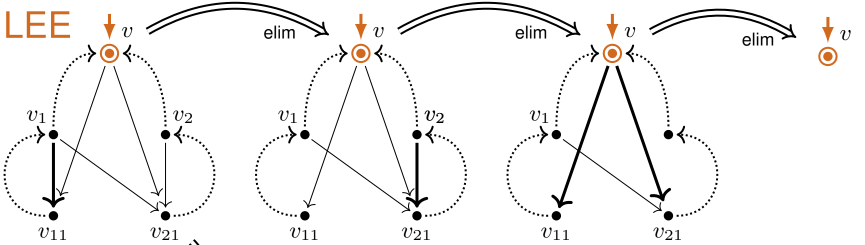
LLEE-1-chart interpretation (example) *(TERMGRAPH, 2020)*

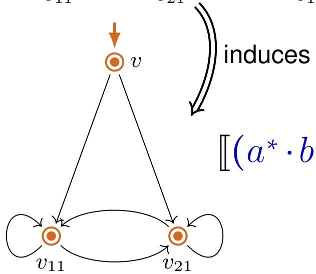


$\underline{\mathcal{C}}(\ (a^* \cdot b^*)^*\ )$ $\underline{\leftrightarrow}^{(1)}$ $\underline{\mathcal{C}}(\ (a+b)^*\ )$

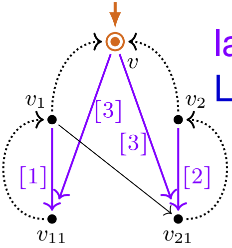
LEE, and LLEE-witness, induced process graph

LEE





$[\![(a^* \cdot b^*)^*]\!]_{P}$



layered LEE-witness

LEE-charts: properties and results

Lemmas

(I) Chart interpretations of 1-free star expressions satisfy LEE.

(SU) LEE-charts have unique provable solutions up to Mil-provability.

(C) LEE is preserved under bisimulation collapse.

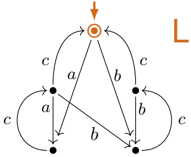
Theorem *(G/Fokkink, LICS 2020)*

The adaptation BBP of Mil to 1-free star expressions is complete.

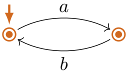
Consequence of lemmas used

(E) A chart $\mathcal{C}$ is expressible by a 1-free star expr. modulo bisimilarity
$\Longleftrightarrow$ the bisimulation collapse of $\mathcal{C}$ is a LEE-chart.

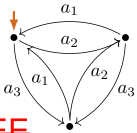
Hence expressible | not expressible by 1-free star expressions:



LEE



~~LEE~~



~~LEE~~