Modeling Terms by Graphs with Structure Constraints

(Two Illustrations)

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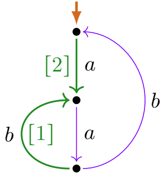

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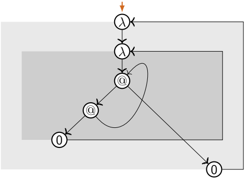

Overview

1. Process interpretation of regular expressions
   - LEE-witnesses: graph labelings based on a loop–condition LEE
2. Maximal sharing of functional programs
   - higher-order λ-term graphs

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   - Milner's questions, known results
   - structure-constrained process graphs:
     - LEE-witnesses: graph labelings based on a loop–condition LEE
     - preservation under bisimulation collapse
   - readback: from graph labelings to regular expressions
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   - from terms in the λ-calculus with letrec to:
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     - λ-NFAs, and λ-DFAs
   - minimization / readback / efficiency / Haskell implementation

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Regular expressions under process semantics (bisimilarity $\underline{\leftrightarrow}$)

Given: process graph interpretation $[\![\cdot]\!]_{P}$, studied under $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, modulo $\underline{\leftrightarrow}$ incomplete

λ-calculus with letrec under unfolding semantics

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(at least with ‘sufficiently many’)

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Not available: term graph interpretation that is studied under $\underline{\leftrightarrow}$

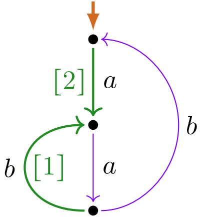
- graph representations used by compilers were not intended for use under $\underline{\leftrightarrow}$

Desired: term graph interpretation that:

- natural correspondence with terms in $\boldsymbol{\lambda}_{\text{letrec}}$
- supports compactification under $\underline{\leftrightarrow}$
- efficient translation and readback

Process interpretation of regular expressions

(current work with Wan Fokkink)



Regular Expressions

Definition

The set $\mathsf{Reg}(A)$ of regular expressions over alphabet A is defined by the grammar:

$$e, f \;::=\; 0 \mid 1 \mid a \mid (e + f) \mid (e \cdot f) \mid (e^{\star}) \qquad \text{(for } a \in A\text{).}$$

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Note, here:

- symbol 0 instead of $\varnothing$
- symbol 1 used (often dropped, definable as $0^{\star}$)
- no complementation operation $\overline{e}$
  - is not expressible under language interpretation

Language interpretation $[\![\cdot]\!]_{L}$ *(S.C. Kleene, 1951)*

$\mathbf{0} \xmapsto{[\![\cdot]\!]_{L}}$ empty language $\varnothing$

$\mathbf{1} \xmapsto{[\![\cdot]\!]_{L}} \{\epsilon\}$ (ϵ the empty word)

$a \xmapsto{[\![\cdot]\!]_{L}} \{a\}$

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$e + f \xmapsto{[\![\cdot]\!]_{L}}$ union of $[\![e]\!]_{L}$ and $[\![f]\!]_{L}$

$e \cdot f \xmapsto{[\![\cdot]\!]_{L}}$ element-wise concatenation of $[\![e]\!]_{L}$ and $[\![f]\!]_{L}$

$e^{*} \xmapsto{[\![\cdot]\!]_{L}}$ set of words formed by concatenating words in $[\![e]\!]_{L}$

Process interpretation $[\![\cdot]\!]_P$ *(R. Milner, 1984)*

$\mathbf{0} \xmapsto{[\![\cdot]\!]_P}$ deadlock $\boldsymbol{\delta}$, no termination

$\mathbf{1} \xmapsto{[\![\cdot]\!]_P}$ empty process $\boldsymbol{\epsilon}$, then terminate

$a \xmapsto{[\![\cdot]\!]_P}$ atomic action a, then terminate

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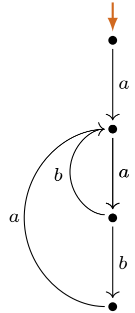
$a \xmapsto{[\![\cdot]\!]_{P}}$ atomic action a, then terminate

$e + f \xmapsto{[\![\cdot]\!]_{P}}$ alternative composition between $[\![e]\!]_{P}$ and $[\![f]\!]_{P}$

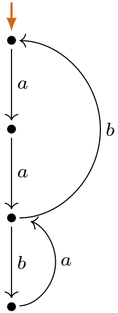
$e \cdot f \xmapsto{[\![\cdot]\!]_{P}}$ sequential composition of $[\![e]\!]_{P}$ and $[\![f]\!]_{P}$

$e^{*} \xmapsto{[\![\cdot]\!]_{P}}$ unbounded iteration of $[\![e]\!]_{P}$, option to terminate

Process interpretation of regular expressions



$$a(a(b+ba))^*0$$



$$(aa(ba)^*b)^*0$$

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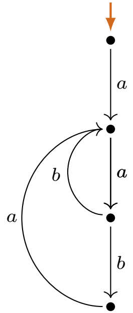
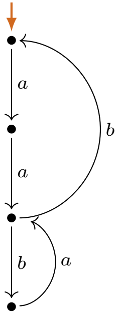


$a \cdot (a \cdot (b + b \cdot a))^* \cdot 0$



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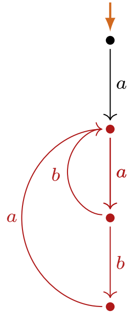

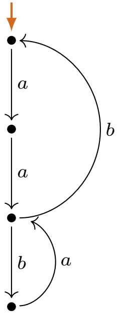

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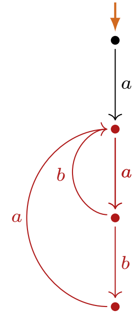

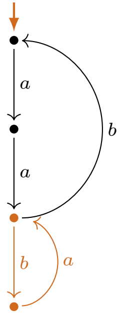

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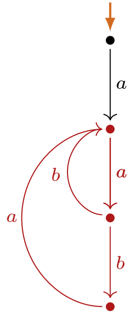

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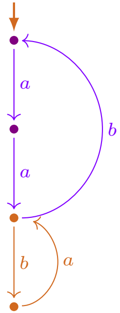

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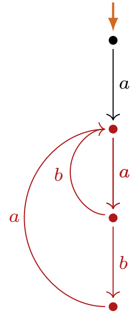

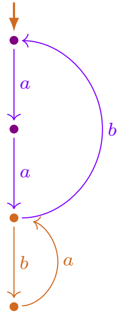

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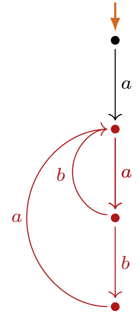

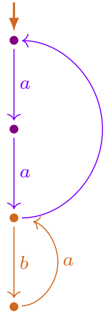

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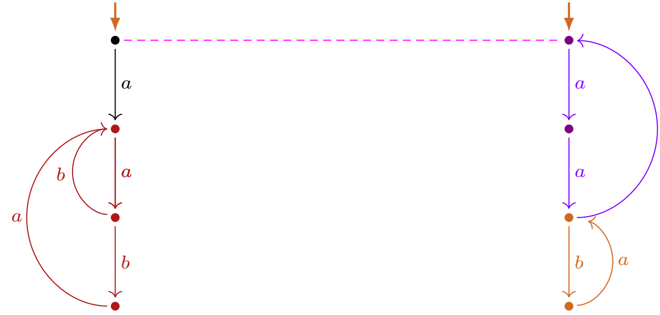

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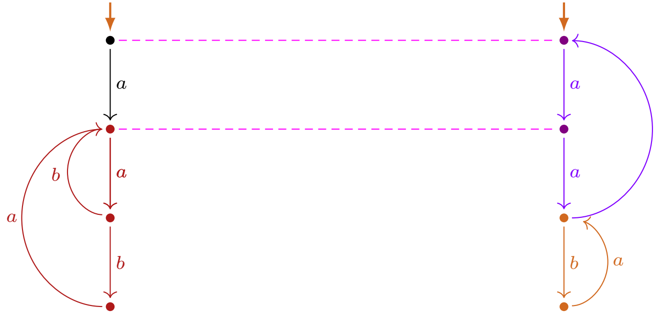

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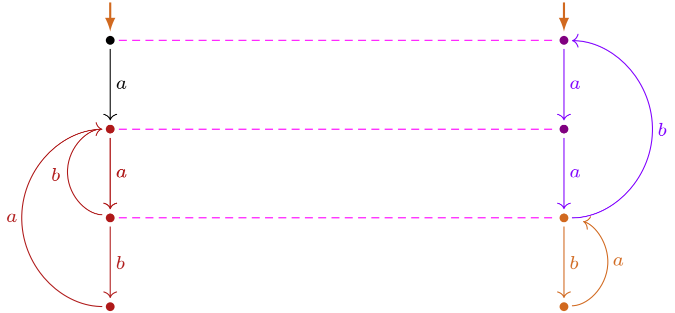

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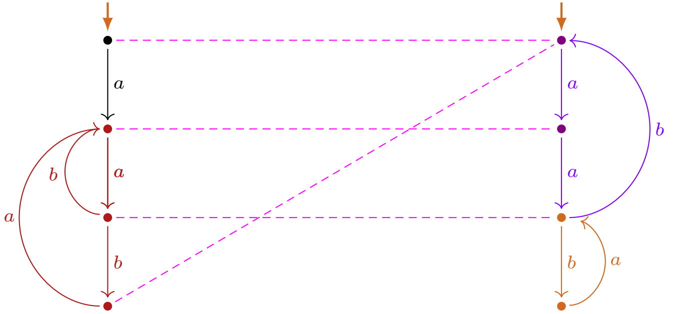

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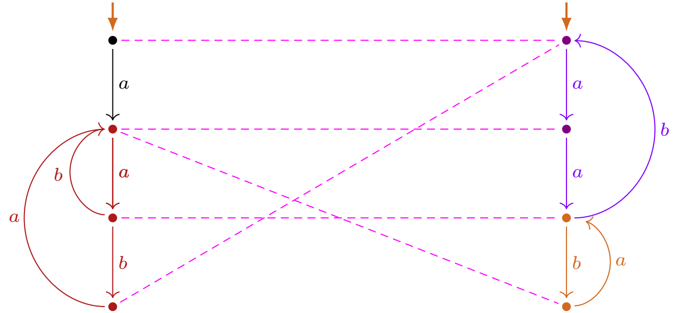

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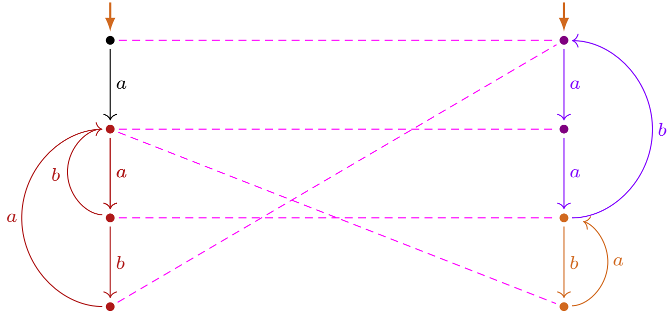

$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}} \quad \underline{\leftrightarrow} \quad [\![(aa(ba)^{*}b)^{*}0]\!]_{\boldsymbol{P}}$$

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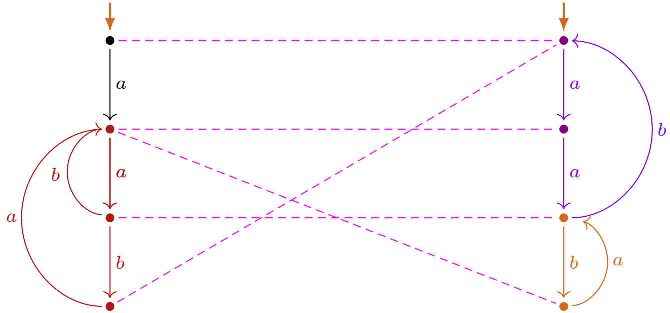

$$a(a(b+ba))^{*}0 \qquad \underline{\leftrightarrow}_{\boldsymbol{P}} \qquad (aa(ba)^{*}b)^{*}0$$

Process graphs and NFAs

Definition

A process graph over actions in A is a tuple $G = \langle V, v_{\mathsf{s}}, T, E\rangle$ where:

- V is a set of *vertices*,
- $v_{\mathsf{s}} \in V$ is the *start vertex*,
- $T \subseteq V \times A \times V$ the set of *transitions*,
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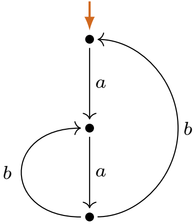
Antimirov (1996): NFA-definition of $[\![\cdot]\!]_{\boldsymbol{P}}$ via partial derivatives.

Expressible process graphs (under bisimulation $\underline{\leftrightarrow}$)



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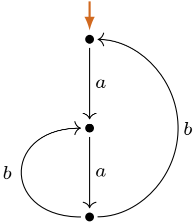

$$\notin im(\llbracket\cdot\rrbracket_{\boldsymbol{P}})$$

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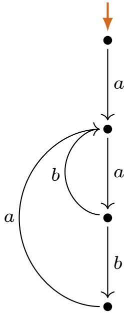

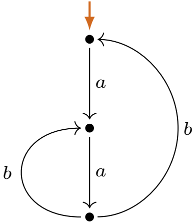

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$\llbracket\cdot\rrbracket_{\boldsymbol{P}}$-expressible

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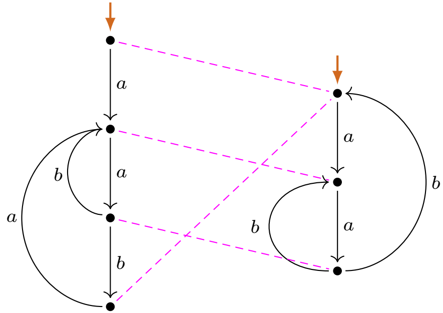

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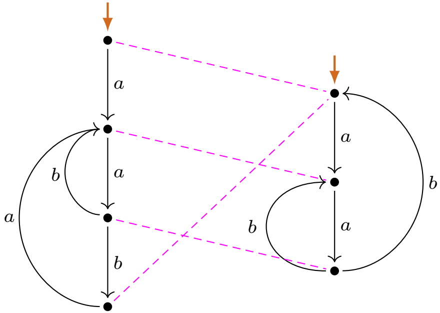

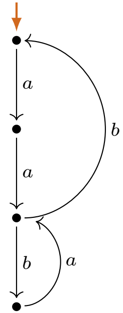

$\in im(\llbracket\cdot\rrbracket_{\boldsymbol{P}})$ $\notin im(\llbracket\cdot\rrbracket_{\boldsymbol{P}})$ $\in im(\llbracket\cdot\rrbracket_{\boldsymbol{P}})$

$\llbracket\cdot\rrbracket_{\boldsymbol{P}}$-expressible $\llbracket\cdot\rrbracket_{\boldsymbol{P}}$-expressible

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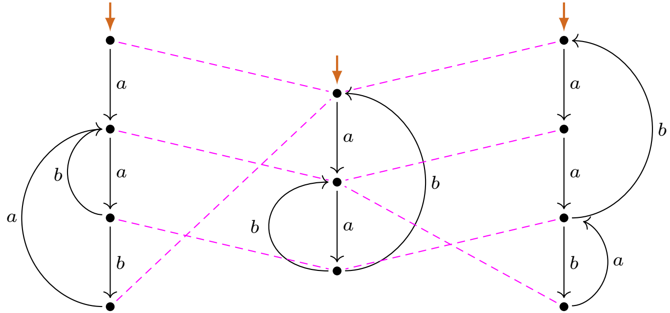

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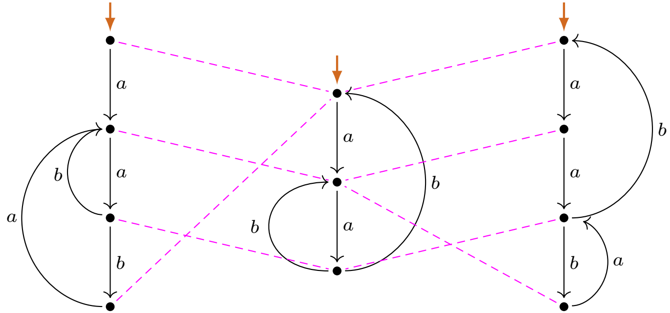

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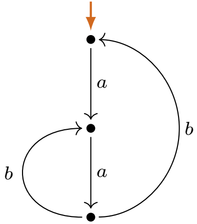
$\llbracket\cdot\rrbracket_{P}$-expressible $\llbracket\cdot\rrbracket_{P}$-expressible modulo $\underline{\leftrightarrow}$ $\llbracket\cdot\rrbracket_{P}$-expressible

Properties of $\boldsymbol{P}$

- Not every finite-state process is $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible.



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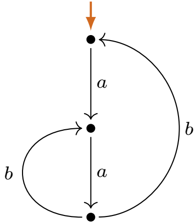

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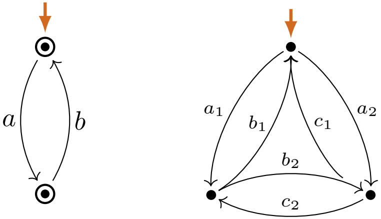

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- Fewer identities hold for $\underline{\leftrightarrow}_{\boldsymbol{P}}$ than for $=_{\boldsymbol{L}}$: $\quad \underline{\leftrightarrow}_{\boldsymbol{P}} \subsetneqq =_{\boldsymbol{L}}$.

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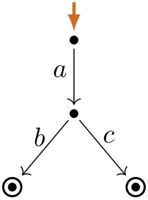
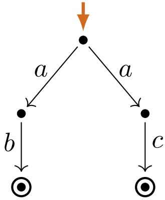


$=_{\boldsymbol{L}}$



$$a \cdot (b + c) \quad =_{\boldsymbol{L}} \quad a \cdot b + a \cdot c$$

Properties of $\boldsymbol{P}$

- Not every finite-state process is $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible.
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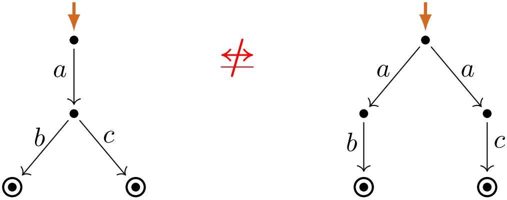

$$a \cdot (b + c) \quad \not\underline{\leftrightarrow}_{\boldsymbol{P}} \quad a \cdot b + a \cdot c$$

Salomaa's axiomatization of $=_{L}$ (products commuted)

Axioms:

(B1) $e + (f + g) = (e + f) + g$

(B2) $(e \cdot f) \cdot g = e \cdot (f \cdot g)$

(B3) $e + f = f + e$

(B4) $(e + f) \cdot g = e \cdot g + f \cdot g$

(B5) $e \cdot (f + g) = e \cdot f + e \cdot g$

(B6) $e + e = e$

(B7) $e \cdot 1 = e$

(B8) $e \cdot 0 = 0$

(B9) $e + 0 = e$

(B10) $e^{*} = 1 + e \cdot e^{*}$

(B11) $e^{*} = (1 + e)^{*}$

Inference rules: equational logic *plus*

$$\frac{e = f \cdot e + g}{e = f^{*} \cdot g}\ \mathsf{FIX} \quad (\text{if } \underbrace{\{\epsilon\} \notin [\![f]\!]_{L}}_{\textit{non-empty-word property}})$$

Sound and unsound axioms with respect to $\underline{\leftrightarrow}_{P}$

Axioms :

(B1)	$e + (f + g) = (e + f) + g$	(B7)	$e \cdot 1 = e$
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(B3)	$e + f = f + e$	(B9)	$e + 0 = e$
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(B5)	$e \cdot (f + g) = e \cdot f + e \cdot g$	(B11)	$e^{*} = (1 + e)^{*}$
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Milner's questions

Q2. Is Mil complete for $\underline{\leftrightarrow}_{P}$?

