Proving equality for recursive types — A duality between 'syntactic-matching' tableaux and Brandt-Henglein derivations

(Following an observation by J.W. Klop)

Clemens Grabmayer

clemens@cs.vu.nl
www.cs.vu.nl/~clemens

Theoretical Computer Science group,
Vrije Universiteit Amsterdam

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Overview of the Talk

- **Recursive Types: Definitions and some necessary facts.**
  - Tree-unfolding, strong recursive type equivalence $=_\mu$.
  - The sound and complete proof-systems $\mathbf{AC}^{=}$, $\mathbf{HB}^{=}$ and $\mathbf{SM}^{=}$ for $=_\mu$.
  - Introduction of the variant-systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and $\mathbf{SM}_{\mathbf{0}}^{=}$.
- **Observation by J.W. Klop:** A striking relationship between "checking consistency in $\mathbf{SM}_{\mathbf{0}}^{=}$" and "trying to find a proof in $\mathbf{HB}_{\mathbf{0}}^{=}$".
- **Approach I: Formalization with "Consistency Unfoldings"**
  - A conservative extension $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}_{\mathbf{0}}^{=}$.
  - Rather technical formal definitions and necessary notions (. . . , *consistency-unfoldings*).
  - Main Theorem: A *Duality* between the systems $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ and $\mathbf{SM}_{\mathbf{0}}^{=}$.
  - "*Developing the* duality *on fine-grained film*".

- **Approach II: Formalization with "Syntactic-Matching Tableaux"**
  - Relation of consistency-unfoldings to "analytic tableaux".
  - The annotated tableau-system **ann-t-SM$_0^{=}$**.
  - *Syntactic-matching tableaux.*
  - New Duality: A view on the main theorem in Approach I from the world of tableaux.
  - Specializing the new duality.
- **What can the *duality* show?**
- **Summary.**

Recursive Types (with only type-constructor $\to$)

We let $TVar$ be a countable set of *type-variables*.

Definition 1. (Recursive Types μTp).

$$\tau ::= \alpha \mid \bot \mid \top \mid (\tau_1 \to \tau_2) \mid (\mu\alpha.\,\tau_0) \qquad (\alpha \in TVar).$$

(The *leading symbol* of the recursive type τ is coloured in orange).

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Examples: $\mu\beta.\,(\beta \to \bot)$, $\mu\alpha.\,((\alpha \to \alpha) \to \alpha)$, $\underbrace{\mu\gamma.\,\gamma}_{\sim\bot}$.

The operations *Unfolding* and its converse *Folding*

$$\mu\alpha.\,\tau_0 \longrightarrow_{\text{unfold}} \tau_0[\mu\alpha.\,\tau_0/\alpha] \qquad \text{and} \qquad \tau_0[\mu\alpha.\,\tau_0/\alpha] \longrightarrow_{\text{fold}} \mu\alpha.\,\tau_0$$

play an important part in reasoning with and for applying recursive types.

Definition 2. (Recursive Type Equality $=_\mu$ (Strong Recursive Type Equivalence))

For all $\tau, \sigma \in \mu Tp$:

$$\tau =_\mu \sigma \;:\iff\; \mathbf{Tree}(\tau) = \mathbf{Tree}(\sigma)\,.$$

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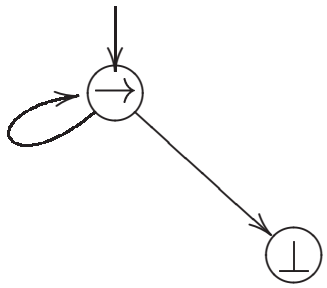
Example 1 (Tree-Unfolding, Strong Recursive Type Equivalence).

The recursive types $\tau := \mu\alpha.\,(\alpha \to \bot)$ and $\sigma := \mu\beta.\,((\beta \to \bot) \to \bot)$ correspond respectively to the different cyclic term-graphs



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