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Milner's questions, and partial results

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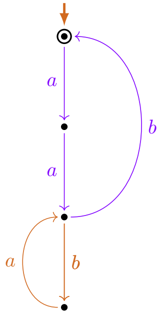
- Mil is complete when restricted to 1-return-less expressions *(Corradini, De Nicola, Labella, 2002)*
- Mil$^-$ + one of two stronger rules (than RSP*) is complete *(G, 2006)*
  - with a coinductive rule (based on *Antimirov's* partial derivatives)
  - with a unique solvability principle USP

Well-behaved form, looping palm trees



$[\![(aa(ba)^{*}b)^{*}]\!]_{\boldsymbol{P}}$

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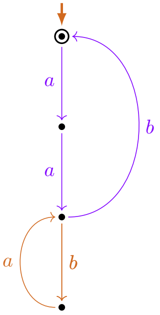

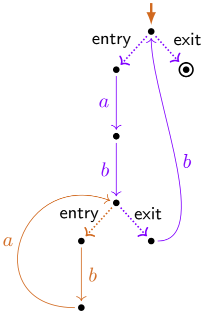

well-behaved form
(Corradini, Baeten)

$[\![(aa(ba)^*b)^*]\!]_{\boldsymbol{P}}$

$[\![(1 \cdot aa(1 \cdot ba)^* 1 \cdot b)^* (1 \cdot 1)]\!]_{\boldsymbol{P}}$

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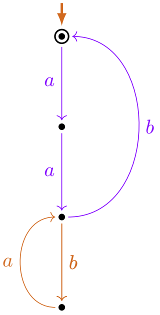

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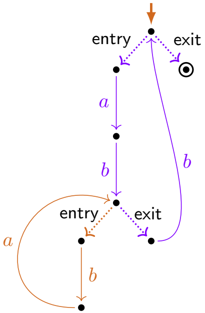

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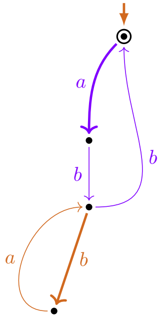

looping palm tree

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Loop chart

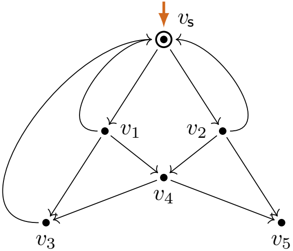
Definition

A process graph is a loop chart if:

L-1.

L-2.

L-3.



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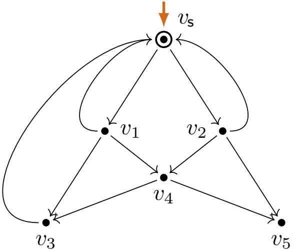

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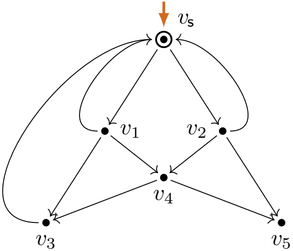

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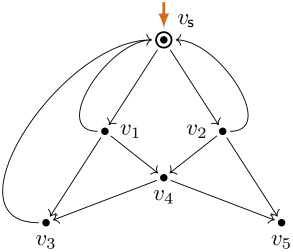

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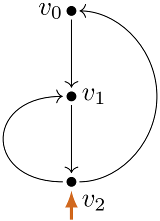

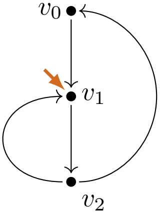

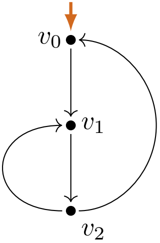

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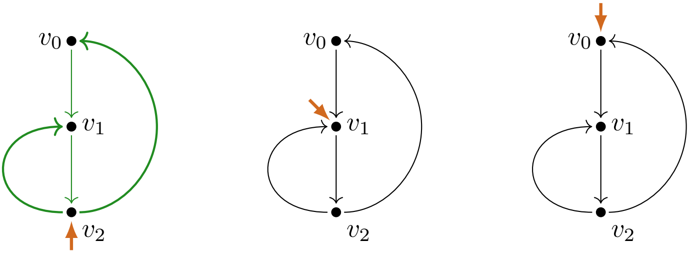

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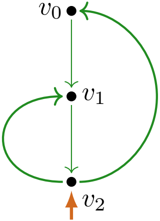

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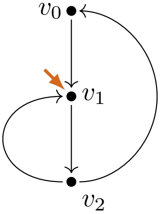

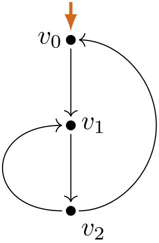

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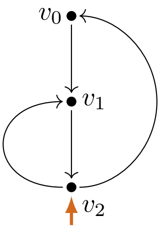

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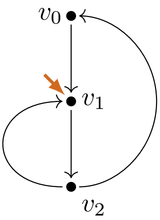

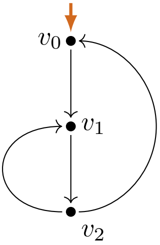

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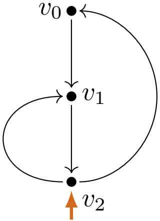

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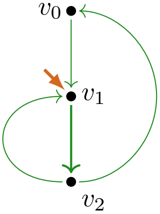

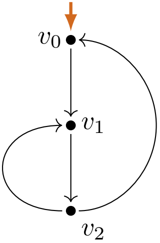

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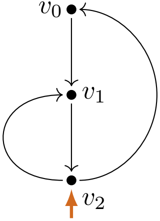

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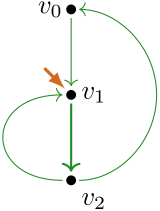

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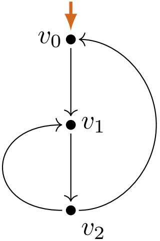

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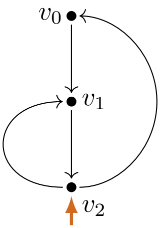

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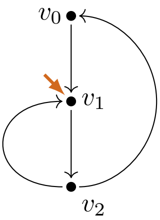

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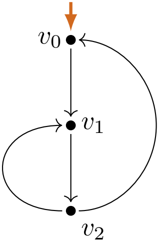

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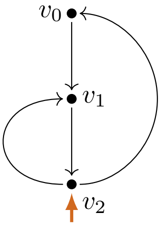

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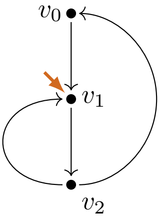

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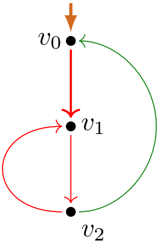

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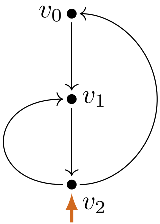

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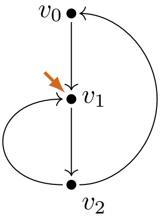
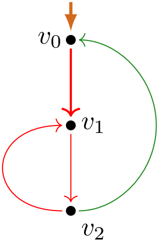


loop chart



no loop chart

Loop chart

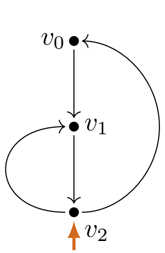
Definition

A process graph is a loop chart if:

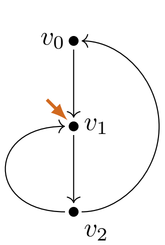
L-1. There is an infinite path from the start vertex.

L-2. Every infinite path from the start vertex returns to it.

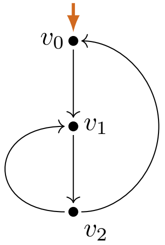
L-3. Termination is only possible at the start vertex.



loop chart

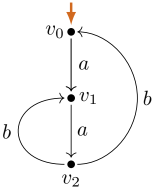


loop chart

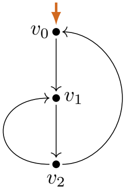


no loop chart

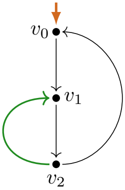
Loop elimination



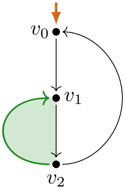
Loop elimination



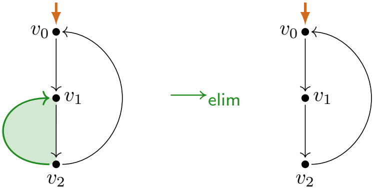
Loop elimination



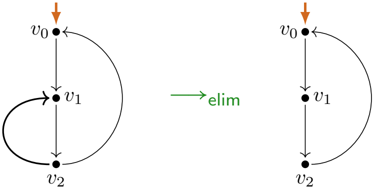
Loop elimination



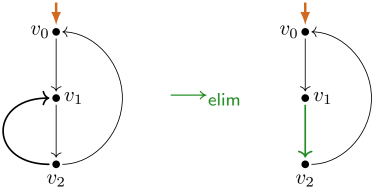
Loop elimination



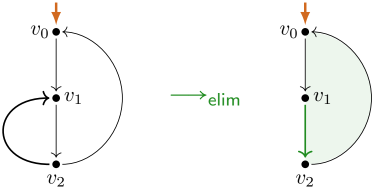
Loop elimination



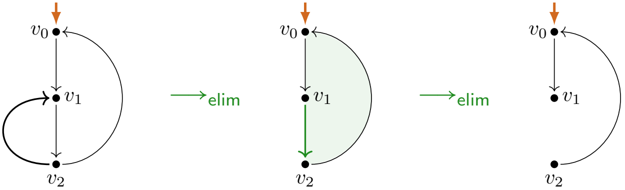
Loop elimination



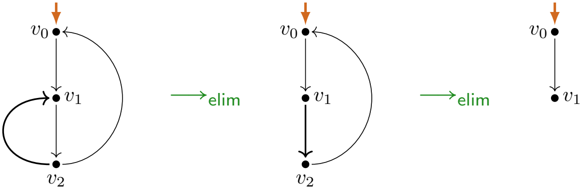
Loop elimination



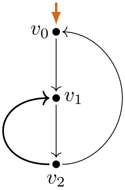
Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2

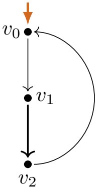
Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1

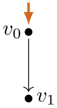
Loop elimination

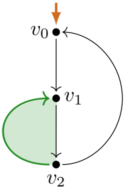


$\longrightarrow_{\text{elim}}$

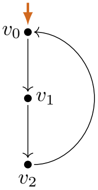


$\longrightarrow_{\text{elim}}$

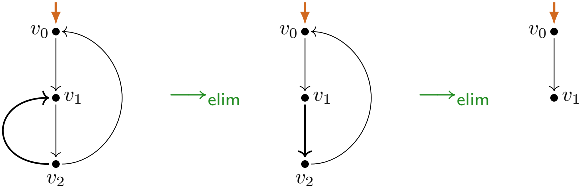


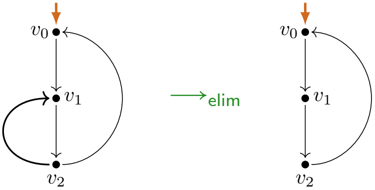


$\longrightarrow_{\text{elim}}$

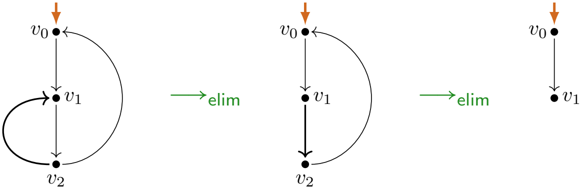


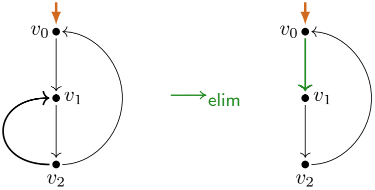
Loop elimination



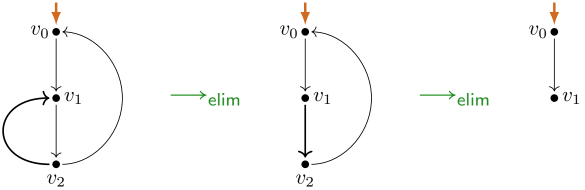


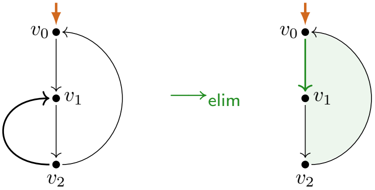
Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1

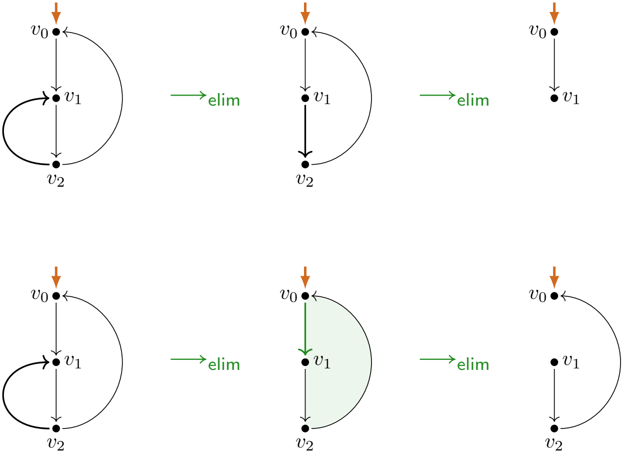


Loop elimination

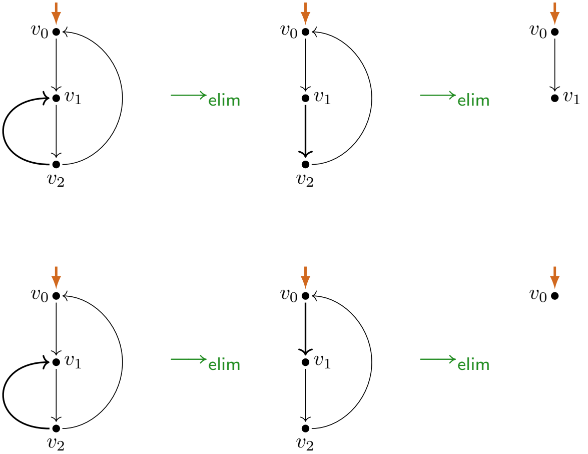

v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1



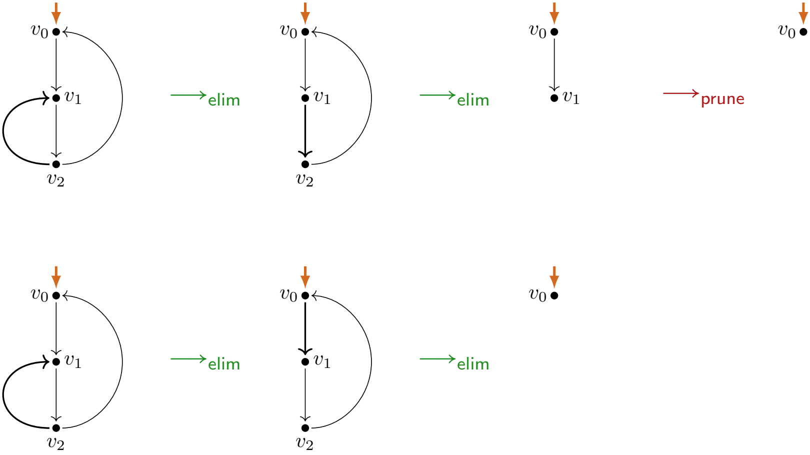
Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2

Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0

Loop elimination


v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0

Loop elimination, and properties

$\xrightarrow{}_{\text{elim}}$: eliminate a transition-induced loop by:

- removing the loop-entry transition(s)
- garbage collection

$\xrightarrow{}_{\text{prune}}$: remove a transition to a deadlocking state

Loop elimination, and properties

$\longrightarrow_{\text{elim}}$: eliminate a transition-induced loop by:

- removing the loop-entry transition(s)
- garbage collection

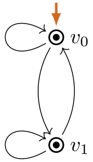
$\longrightarrow_{\text{prune}}$: remove a transition to a deadlocking state

Lemma

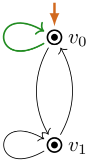
(i) $\longrightarrow_{\text{elim}}$ *is terminating.*

(ii) $\longrightarrow_{\text{elim}} \cup \longrightarrow_{\text{prune}}$ *is terminating and confluent.*

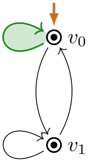
Loop elimination



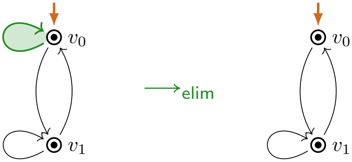
Loop elimination



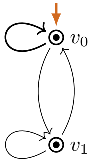
Loop elimination



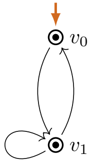
Loop elimination



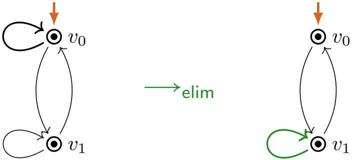
Loop elimination



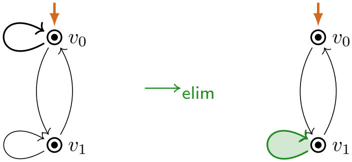
$\longrightarrow_{\text{elim}}$



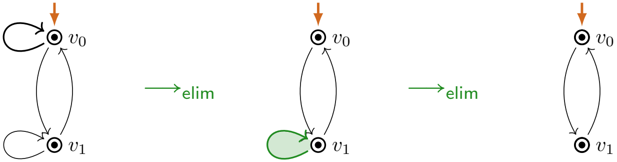
Loop elimination



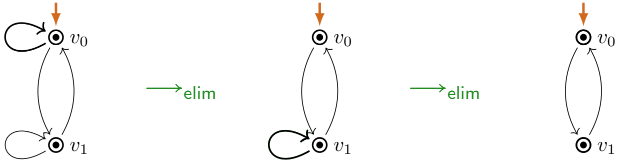
Loop elimination



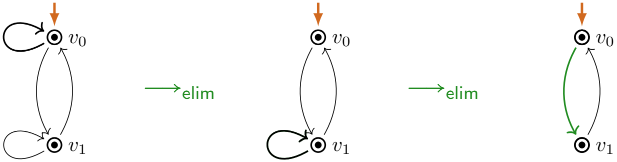
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

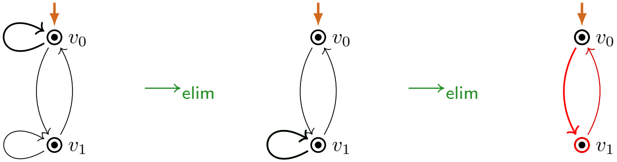
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

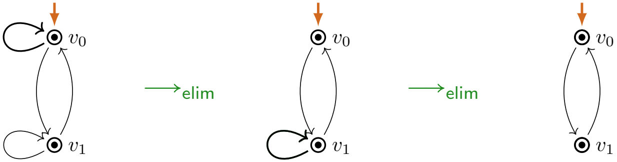
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

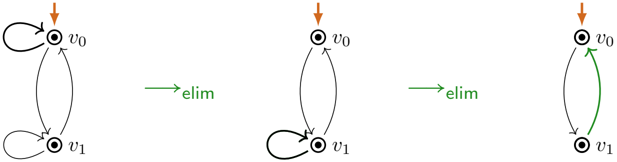
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

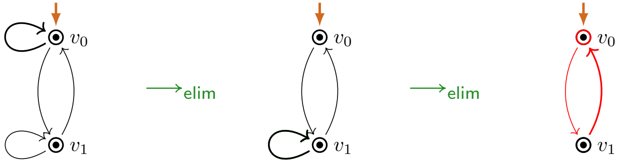
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

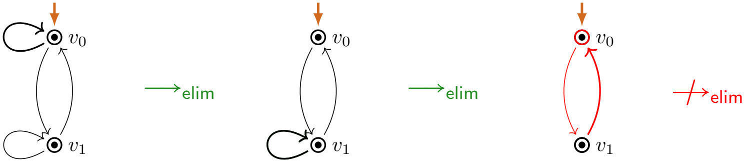
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

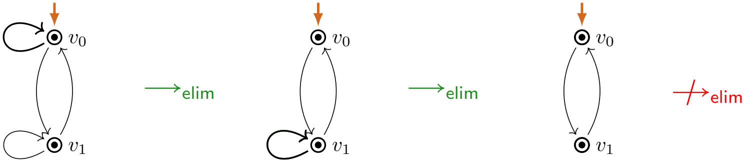
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1

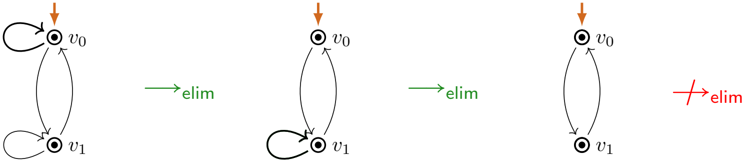
Loop elimination

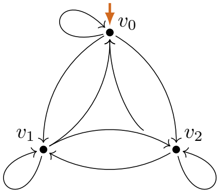

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

Loop elimination

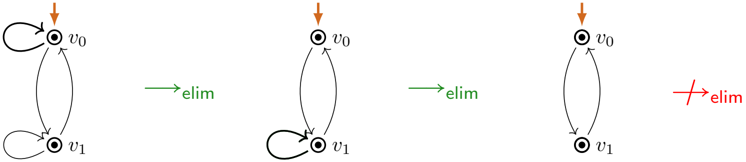

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

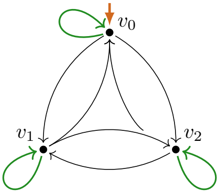
Loop elimination



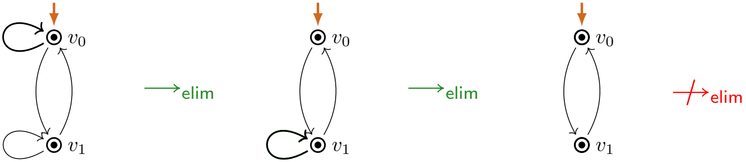


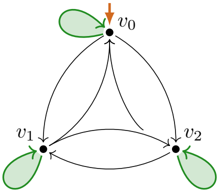
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

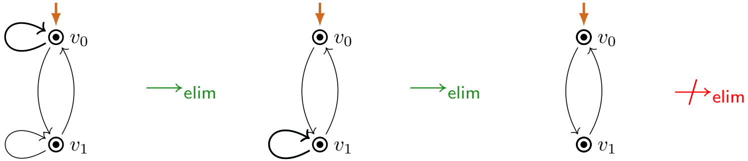


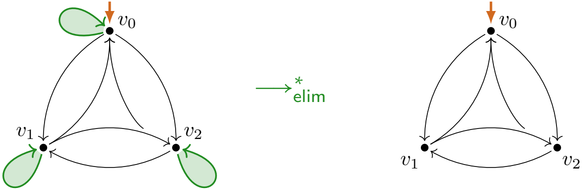
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

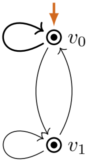


Loop elimination

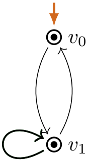

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$



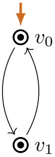
Loop elimination



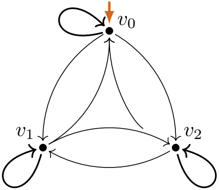
$\longrightarrow_{\text{elim}}$



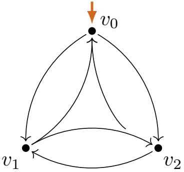
$\longrightarrow_{\text{elim}}$



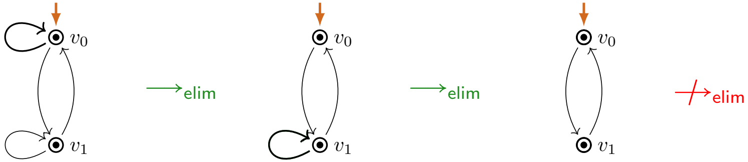
$\not\longrightarrow_{\text{elim}}$

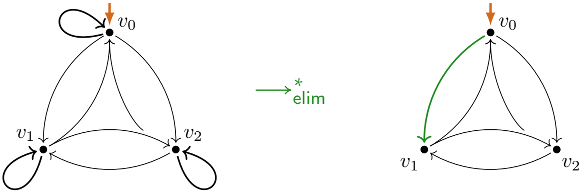


$\longrightarrow_{\text{elim}}^{*}$

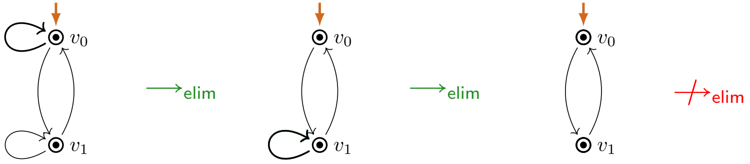


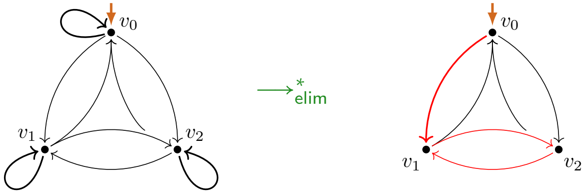
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

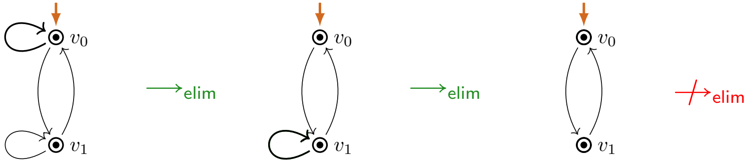


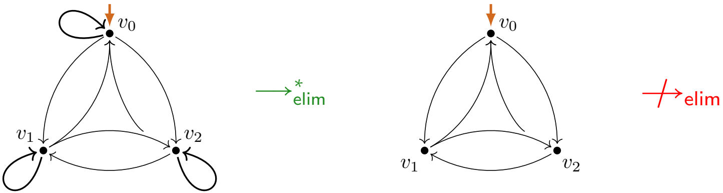
Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$



Loop elimination


v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$


v_0
v_1
v_2
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

$$\exists\, G_0 \left(G \longrightarrow^{*}_{\text{elim}} G_0 \not\longrightarrow_{\text{elim}} \right.$$
$$\left. \wedge\ G_0 \text{ has no infinite trace} \right) .$$

Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

$$\exists\, G_0 \,\big(\, G \longrightarrow^{*}_{\text{elim}} G_0 \not\longrightarrow_{\text{elim}} \\ \wedge\; G_0 \text{ has no infinite trace} \,\big)\;.$$

Lemma (by using confluence properties)

For every process graph G the following are equivalent:

(i) LEE(G).

(ii) *There is an $\longrightarrow_{\text{elim}}$ normal form without an infinite trace.*

Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

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Lemma (by using confluence properties)

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(iii) *There is an $\longrightarrow_{\text{elim},\text{prune}}$ normal form without an infinite trace.*

Structure property LEE

Definition

A process graph G satisfies LEE (*loop existence and elimination*) if:

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Lemma (by using confluence properties)

For every process graph G the following are equivalent:

(i) LEE(G).

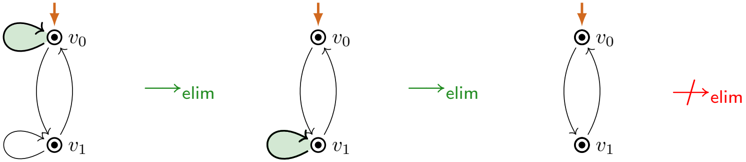
(ii) *There is an $\longrightarrow_{\text{elim}}$ normal form without an infinite trace.*

(iii) *There is an $\longrightarrow_{\text{elim,prune}}$ normal form without an infinite trace.*

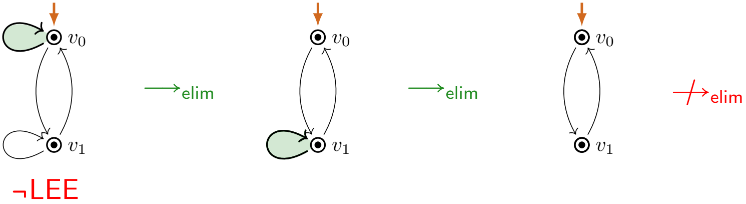
(iv) *Every $\longrightarrow_{\text{elim}}$ normal form is without an infinite trace.*

(v) *Every $\longrightarrow_{\text{elim,prune}}$ normal form is without an infinite trace.*

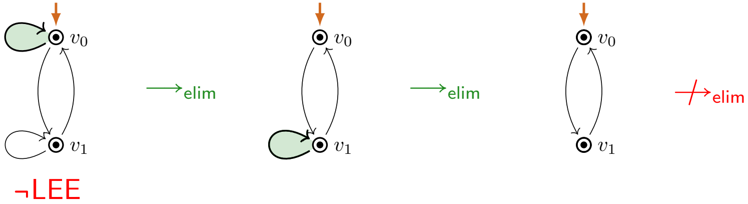
LEE fails

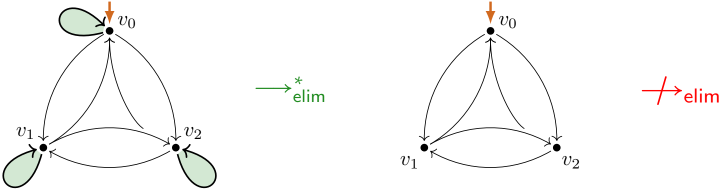

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

LEE fails

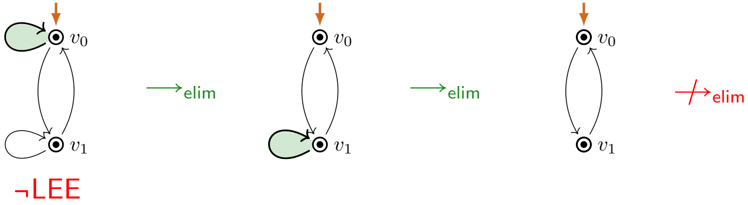

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$
¬LEE

LEE fails

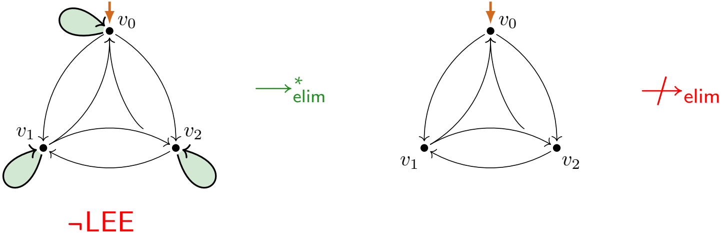

v_0
v_1
¬LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$


v_0
v_1
v_2
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

LEE fails

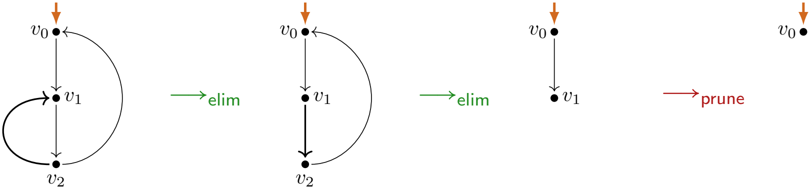

v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\not\longrightarrow_{\text{elim}}$

¬LEE

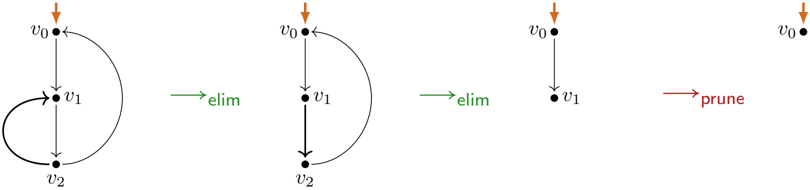

v_0
v_1
v_2
$\longrightarrow^{*}_{\text{elim}}$
v_0
v_1
v_2
$\not\longrightarrow_{\text{elim}}$

¬LEE

LEE holds

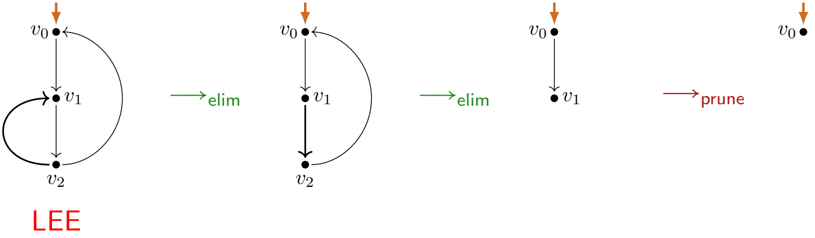

v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

LEE holds

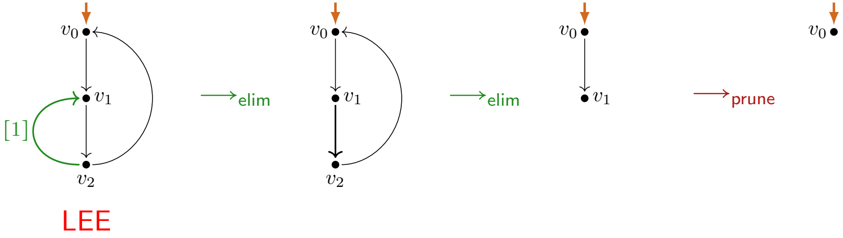

v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

LEE

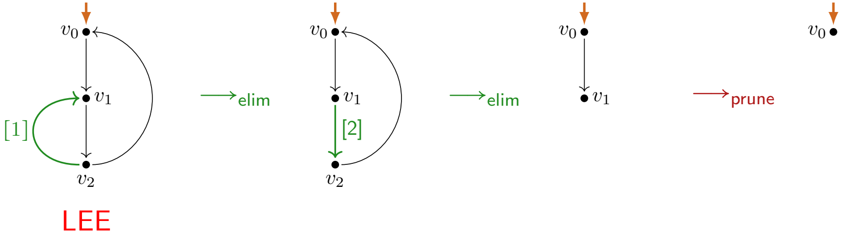
LEE holds / Recording loop elimination



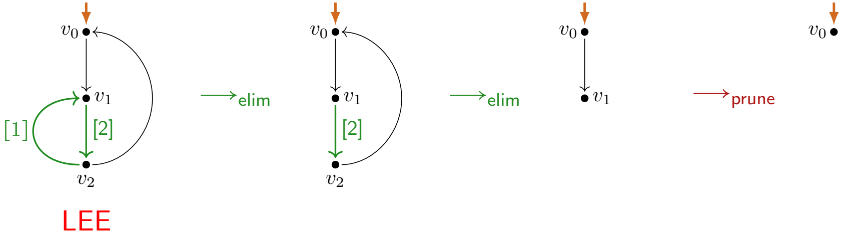
LEE holds / Recording loop elimination


v_0
v_1
v_2
[1]
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

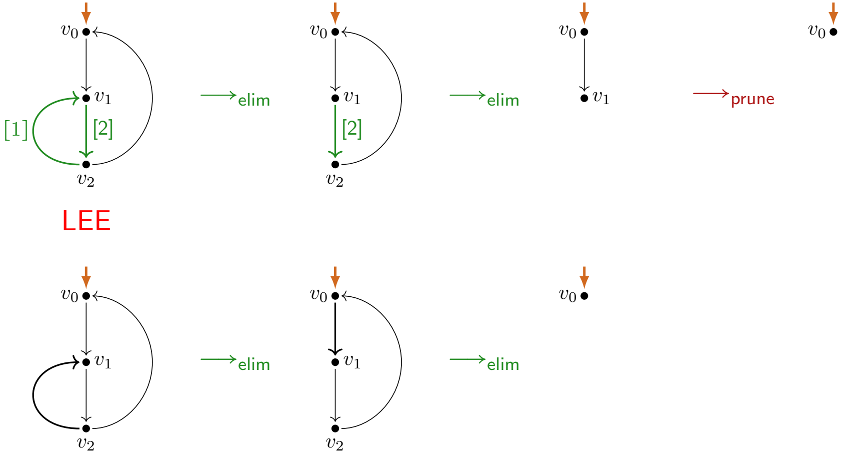
LEE holds / Recording loop elimination


v_0
v_1
v_2
[1]
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
[2]
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

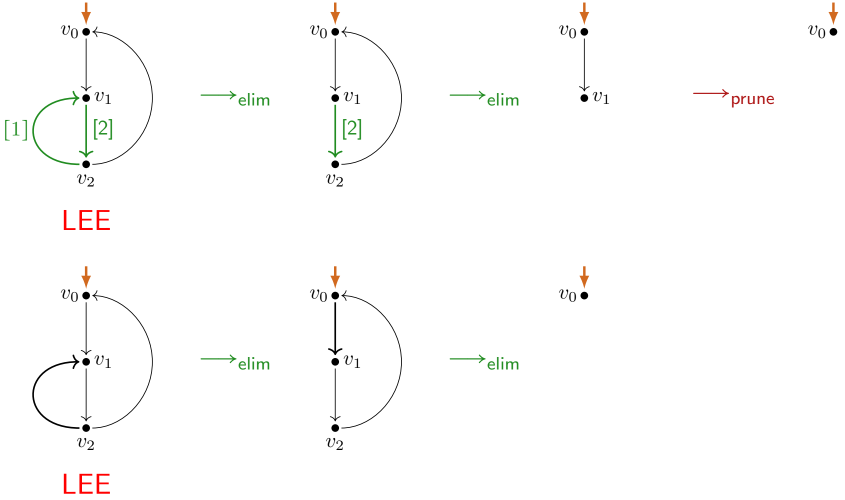
LEE holds / Recording loop elimination


v_0
v_1
v_2
[1]
[2]
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
[2]
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0

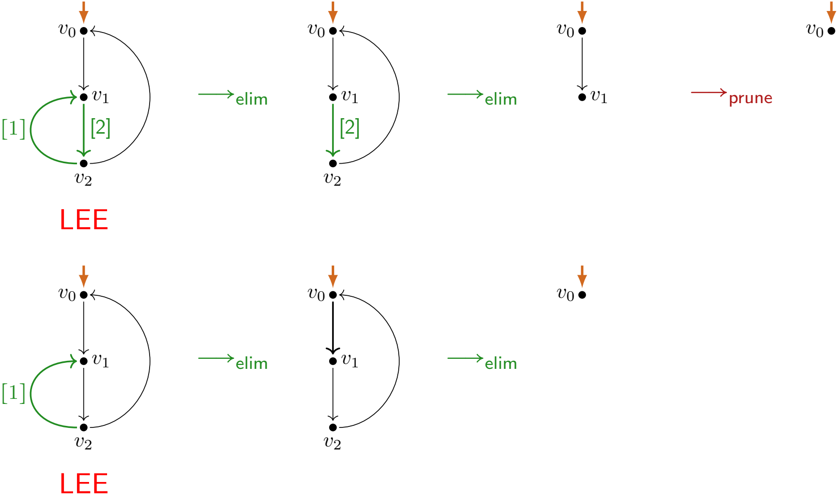
LEE holds / Recording loop elimination



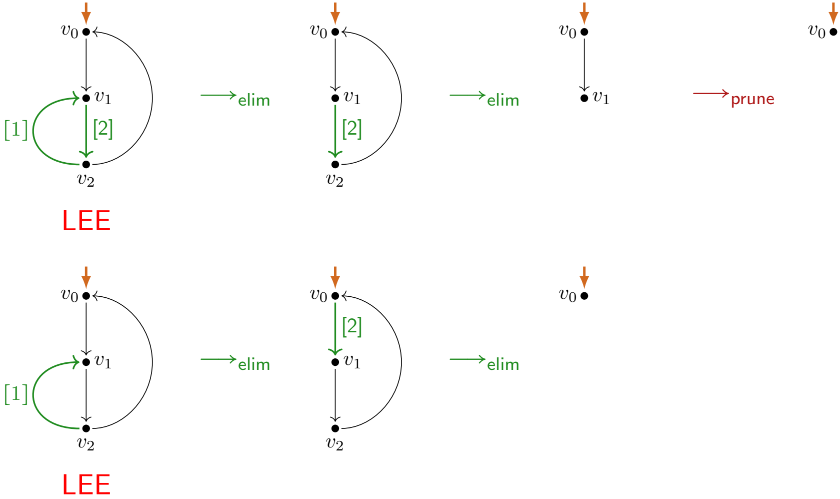
LEE holds / Recording loop elimination



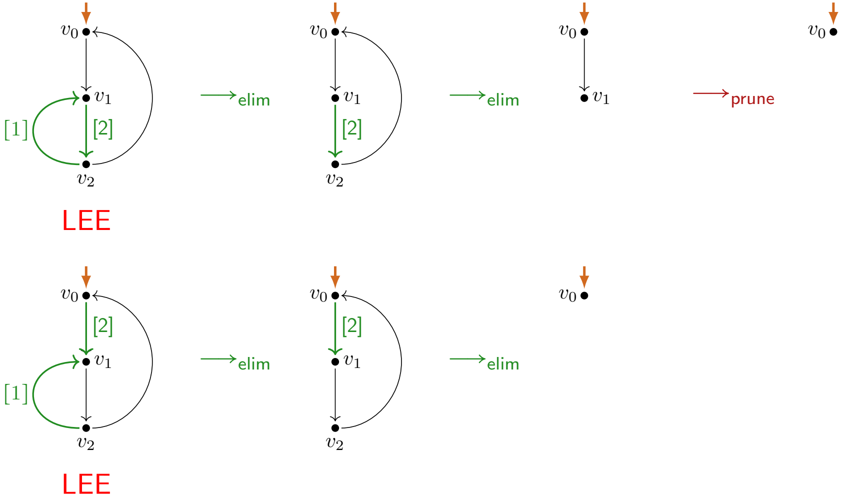
LEE holds / Recording loop elimination


v_0
v_1
[1]
[2]
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
[2]
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0
v_0
v_1
[1]
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0

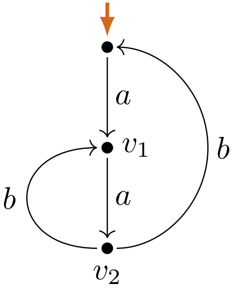
LEE holds / Recording loop elimination


v_0
v_1
[1]
[2]
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
v_1
[2]
v_2
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0
v_0
v_1
[1]
v_2
LEE
$\longrightarrow_{\text{elim}}$
v_0
[2]
v_1
v_2
$\longrightarrow_{\text{elim}}$
v_0

LEE holds / Recording loop elimination

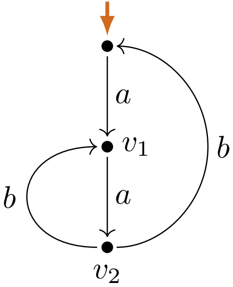

v_0
v_1
v_2
[1]
[2]
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
[2]
$\longrightarrow_{\text{elim}}$
v_0
v_1
$\longrightarrow_{\text{prune}}$
v_0
LEE
v_0
v_1
v_2
[2]
[1]
$\longrightarrow_{\text{elim}}$
v_0
v_1
v_2
[2]
$\longrightarrow_{\text{elim}}$
v_0
LEE

LEE-witness



LEE-witness

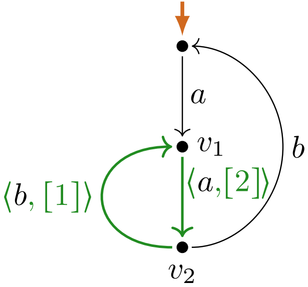
loop–branch labeling: marking transitions $\xrightarrow{a}$ as:



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

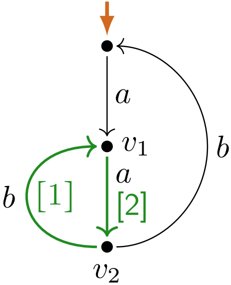
- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$,



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

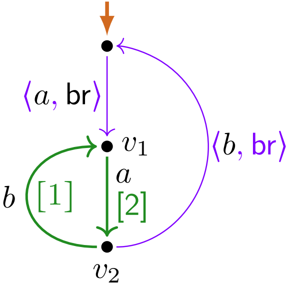
- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

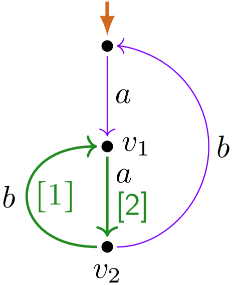
- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$,



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

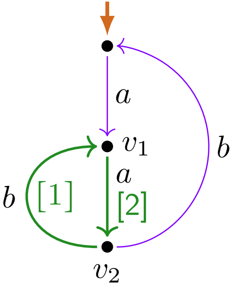
- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.



LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.



Definition

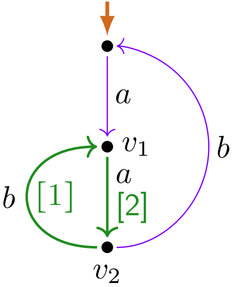
A loop–branch labeling is a LEE-witness, if:

L1.

L2.

L3.

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

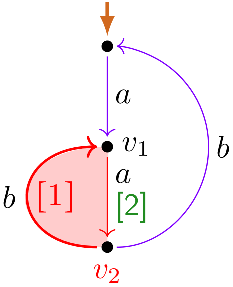
L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

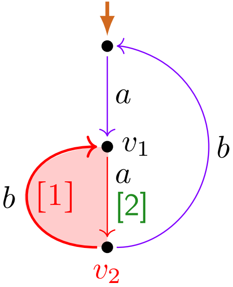
L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br},[>1]})$
is loop subchart

Definition

A loop–branch labeling is a LEE-witness, if:

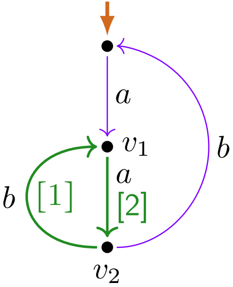
L1.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness



loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

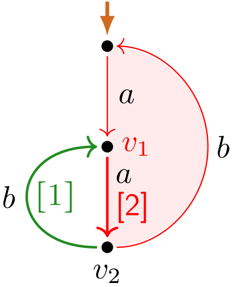
L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.



$\mathcal{L}(v_1, \rightarrow_{[2]}, \rightarrow_{\text{br},[>2]})$

Definition

A loop–branch labeling is a LEE-witness, if:

L1.

L2.

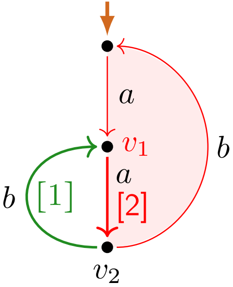
L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\text{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\text{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.



$\mathcal{L}(v_1, \to_{[2]}, \to_{\mathsf{br},[>2]})$
is loop subchart

Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \to_{[n]} \Rightarrow \mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br},[>n]}) \text{ is a loop subchart} \right)$.

L2.

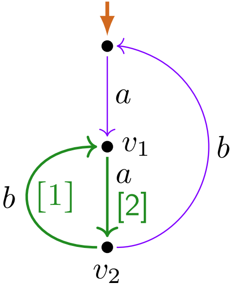
L3.

$\mathcal{L}(v, \to_{[n]}, \to_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\to_{[n]}$ from v
followed by branch steps $\to_{\mathsf{br}}$
or entry steps $\to_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\mathsf{br}\rangle}$, written $\xrightarrow{a}_{\mathsf{br}}$ or $\xrightarrow{a}$.



Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(\begin{array}{r} v \rightarrow_{[n]} \;\Rightarrow\; \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]}) \\ \text{is a loop subchart} \end{array} \right)$.

L2.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
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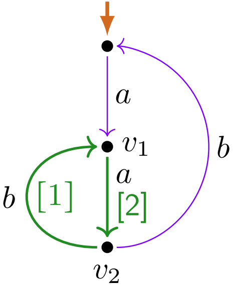

Definition

A loop–branch labeling is a LEE-witness, if:

L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \rightarrow_{[n]} \Rightarrow \begin{array}{c}\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]}) \\ \text{is a loop subchart}\end{array} \right)$.

L2. No infinite $\rightarrow_{\mathsf{br}}$ path from start vertex.

L3.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br},[>n]})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
followed by branch steps $\rightarrow_{\mathsf{br}}$
or entry steps $\rightarrow_{[m]}$ with $m > n$,
until v is reached again

LEE-witness

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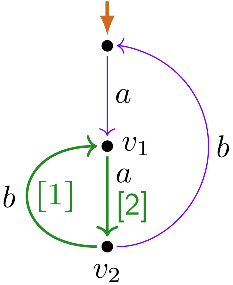

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L3. Overlapping/touching loop subcharts gen. from different vertices have different entry-step levels.

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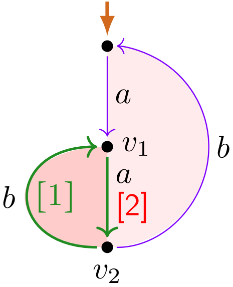

$\mathcal{L}(v_2, \to_{[1]}, \to_{\text{br},[>1]})$

$\mathcal{L}(v_1, \to_{[2]}, \to_{\text{br},[>2]})$

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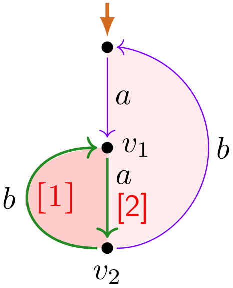

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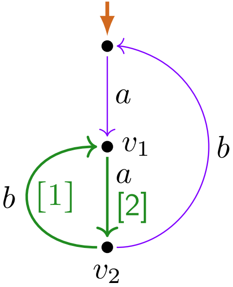

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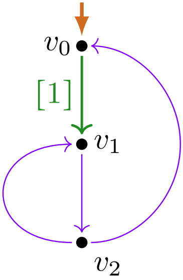
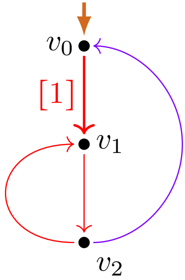


LEE-witness ?



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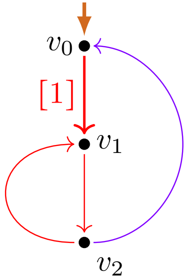
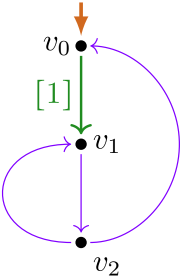


no!

(L1.) violated:

$\mathcal{L}(v_0, \rightarrow_{[1]}, \rightarrow_{\text{br},[>1]})$
not a loop chart

LEE-witness ?



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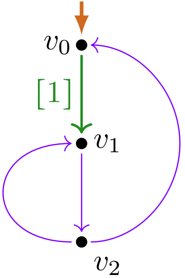

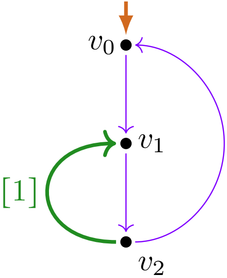

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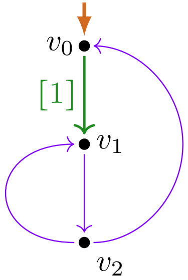

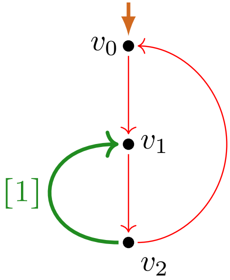

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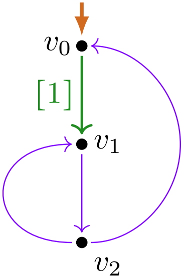
LEE-witness ?



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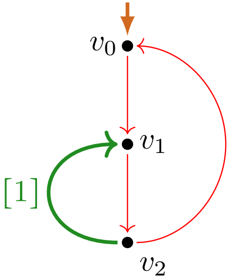
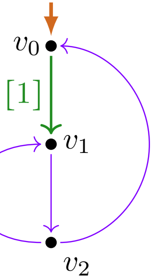


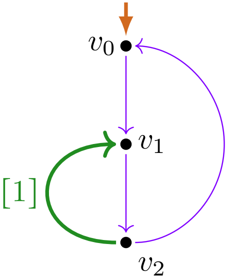
no!

(L2.) violated:

infinite $\to_{\mathsf{br}}$ path
from start vertex

LEE-witness ?





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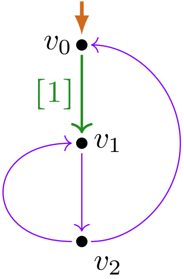

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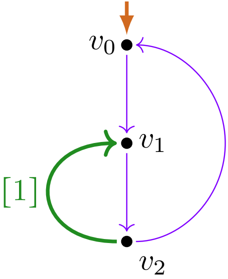

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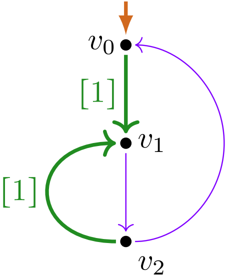

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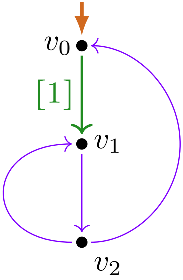

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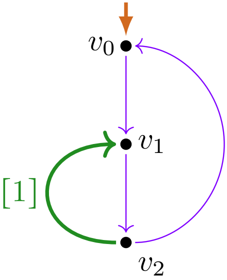

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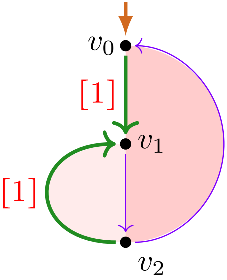
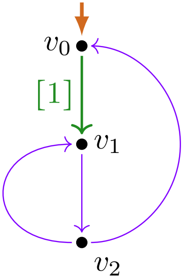


LEE-witness ?



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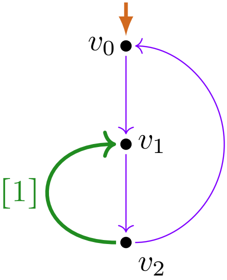

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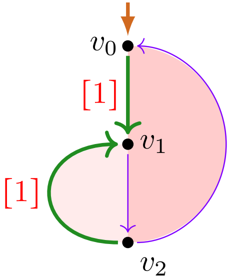
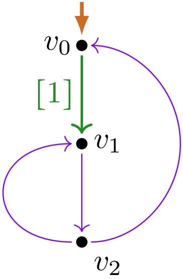


no!

(L3.) violated:

overlapping loop charts have same level

LEE-witness ?

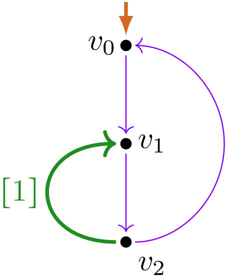


no!

(L1.) violated:

$\mathcal{L}(v_0, \to_{[1]}, \to_{\mathsf{br},[>1]})$

not a loop chart



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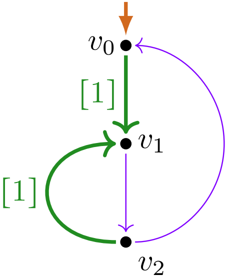
from start vertex



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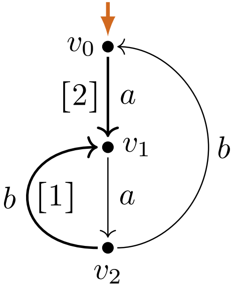

LEE-witness

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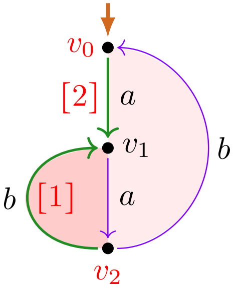

$\mathcal{L}(v_2, \to_{[1]}, \to_{\mathsf{br},[>1]})$
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LEE-witness

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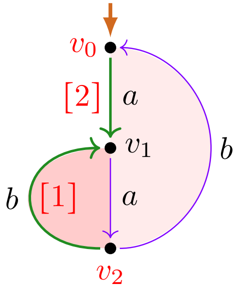
Layered LEE-witness



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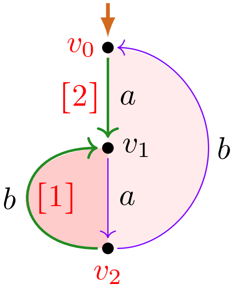

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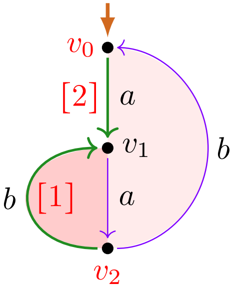
Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br}})$

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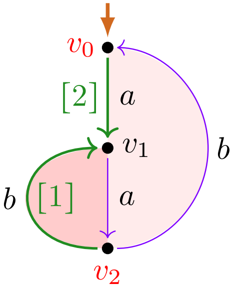
followed by branch steps $\rightarrow_{\mathsf{br}}$

until v is reached again

Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\mathsf{br}})$
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l-L1. $\forall n \in \mathbb{N} \forall v \in V \left(v \rightarrow_{[n]} \Rightarrow \mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br}}) \text{ is a loop subchart} \right).$

l-L2. No infinite $\rightarrow_{\mathsf{br}}$ path from start vertex.

l-L3. A loop subchart generated by a vertex contained in another generated loop subchart has lower level.

$\mathcal{L}(v, \rightarrow_{[n]}, \rightarrow_{\mathsf{br}})$:= subchart induced
by entry steps $\rightarrow_{[n]}$ from v
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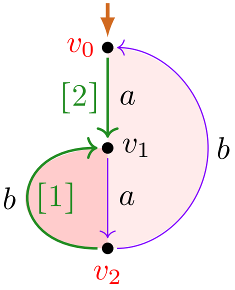
Layered LEE-witness



$\mathcal{L}(v_2, \rightarrow_{[1]}, \rightarrow_{\text{br}})$

$\mathcal{L}(v_0, \rightarrow_{[2]}, \rightarrow_{\text{br}})$

layered LEE-witness

loop–branch labeling: marking transitions $\xrightarrow{a}$ as:

- entry steps $\xrightarrow{\langle a,[n]\rangle}$ for $n \in \mathbb{N}$, written $\xrightarrow{a}_{[n]}$,
- branch steps $\xrightarrow{\langle a,\text{br}\rangle}$, written $\xrightarrow{a}_{\text{br}}$ or $\xrightarrow{a}$.

Definition

A loop–branch labeling is a layered LEE-witness, if:

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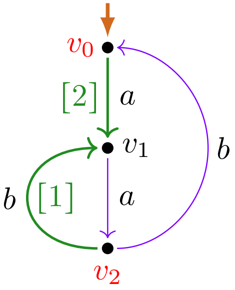
Layered LEE-witness



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LEE versus LEE-witness

Theorem

For every process graph G :

$$\text{LEE}(G) \iff G \textit{ has a } \text{LEE}\textit{-witness.}$$

LEE versus LEE-witness

Theorem

For every process graph G *:*

$$\mathsf{LEE}(G) \iff G \text{ has a } \mathsf{LEE}\text{-witness.}$$

Proof.

$\Rightarrow$: record loop elimination

LEE versus LEE-witness

Theorem

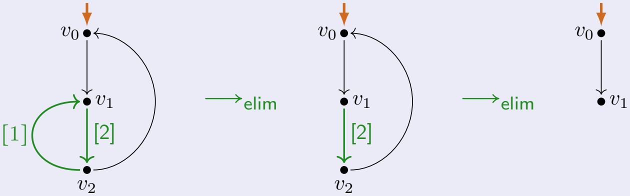
For every process graph G :

$$\mathsf{LEE}(G) \iff G \textit{ has a } \mathsf{LEE}\textit{-witness.}$$

Proof.

⇒ : record loop elimination

⇐ : carry out loop-elimination as indicated in the LEE-witness, in *inside–out* direction, e.g.:



LEE and (layered) LEE-witness

Lemma

Every layered LEE-witness is a LEE-witness.

Lemma

Every LEE-witness $\widehat{G}$ of a process graph G can be transformed by an effective procedure (cut-elimination-like) into a layered LEE-witness $\widehat{G}'$ of G.

LEE and (layered) LEE-witness

Lemma

Every layered LEE*-witness is a* LEE*-witness.*

Lemma

Every LEE*-witness* $\widehat{G}$ *of a process graph* G *can be transformed by an effective procedure (cut-elimination-like) into a layered* LEE*-witness* $\widehat{G}'$ *of* G*.*

Lemma

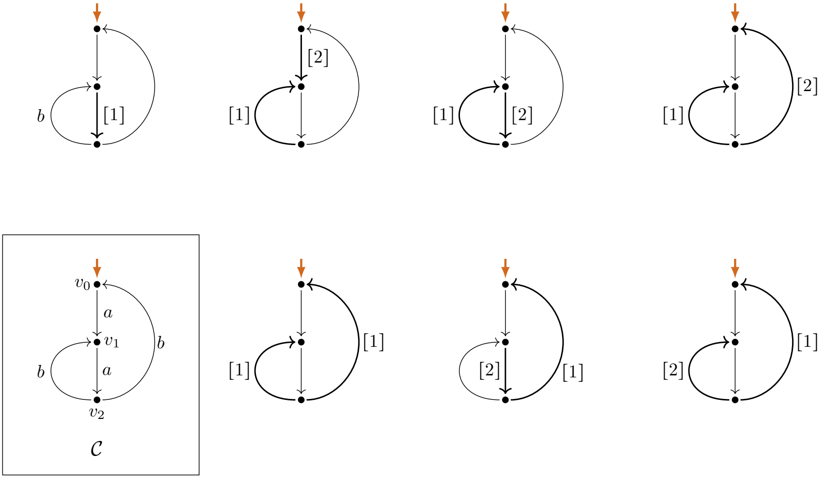
For every process graph G *the following are equivalent:*

(i) LEE(G).

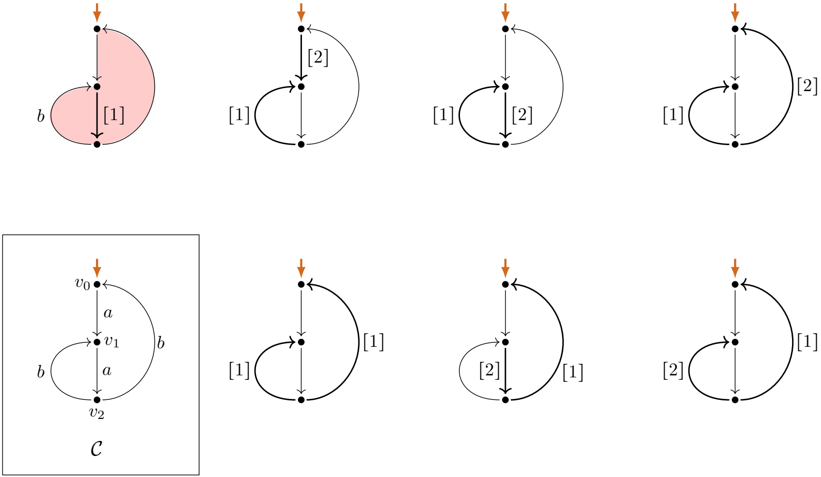
(ii) G *has a* LEE*-witness.*

(iii) G *has a layered* LEE*-witness.*

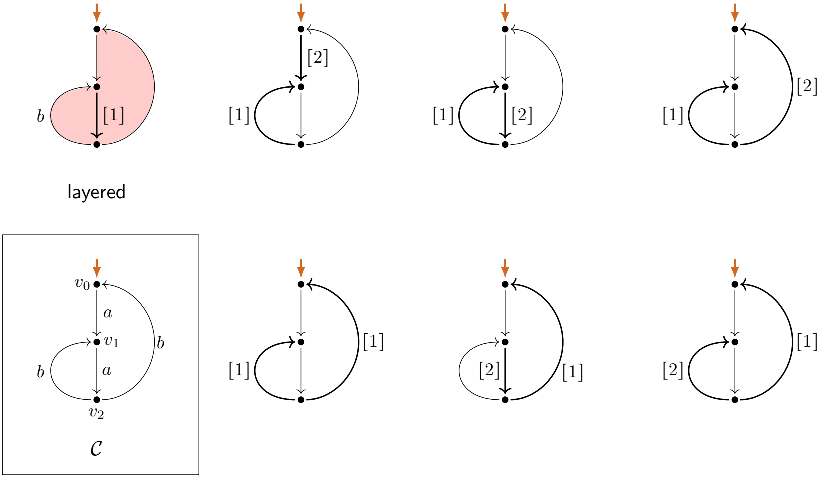
7 LEE-witnesses



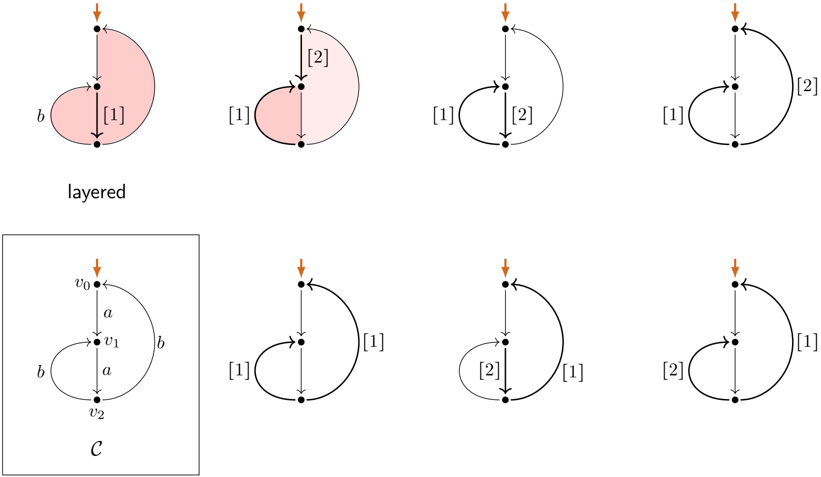
7 LEE-witnesses



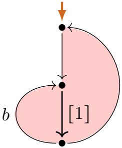
7 LEE-witnesses



7 LEE-witnesses



7 LEE-witnesses



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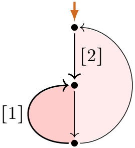

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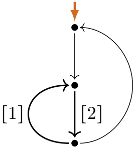

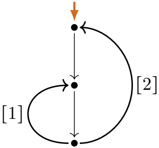

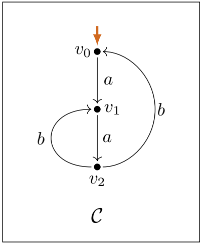

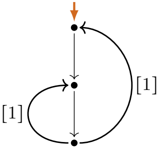

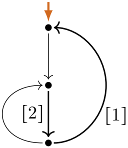

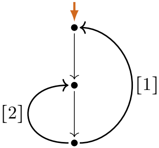

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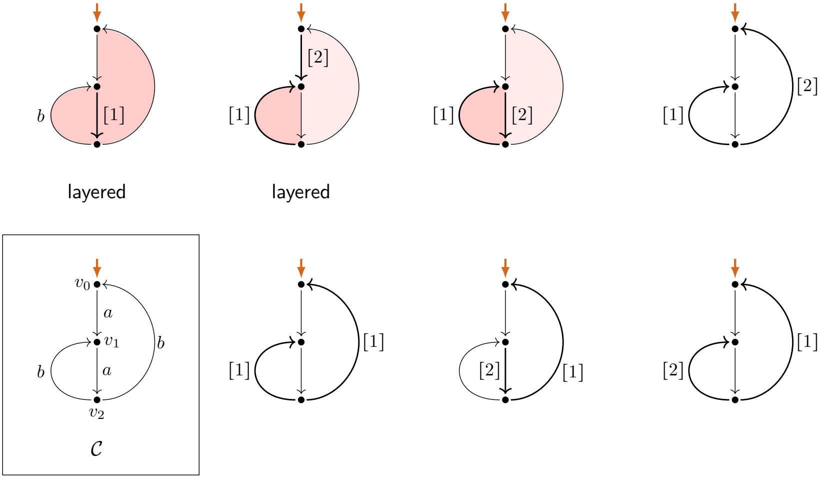

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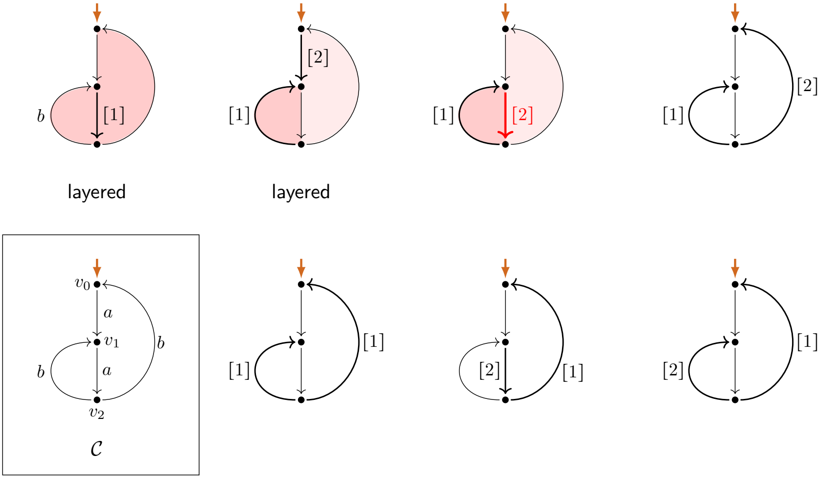

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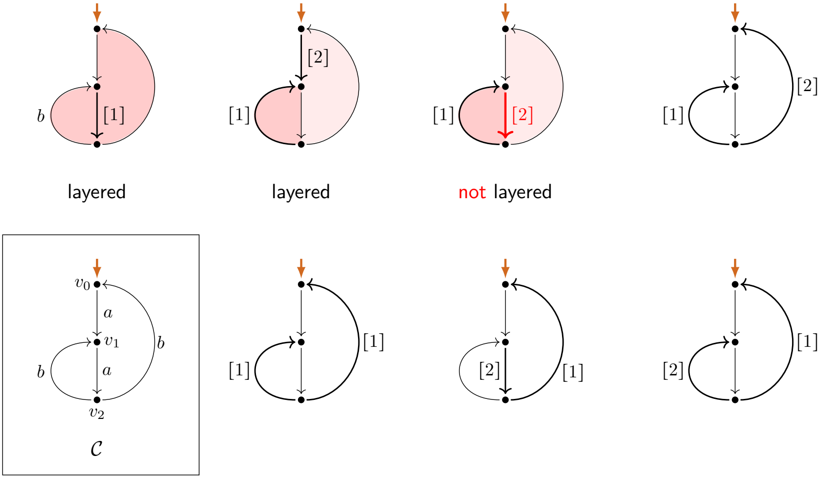
7 LEE-witnesses



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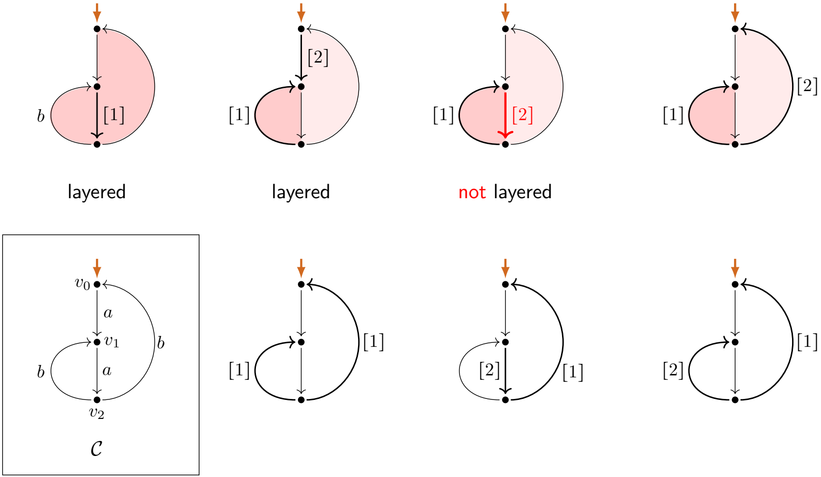
7 LEE-witnesses



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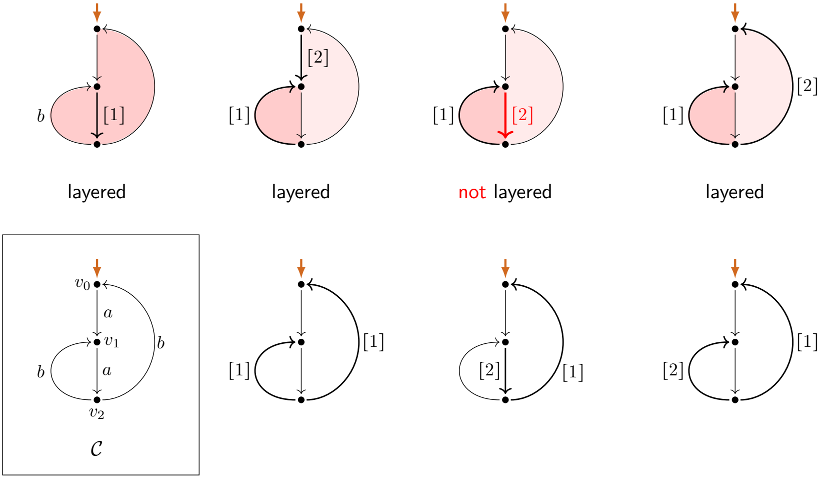
7 LEE-witnesses



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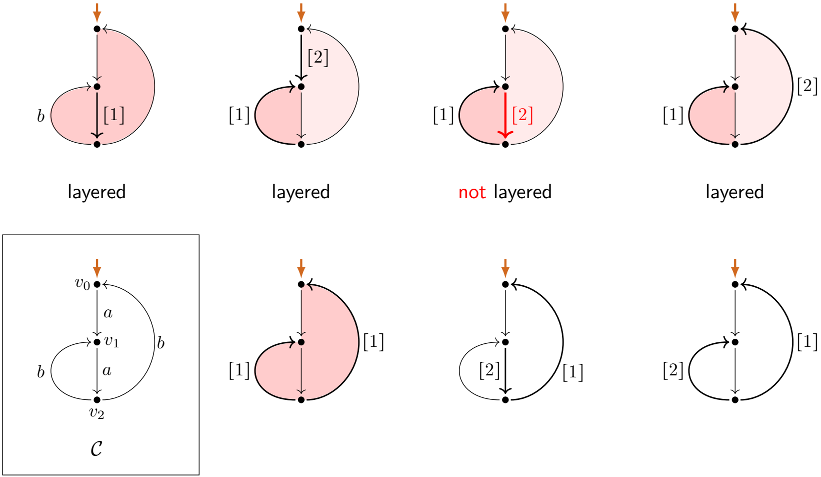
7 LEE-witnesses



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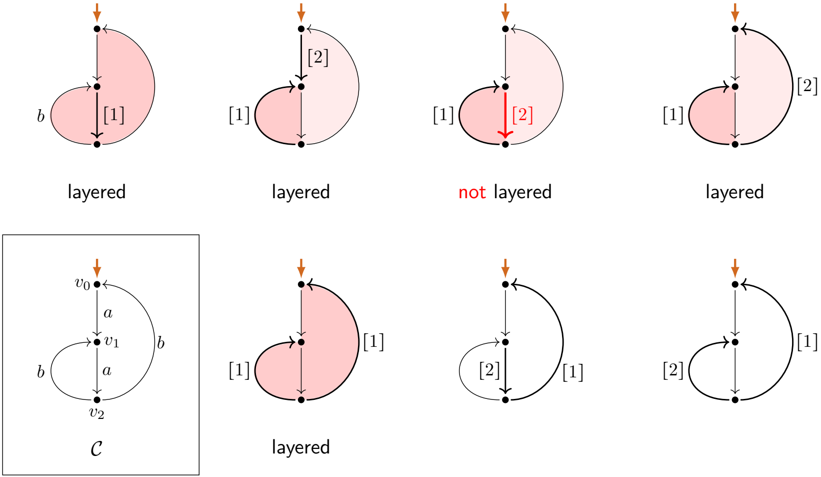
7 LEE-witnesses



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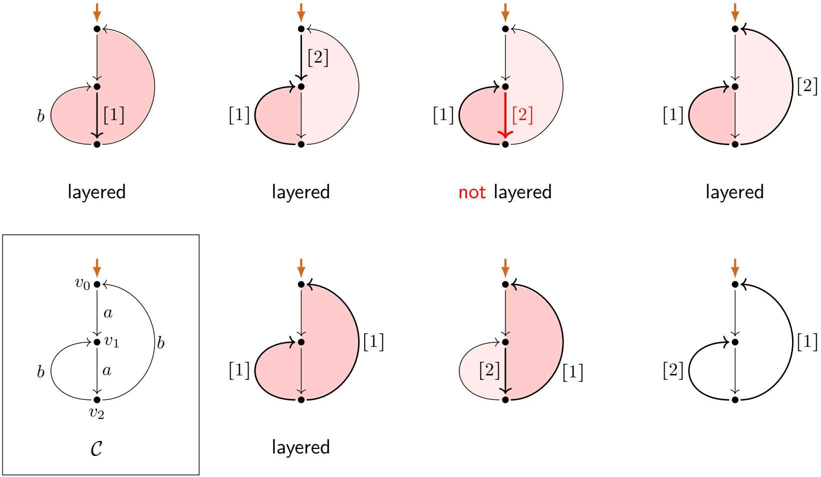
7 LEE-witnesses



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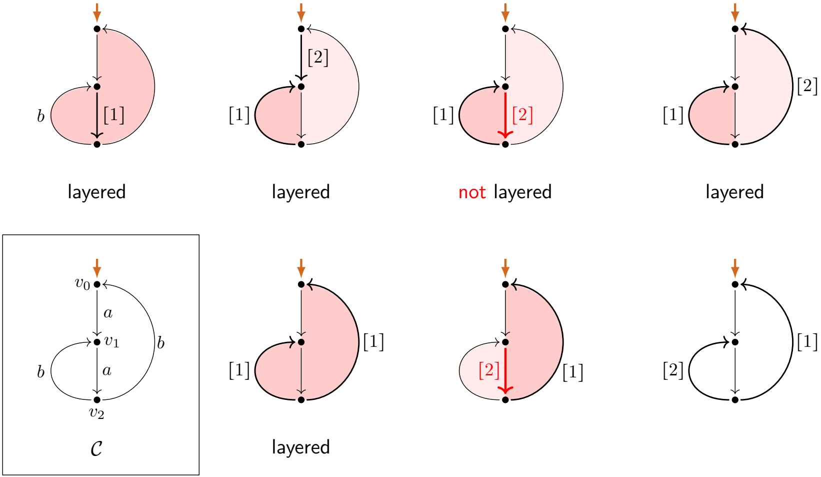

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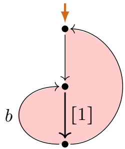

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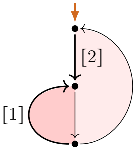

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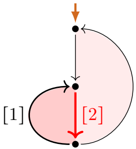

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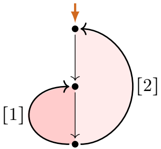

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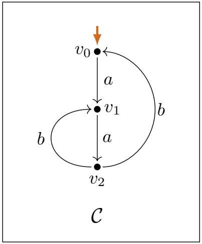

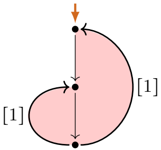

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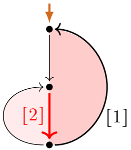

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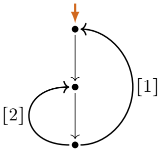

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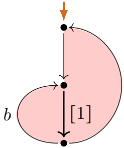

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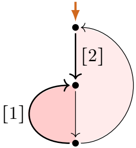

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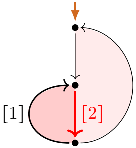

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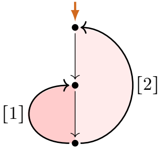

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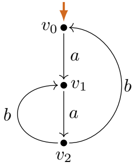

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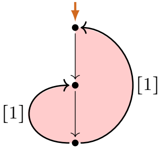

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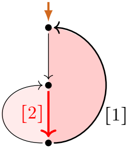

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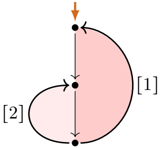

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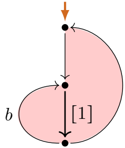

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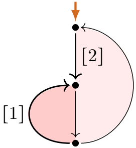

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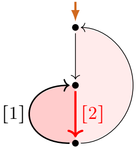

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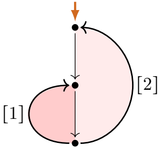

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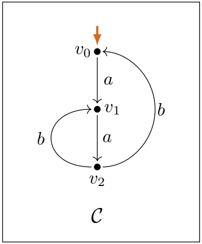

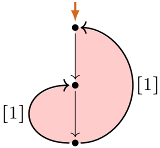

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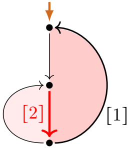

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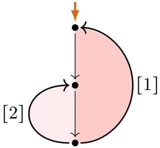

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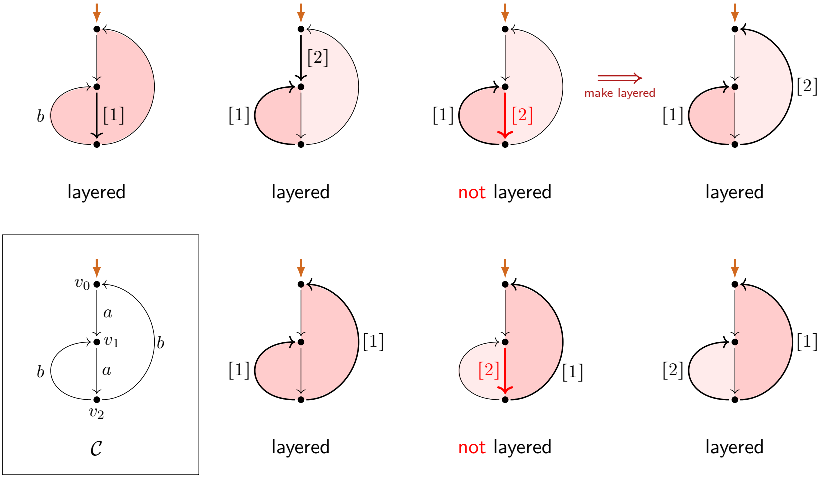
7 LEE-witnesses



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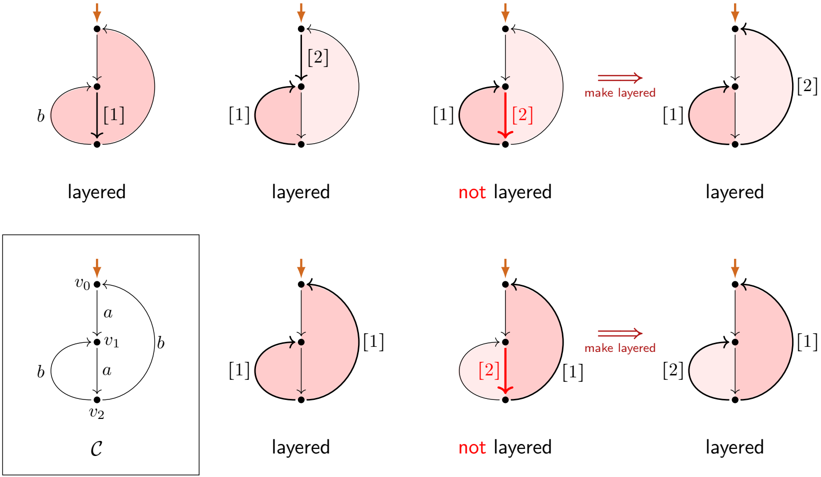
7 LEE-witnesses



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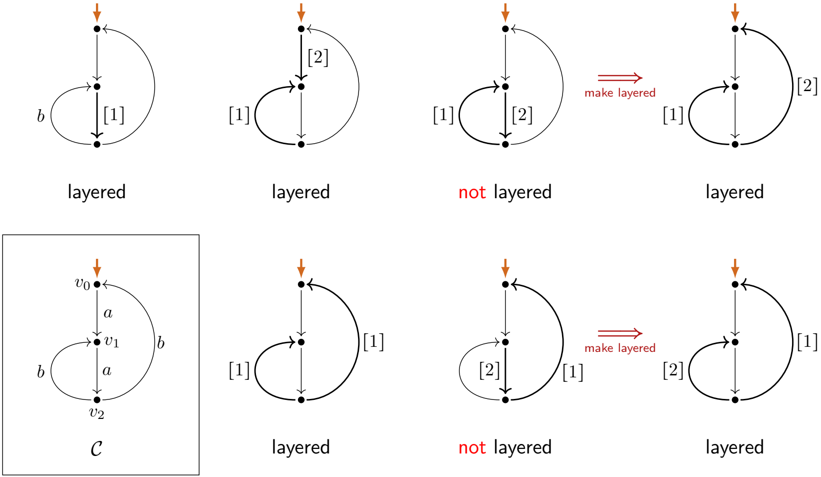
7 LEE-witnesses



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LEE under bisimulation?

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.

LEE under bisimulation

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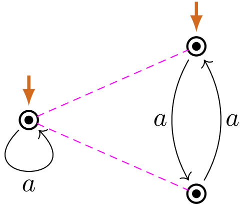
- LEE is not invariant under bisimulation.

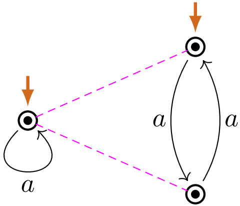


LEE ¬LEE

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.



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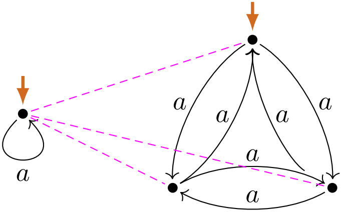
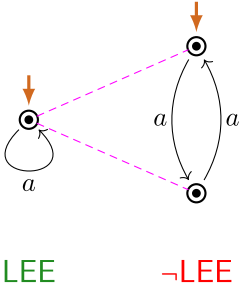


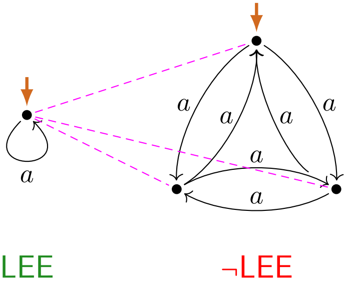
LEE ¬LEE

LEE under bisimulation

Observation

- LEE is not invariant under bisimulation.
- LEE is not preserved by converse functional bisimulation.





LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\to}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

LEE under functional bisimulation

Lemma

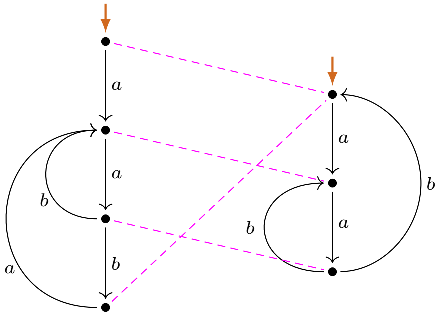
(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \trianglerighteq G_2 \implies \mathsf{LEE}(G_2) \; .$$

Proof (Idea).

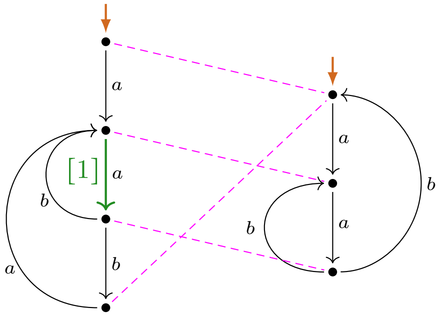
Use loop elimination in G_1 to carry out loop elimination in G_2.

Collapsing LEE-witnesses



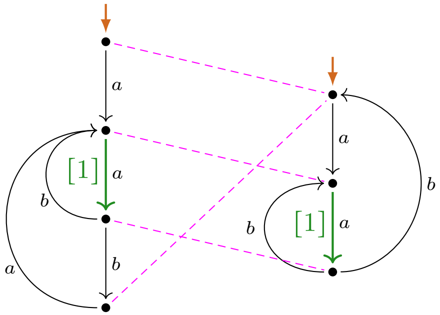
$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses



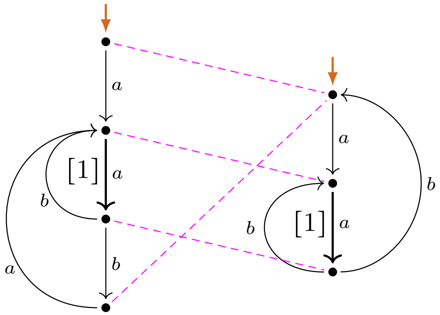
$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses



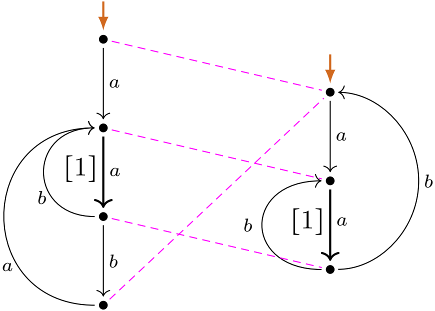
$$[\![a(a(b+ba))^* 0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses

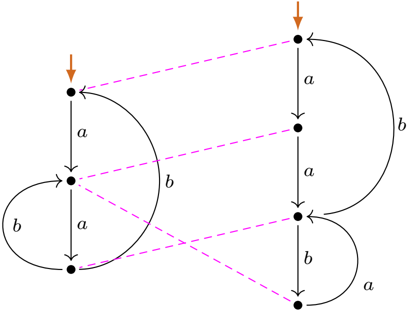


$$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$$

Collapsing LEE-witnesses

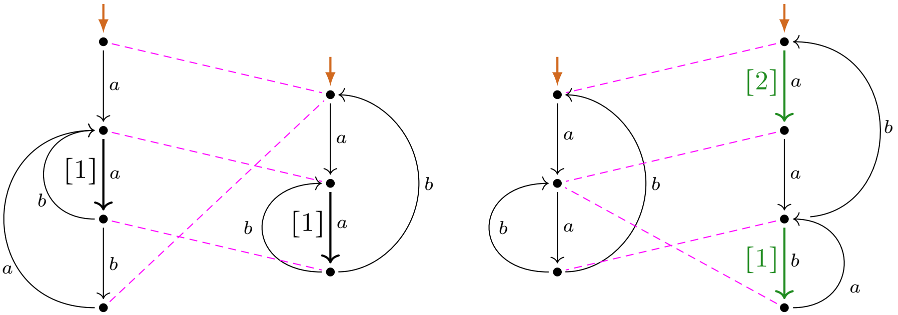


$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$



$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

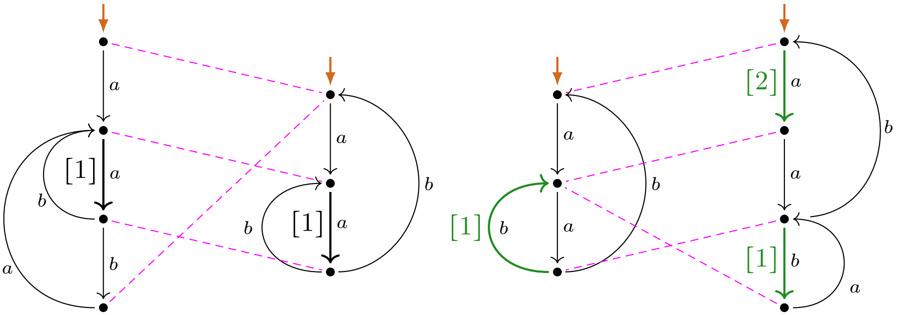
Collapsing LEE-witnesses



$[\![a(a(b+ba))^{*}0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^{*}b)^{*}0]\!]_{\boldsymbol{P}}$

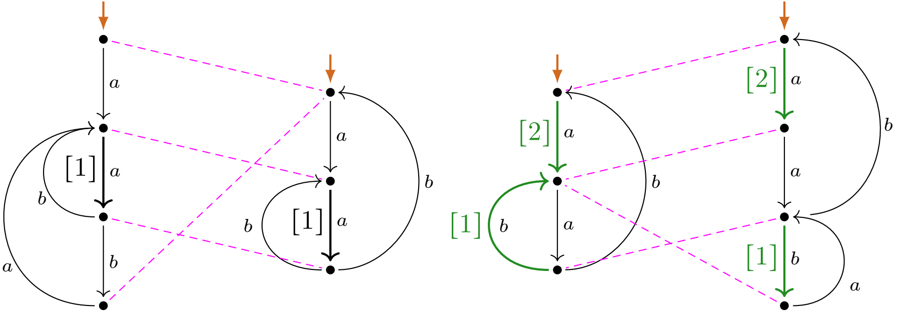
Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

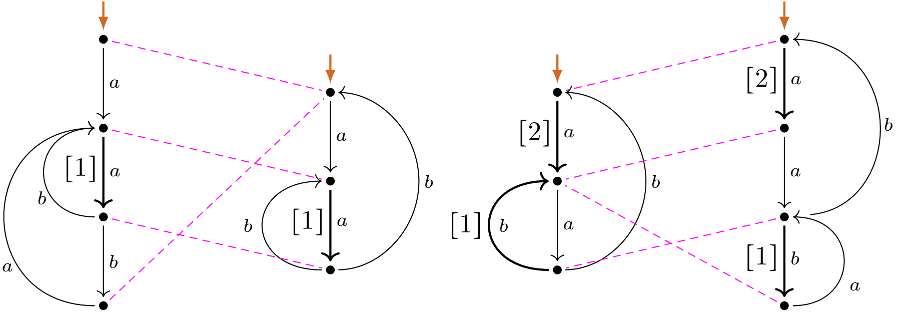
Collapsing LEE-witnesses



$\llbracket a(a(b+ba))^{*}0 \rrbracket_{\boldsymbol{P}}$

$\llbracket (aa(ba)^{*}b)^{*}0 \rrbracket_{\boldsymbol{P}}$

Collapsing LEE-witnesses



$[\![a(a(b+ba))^*0]\!]_{\boldsymbol{P}}$

$[\![(aa(ba)^*b)^*0]\!]_{\boldsymbol{P}}$

LEE under functional bisimulation

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

LEE under functional bisimulation / bisimulation collapse

Lemma

(i) LEE *is preserved by functional bisimulations:*

$$\mathsf{LEE}(G_1) \wedge G_1 \mathrel{\underline{\rightarrow}} G_2 \implies \mathsf{LEE}(G_2)\ .$$

(ii) LEE *is preserved from a process graph to its bisimulation collapse:*

$$\mathsf{LEE}(G) \wedge C \textit{ is bisimulation collapse of } G \implies \mathsf{LEE}(C)\ .$$

Idea of Proof for (i)

Use loop elimination in G_1 to carry out loop elimination in G_2.

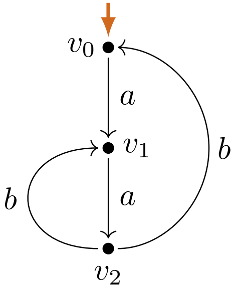
Readback

Lemma

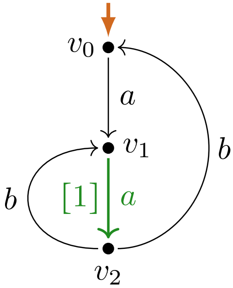
Process graphs with LEE are $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathsf{Reg}(A) \left(G \leftrightarrow [\![e]\!]_{\boldsymbol{P}} \right) .$$

Readback from layered LEE-witness (example)

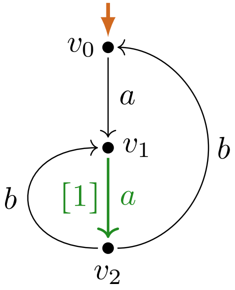


Readback from layered LEE-witness (example)



layered
LEE-witness

Readback from layered LEE-witness (example)

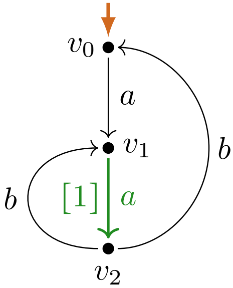


layered
LEE-witness

$$
\begin{aligned}
s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot s(v_1) \\
&=_{\mathsf{Mil}^-} a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
&=_{\mathsf{Mil}^-} \big(a \cdot (b + b \cdot a)\big)^* \cdot 0 \\
s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\
&=_{\mathsf{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \\
&=_{\mathsf{Mil}^-} b + b \cdot a \\
s(v_1, v_1) &= 1 \\
s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\
&= 0^* \cdot a \cdot 1 \\
&=_{\mathsf{Mil}^-} a
\end{aligned}
$$

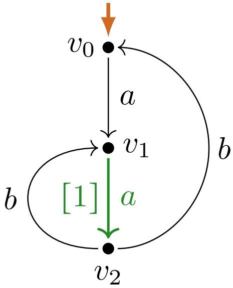
Readback from layered LEE-witness (example)

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$



layered
LEE-witness

Readback from layered LEE-witness (example)

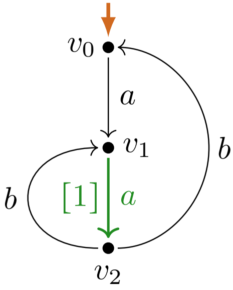


layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

Readback from layered LEE-witness (example)



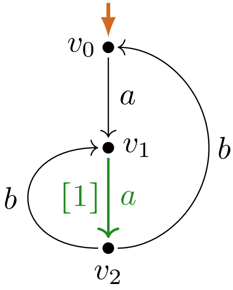
layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$s(v_2, v_1) = \quad 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

Readback from layered LEE-witness (example)



layered
LEE-witness

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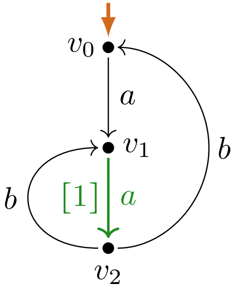
$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

$$s(v_1, v_1) = 1$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = \quad 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \quad \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

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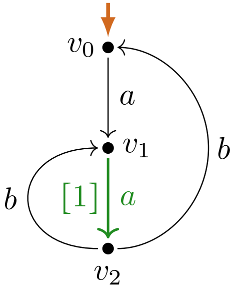
$$s(v_0, v_1) = \quad 0^* \cdot a \cdot s(v_1, v_1)$$

Readback from layered LEE-witness (example)



layered
LEE-witness

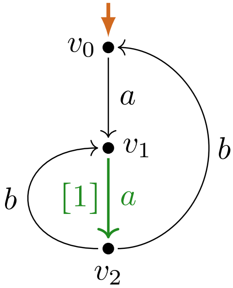
$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$s(v_1) = \big(a \cdot s(v_2, v_1)\big)^* \cdot 0$$

$$s(v_2, v_1) = 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big)$$

$$\begin{aligned} s(v_1, v_1) &= 1 \\ s(v_0, v_1) &= 0^* \cdot a \cdot s(v_1, v_1) \\ &= 0^* \cdot a \cdot 1 \end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

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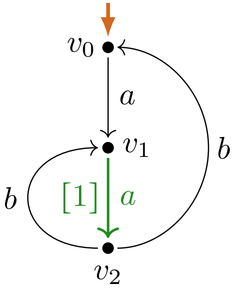
Readback from layered LEE-witness (example)



layered
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$$\begin{aligned} s(v_2, v_1) &= 0^* \cdot \big(b \cdot s(v_1, v_1) + b \cdot s(v_0, v_1)\big) \\ &=_{\text{Mil}^-} 0^* \cdot \big(b \cdot 1 + b \cdot a\big) \end{aligned}$$

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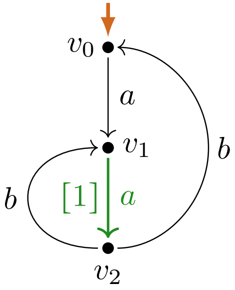

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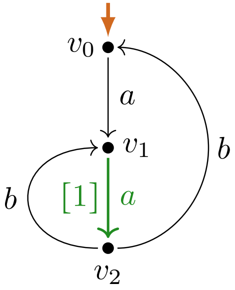
\end{aligned}$$

Readback from layered LEE-witness (example)



layered
LEE-witness

$$s(v_0) = 0^* \cdot a \cdot s(v_1)$$

$$\begin{aligned}
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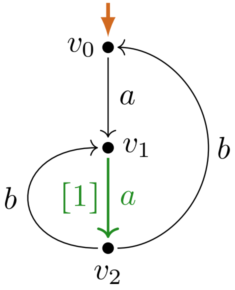
Readback from layered LEE-witness (example)



layered
LEE-witness

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s(v_0) &= 0^* \cdot a \cdot s(v_1) \\
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$$\begin{aligned}
s(v_1) &= \big(a \cdot s(v_2, v_1)\big)^* \cdot 0 \\
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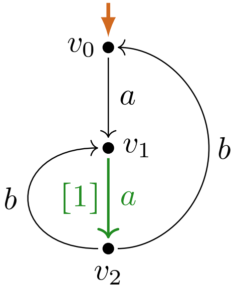
Readback from layered LEE-witness (example)



layered
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\end{aligned}
$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{\boldsymbol{P}}$-expressible:

$$\mathsf{LEE}(G) \implies \exists e \in \mathrm{Reg}(A) \left(G \leftrightarroweq [\![e]\!]_{\boldsymbol{P}} \right) .$$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{\mathbf{1r}}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathsf{Reg}^{\cancel{\mathbf{1r}}\backslash\star}(A)\,\big(\, G \,\underline{\leftrightarrow}\, [\![e]\!]_{\boldsymbol{P}} \,\big)\,.$$

1-return-less regular expressions

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Definition (Corradini, De Nicola, Labella (here intuitive version))

A regular expression e is 1-return-less(-under-$\star$) ($e \in \mathrm{Reg}^{\cancel{\mathbf{1r}}\backslash\star}(A)$) if:

1-return-less regular expressions

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- for no iteration subexpression f^* of e does $[\![f]\!]_{\boldsymbol{P}}$ proceed to a process p such that:
  - p has the option to immediately terminate, and
  - p has the option to do a proper step, and terminate later.

Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$

1-return-less regular expressions

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Non-/Examples of 1-return-less regular expressions

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1-return-less regular expressions

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Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$ ✗
- $(a \cdot (0^* + b))^*$

1-return-less regular expressions

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- $(a \cdot (1 + b))^*$ ✗
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1-return-less regular expressions

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Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$ ✗
- $(a \cdot (0^* + b))^*$ ✗
- $a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0$

1-return-less regular expressions

Lemma

Process graphs with LEE are $[\![\cdot]\!]_{P}^{\mathbf{1r}\backslash\star}$-expressible:

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- for no iteration subexpression f^* of e does $[\![f]\!]_{P}$ proceed to a process p such that:
  - p has the option to immediately terminate, and
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Non-/Examples of 1-return-less regular expressions

- $(a \cdot (1 + b))^*$ ✗
- $(a \cdot (0^* + b))^*$ ✗
- $a \cdot \big(a \cdot (b + b \cdot a)\big)^* \cdot 0$ ✓

1-return-less regular expressions

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Process graphs with LEE are $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{\mathbf{1r}}\backslash\star}$-expressible:

$$\mathsf{LEE}(G) \implies \exists\, e \in \mathrm{Reg}^{\cancel{\mathbf{1r}}\backslash\star}(A)\Big(\, G \leftrightarrows [\![e]\!]_{\boldsymbol{P}} \,\Big)\;.$$

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Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$ modulo $\underline{\leftrightarrow}$

Theorem

For every process graph G with bisimulation collapse C the following are equivalent:

(i) *G is $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{\mathbf{1r}}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$.*

(ii) $\mathsf{LEE}(C)$.

(iii) *C has a* LEE*-witness.*

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Milners characterization question restricted:

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Milners characterization question restricted, and adapted:

$\mathrm{Q1}''$. Which structural property of collapsed finite process graphs characterizes $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility modulo $\underline{\leftrightarrow}$?

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Answering Milners characterization question restricted, and adapted:

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- The loop-existence and elimination property LEE.

Characterization of expressibility$^{\cancel{\mathbf{1r}}\backslash\star}$ modulo $\underline{\leftrightarrow}$

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For every process graph G with bisimulation collapse C the following are equivalent:

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- The loop-existence and elimination property LEE.

Also yields: efficient decision method of $\llbracket\cdot\rrbracket_{P}^{\cancel{\mathbf{1r}}\backslash\star}$-expressibility modulo $\underline{\leftrightarrow}$.

Structure constrained finite process graphs

graphs with LEE / a (layered) LEE-witness

Benefits of the class of process graphs with LEE:

- is closed under $\Rightarrow$
- forth-/back-correspondence with 1-return-less regular expressions

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$\subsetneqq$ graphs whose collapse satisfies LEE

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Structure constrained finite process graphs

loop–exit palm trees $\subsetneqq$ $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{\mathbf{1r}}\backslash\star}$-expressible graphs

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Application to Milner's questions yields partial results:

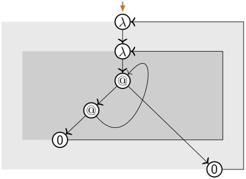
Q1: characterization/efficient decision of $[\![\cdot]\!]_{P}^{\cancel{1r}\backslash\star}$-expressibility modulo $\underline{\leftrightarrow}$

Q2: alternative compl. proof of Mil on 1-return-less expressions (C/DN/L)

Maximal sharing of functional programs

(joint work with Jan Rochel)

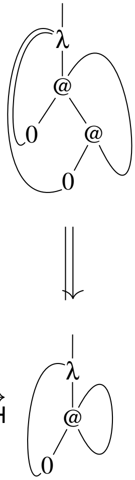
$$\begin{tikzcd}
L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"]
\end{tikzcd}$$



maximal sharing: example (fix)

$$\begin{tikzcd}
L \arrow[rr, mapsto, bend left, "\text{interpret}"] \arrow[d, mapsto, "\text{unfold}"'] & & G \arrow[dd, Rrightarrow, "\text{collapse}"] \\
M & & \\
L_0 \arrow[u, mapsto, "\text{unfold}"] \arrow[rr, mapsto, bend left, "\text{interpret}"] & & G_0 \arrow[ll, mapsto, bend left, "\text{readback}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda f.\,\text{let } r = f\,(f\ r) \text{ in } r \arrow[rr, mapsto, "[\![\cdot]\!]_{\mathcal{T}}"] \arrow[dr, mapsto, "[\![\cdot]\!]_{\lambda^\infty}"'] & & {} \\
& \lambda f.\,f\,(f\,(\ldots)) & \\
\lambda f.\,\text{let } r = f\ r \text{ in } r \arrow[ur, mapsto, "[\![\cdot]\!]_{\lambda^\infty}"] \arrow[rr, mapsto, shift left, "[\![\cdot]\!]_{\mathcal{T}}"] & & {} \arrow[ll, mapsto, shift left, "\text{readback}"]
\end{tikzcd}$$



maximal sharing: the method

$$\begin{tikzcd} L \arrow[r, mapsto, "{[\![\cdot]\!]_{\mathcal{H}}}"] & \mathcal{G} \end{tikzcd}$$

1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L as:
   a. higher-order term graph
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$

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$$\begin{tikzcd}
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 & & G_0
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2. bisimulation collapse $\downarrow\!\!\downarrow$
   of f-o term graph G into G_0

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L \arrow[r, mapsto, "{[\![\cdot]\!]_{\mathcal{H}}}"] \arrow[rr, mapsto, bend left, "{[\![\cdot]\!]_{\mathcal{T}}}"] & \mathcal{G} \arrow[r, mapsto] \arrow[d, Rightarrow] & G \arrow[d, Rightarrow] \\
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interpretation

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\end{tikzcd}$$

running example

instead of:

$\lambda f.\,\mathsf{let}\ r = f\,(f\,r)\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r$

we use:

$\color{red}{\lambda x.}\,\lambda f.\,\mathsf{let}\ r = f\,(f\,r\,\color{red}{x})\,\color{red}{x}\ \mathsf{in}\ r \quad \longmapsto_{\text{max-sharing}} \quad \color{red}{\lambda x.}\,\lambda f.\,\mathsf{let}\ r = f\,r\,\color{red}{x}\ \mathsf{in}\ r$

$L \quad \longmapsto_{\text{max-sharing}} \quad L_0$

graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \mathsf{let}\ r = f\, r\, x\ \mathsf{in}\ r$

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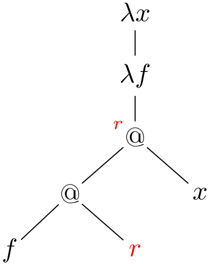

syntax tree

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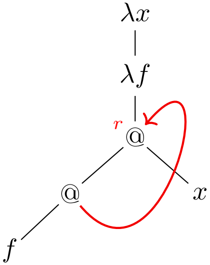

syntax tree (+ recursive backlink)

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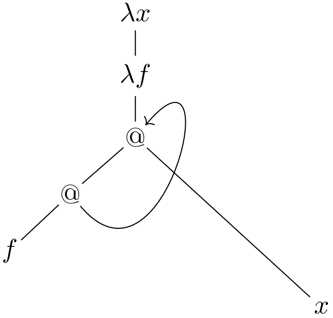

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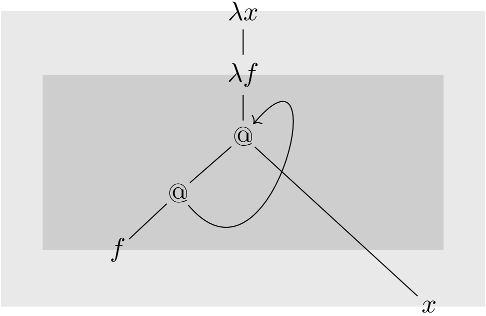

syntax tree (+ recursive backlink, + scopes)

graph interpretation (example 1)

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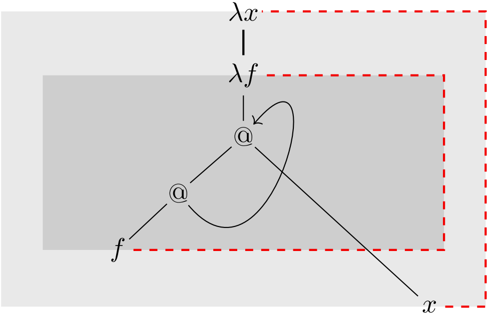

syntax tree (+ recursive backlink, + scopes, + binding links)

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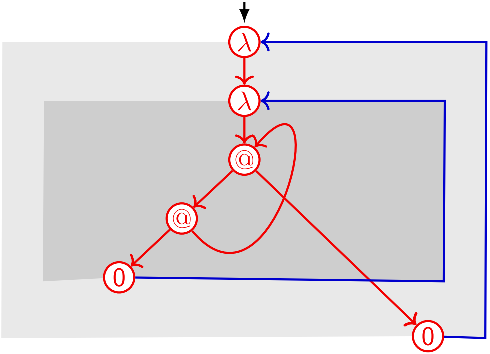
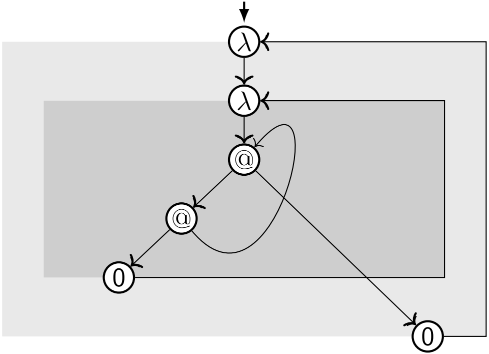


first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

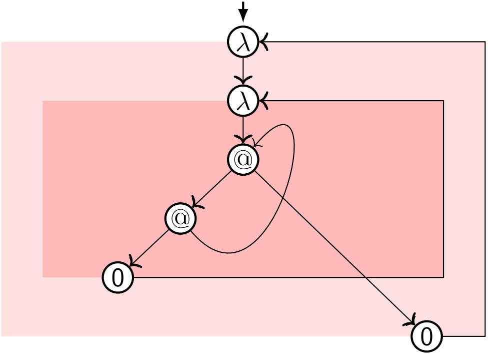
$L_0 = \lambda x.\,\lambda f.\,\text{let } r = f\,r\,x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

$L_0 = \lambda x.\,\lambda f.\,\text{let } r = f\;r\;x \text{ in } r$



first-order term graph (+ scope sets)

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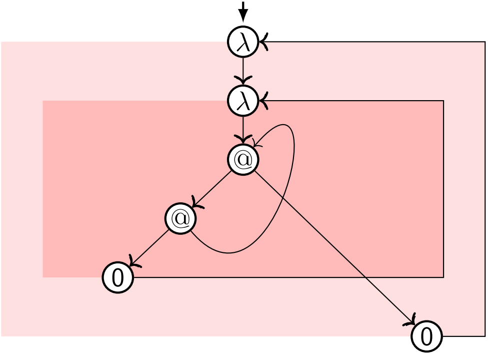
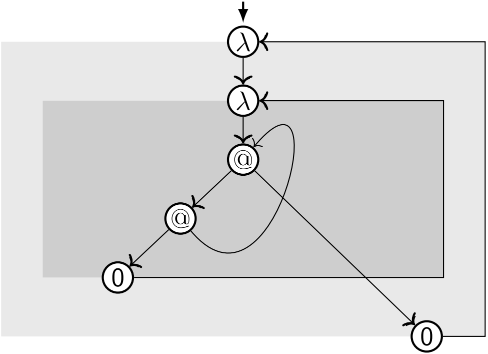
$L_0 = \lambda x.\, \lambda f.\, \mathsf{let}\ r = f\, r\, x\ \mathsf{in}\ r$



higher-order term graph (with scope sets, Blom [2003])

graph interpretation (example 1)

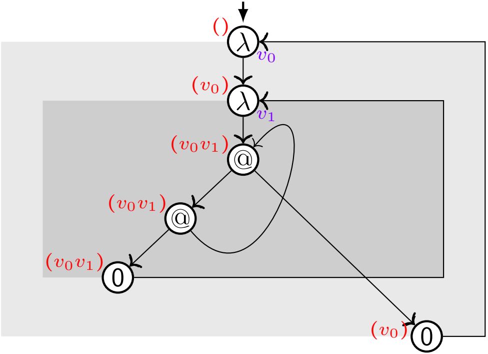
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



higher-order term graph (with scope sets, Blom [2003])

graph interpretation (example 1)

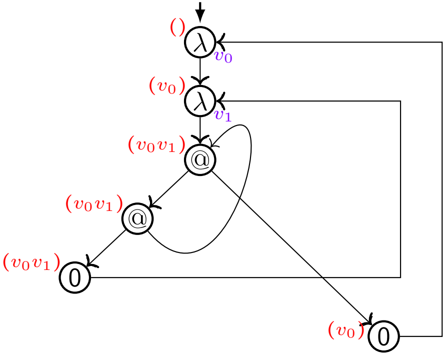
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



higher-order term graph (with scope sets, + abstraction-prefix function)

graph interpretation (example 1)

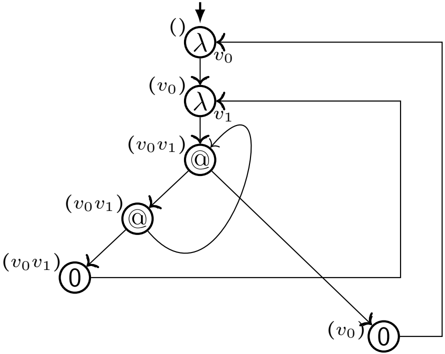
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



higher-order term graph (with abstraction-prefix function)

graph interpretation (example 1)

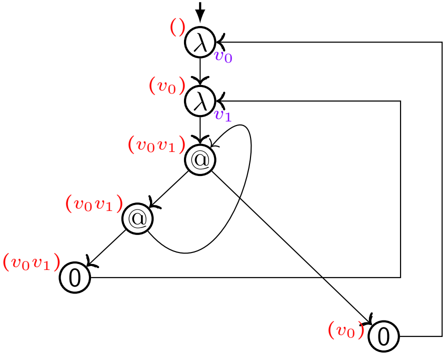
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



λ-higher-order-term-graph $[\![L_0]\!]_{\mathcal{H}}$

graph interpretation (example 1)

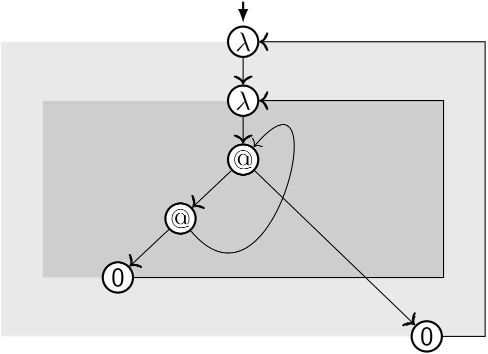
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



first-order term graph (+ abstraction-prefix function)

graph interpretation (example 1)

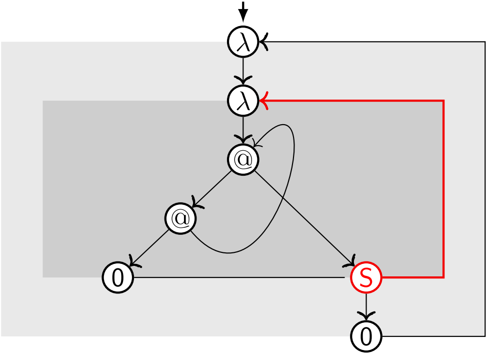
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 1)

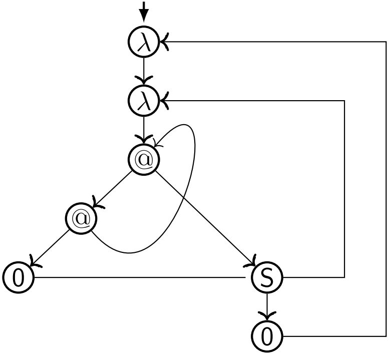
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with scope vertices with backlinks (+ scope sets)

graph interpretation (example 1)

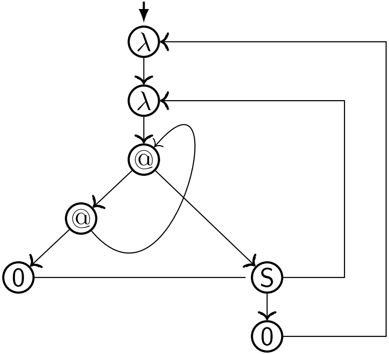
$L_0 = \lambda x.\,\lambda f.\,\text{let } r = f\,r\,x \text{ in } r$



first-order term graph with scope vertices with backlinks

graph interpretation (example 1)

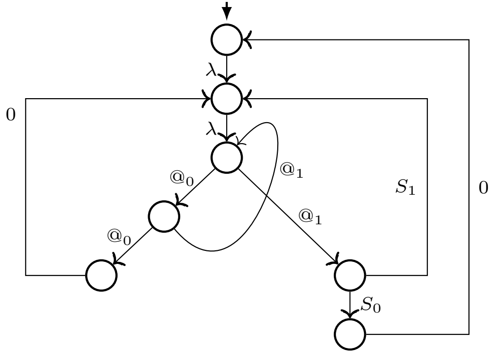
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$

graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



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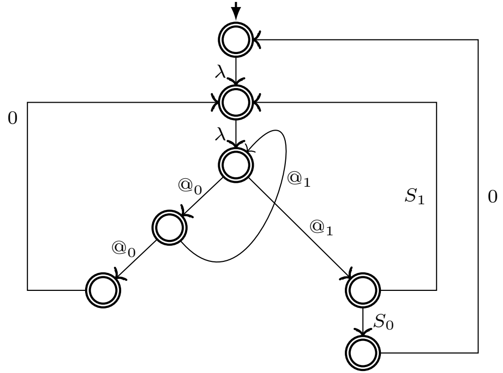
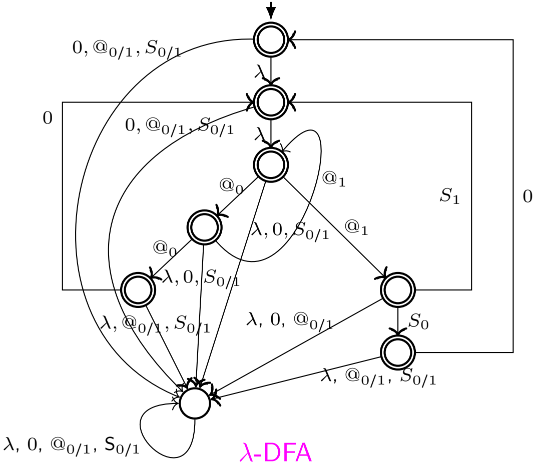


λ-NFA

graph interpretation (example 1)

$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



λ-DFA

graph interpretation (example 2)

$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$

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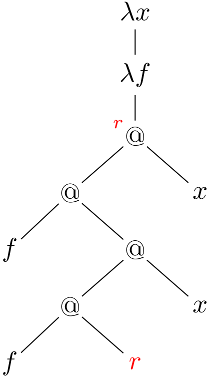
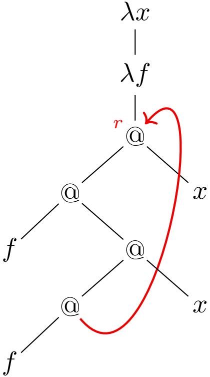
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree

graph interpretation (example 2)

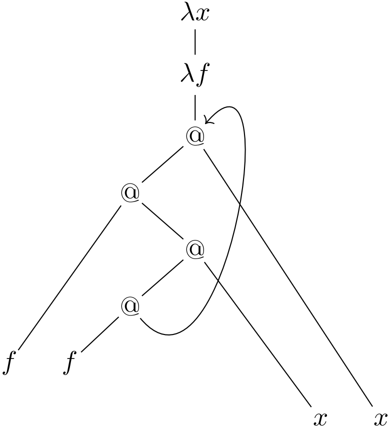
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink)

graph interpretation (example 2)

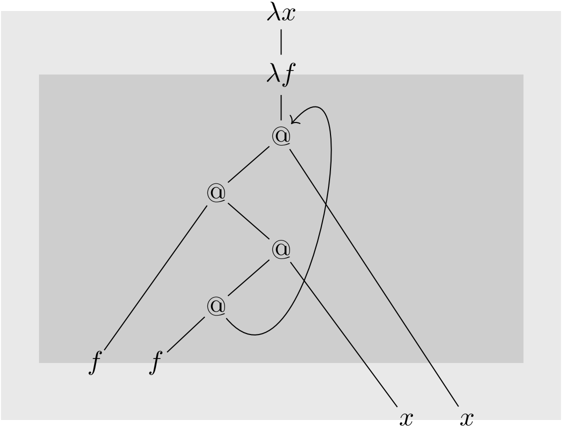
$L = \lambda x.\,\lambda f.\,\text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink)

graph interpretation (example 2)

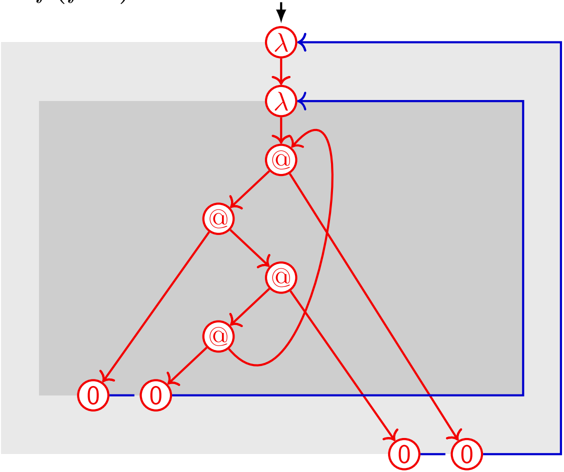
$L = \lambda x.\,\lambda f.\,\text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink, + scopes)

graph interpretation (example 2)

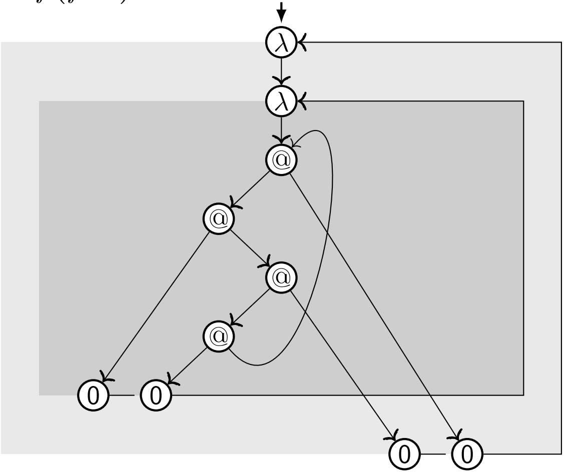
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



first-order term graph with binding backlinks (+ scope sets)

graph interpretation (example 2)

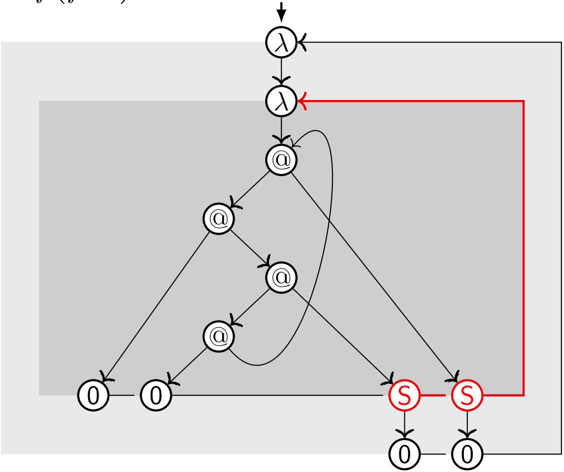
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



λ-higher-order-term-graph $[\![L]\!]_{\mathcal{H}}$

graph interpretation (example 2)

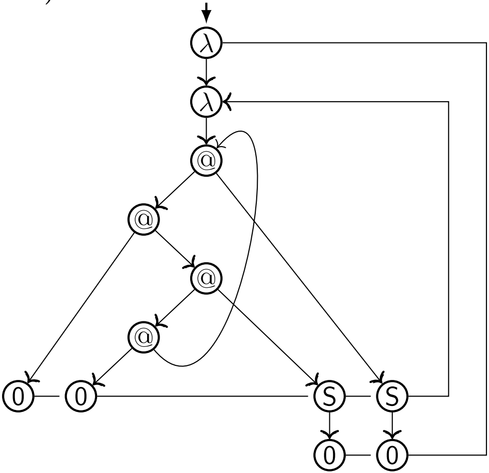
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



first-order term graph with scope vertices with backlinks (+ scope sets)

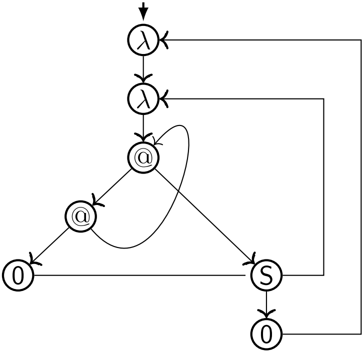
graph interpretation (example 2)

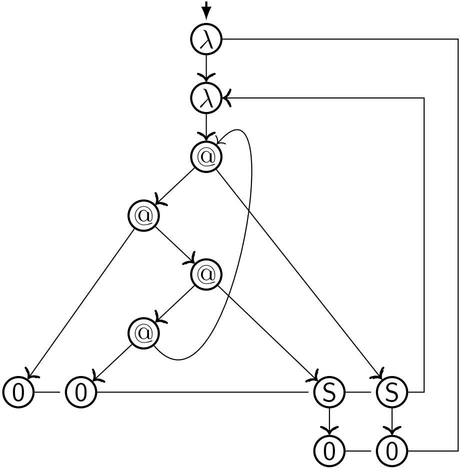
$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



λ-term-graph $[\![L]\!]_{\mathcal{T}}$

graph interpretation (examples 1 and 2)





$[\![L_0]\!]_{\mathcal{T}}$

$[\![L]\!]_{\mathcal{T}}$

interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto$ λ-term-graph $[\![L]\!]_{\mathcal{T}}$

- defined by induction on structure of L
- similar analysis as fully-lazy lambda-lifting
- yields eager-scope λ-term-graphs: $\sim$ minimal scopes

Theorem

For $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 it holds: Equality of infinite unfolding coincides with bisimilarity of λ-term-graph interpretations:

$$[\![L_1]\!]_{\lambda^\infty} = [\![L_2]\!]_{\lambda^\infty} \iff [\![L_1]\!]_{\mathcal{T}} \leftrightarrows [\![L_2]\!]_{\mathcal{T}}$$

interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

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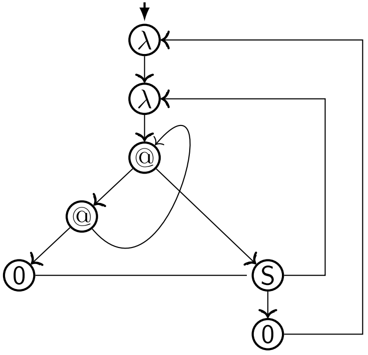
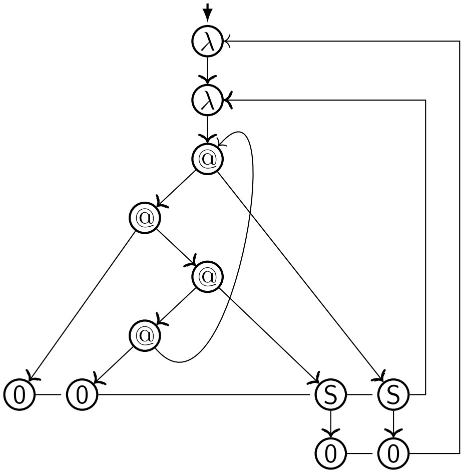
collapse

$$\begin{tikzcd}
L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"]
\end{tikzcd}$$

bisimulation check between λ-term-graphs

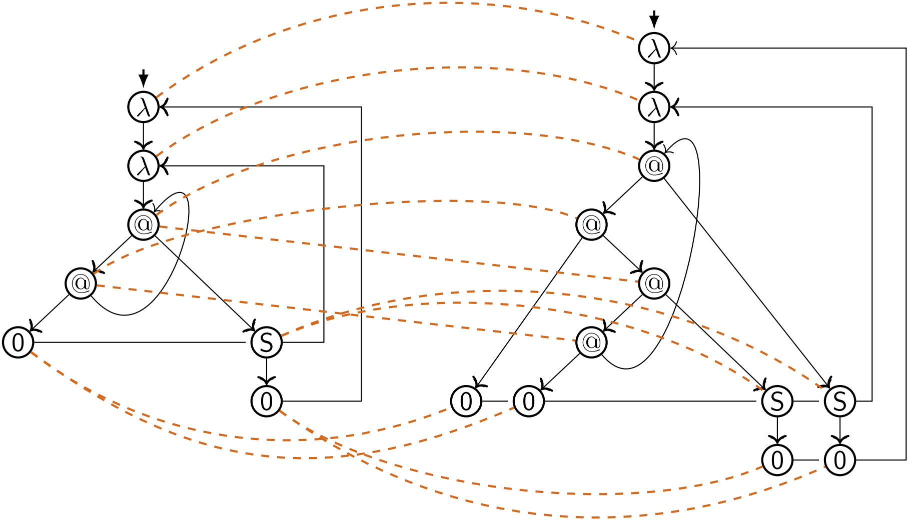


$[\![L_0]\!]_{\mathcal{T}}$



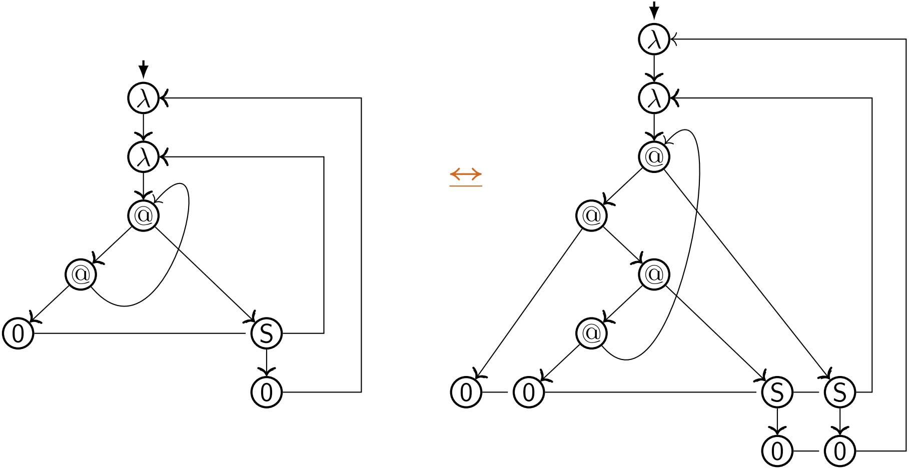
$[\![L]\!]_{\mathcal{T}}$

bisimulation between λ-term-graphs



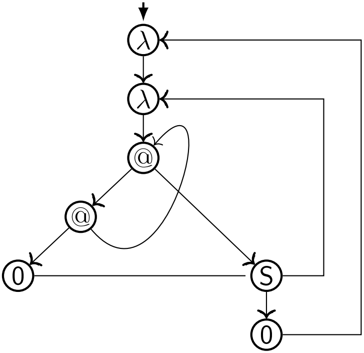
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

bisimilarity between λ-term-graphs

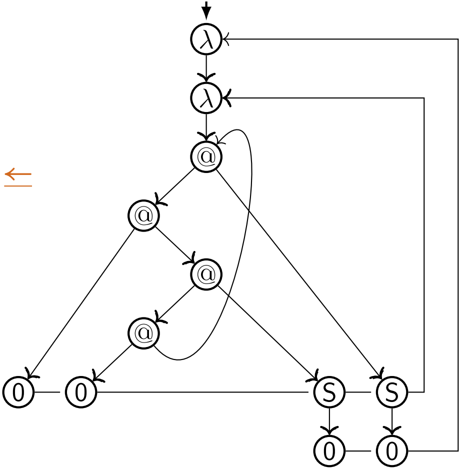


$$[\![L_0]\!]_{\mathcal{T}} \quad \underline{\leftrightarrow} \quad [\![L]\!]_{\mathcal{T}}$$

functional bisimilarity and bisimulation collapse



$\underline{\leftarrow}$



$[\![L_0]\!]_{\mathcal{T}}$ $\underline{\leftarrow}$ $[\![L]\!]_{\mathcal{T}}$

bisimulation collapse: property

Theorem

The class of eager-scope λ-term-graphs is closed under functional bisimilarity $\mathrel{\underline{\to}}$.

$\Longrightarrow$ For a $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L the bisimulation collapse of $[\![L]\!]_{\mathcal{T}}$ is again an eager-scope λ-term-graph.

readback

$$\begin{tikzcd}
L \arrow[r, mapsto, bend left, "\text{interpret}"] & G \arrow[d, Rightarrow, "\text{collapse}"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\text{readback}"]
\end{tikzcd}$$

readback

defined with property:

$$\begin{tikzcd} L & G \arrow[l, mapsto, bend left, "\text{rb}"] & \text{eager-scope} \end{tikzcd}$$

readback

defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, mapsto, "\llbracket\cdot\rrbracket_{\mathcal{T}}"] & G \arrow[l, bend left, mapsto, "\mathsf{rb}"] \end{tikzcd}$$

eager-scope

readback

defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, mapsto, "\llbracket \cdot \rrbracket_{\mathcal{T}}"] & G \text{ eager-scope} \arrow[l, bend left, mapsto, "\mathsf{rb}"] \end{tikzcd}$$

Theorem

For all eager-scope λ-term-graphs G:

$$(\llbracket \cdot \rrbracket_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $\llbracket \cdot \rrbracket_{\mathcal{T}}$ modulo isomorphism $\simeq$.

readback

defined with property:

$$\begin{tikzcd} L \arrow[r, bend left, mapsto, "\llbracket\cdot\rrbracket_{\mathcal{T}}"] & G \text{ eager-scope} \arrow[l, bend left, mapsto, "\mathsf{rb}"] \end{tikzcd}$$

Theorem

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$$(\llbracket\cdot\rrbracket_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $\llbracket\cdot\rrbracket_{\mathcal{T}}$ modulo isomorphism $\simeq$.

idea:

1. construct a spanning tree T of G
2. using local rules, in a bottom-up traversal of T synthesize $L = \mathsf{rb}(G)$

maximal sharing: complexity

$$\begin{tikzcd}
L \arrow[r, mapsto, bend left, "{[\![\cdot]\!]_{\mathcal{T}}}"] & G \arrow[d, "|\!\downarrow"] \\
L_0 & G_0 \arrow[l, mapsto, bend left, "\mathrm{rb}"]
\end{tikzcd}$$

1. interpretation

   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L with $|L| = n$

   as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$

   ▶ in time $O(n^2)$, size $|G| \in O(n^2)$.

2. bisimulation collapse $|\!\downarrow$

   of f-o term graph G into G_0

   ▶ in time $O(|G| \log |G|) = O(n^2 \log n)$

3. readback rb

   of f-o term graph G_0

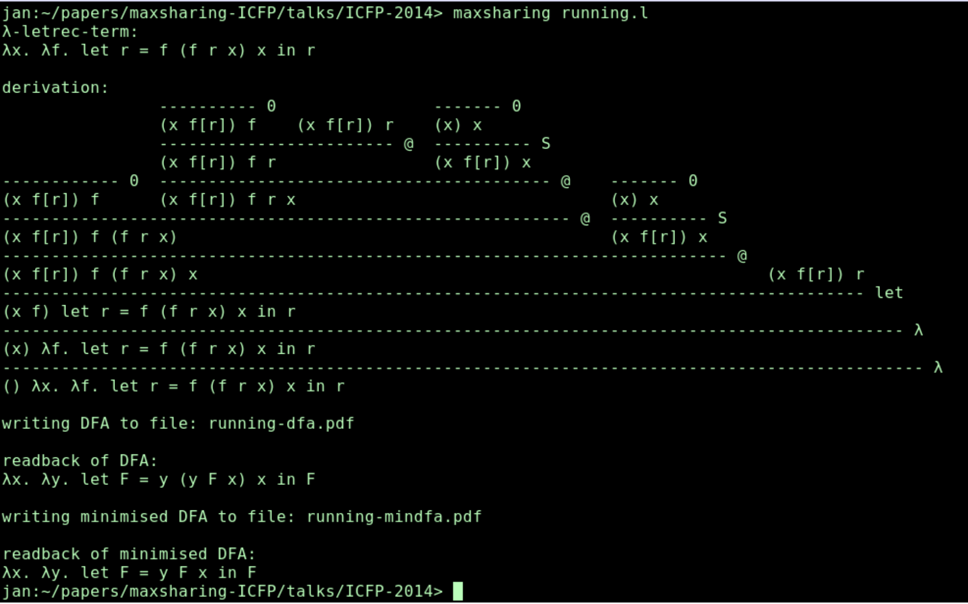
   yielding $\boldsymbol{\lambda}_{\text{letrec}}$-term $L_0 = \mathrm{rb}(G_0)$.

   ▶ in time $O(|G| \log |G|) = O(n^2 \log n)$

Theorem

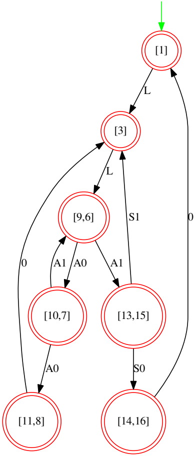
Computing a maximally compact form $L_0 = (\mathrm{rb} \circ |\!\downarrow \circ [\![\cdot]\!]_{\mathcal{T}})(L)$ of L for a $\boldsymbol{\lambda}_{\text{letrec}}$-term L requires time $O(n^2 \log n)$, where $|L| = n$.

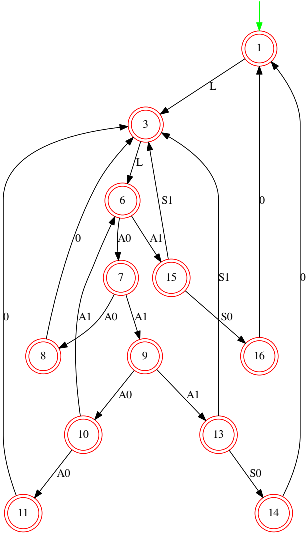
Demo: console output



```
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> maxsharing running.l
λ-letrec-term:
λx. λf. let r = f (f r x) x in r

derivation:
                ---------- 0                ------- 0
                (x f[r]) f    (x f[r]) r    (x) x
                ------------------------ @  ---------- S
                (x f[r]) f r                (x f[r]) x
------------ 0  ---------------------------------------- @    ------- 0
(x f[r]) f      (x f[r]) f r x                                (x) x
---------------------------------------------------------- @  ---------- S
(x f[r]) f (f r x)                                            (x f[r]) x
-------------------------------------------------------------------------- @
(x f[r]) f (f r x) x                                                          (x f[r]) r
---------------------------------------------------------------------------------------- let
(x f) let r = f (f r x) x in r
-------------------------------------------------------------------------------------------- λ
(x) λf. let r = f (f r x) x in r
---------------------------------------------------------------------------------------------- λ
() λx. λf. let r = f (f r x) x in r

writing DFA to file: running-dfa.pdf

readback of DFA:
λx. λy. let F = y (y F x) x in F

writing minimised DFA to file: running-mindfa.pdf

readback of minimised DFA:
λx. λy. let F = y F x in F
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> █
```

Demo: generated λ-NFAs





Resources (maximal sharing)

- tool maxsharing on `hackage.haskell.org`
- papers and reports
  - Maximal Sharing in the Lambda Calculus with Letrec
    - ICFP 2014 paper
    - accompanying report arXiv:1401.1460
  - Term Graph Representations for Cyclic Lambda Terms
    - TERMGRAPH 2013 proceedings
    - extended report arXiv:1308.1034
  - Vincent van Oostrom, CG: Nested Term Graphs
    - TERMGRAPH 2014 post-proceedings in EPTCS 183
- thesis Jan Rochel
  - Unfolding Semantics of the Untyped λ-Calculus with letrec
    - Ph.D. Thesis, Utrecht University, 2016

Comparison results: structure-constrained graphs

Regular expressions under $\underline{\leftrightarrow}_{\boldsymbol{P}}$

Given: graph interpretation $[\![\cdot]\!]_{\boldsymbol{P}}$, studied under bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, incomplete under $\underline{\leftrightarrow}$

λ-calculus with letrec under $=_{\boldsymbol{\lambda}^{\infty}}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Comparison results: structure-constrained graphs

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Given: graph interpretation $[\![\cdot]\!]_{\boldsymbol{P}}$, studied under bisimulation $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftarrow}$, incomplete under $\underline{\leftrightarrow}$

Defined: class of process graphs with LEE / (layered) LEE-witness

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G $\Longleftrightarrow$ G is $[\![\cdot]\!]_{\boldsymbol{P}}^{\cancel{1r}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$

λ-calculus with letrec under $=_{\boldsymbol{\lambda}^\infty}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

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λ-calculus with letrec under $=_{\boldsymbol{\lambda}^\infty}$

Not available: graph interpretation that is studied under $\underline{\leftrightarrow}$

Defined: int's $[\![\cdot]\!]_{\mathcal{H}}/[\![\cdot]\!]_{\mathcal{T}}$ as higher-order/first-order λ-term graphs

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with λ-calculus with letrec
  - efficient translation and readback
  - translation is inverse of readback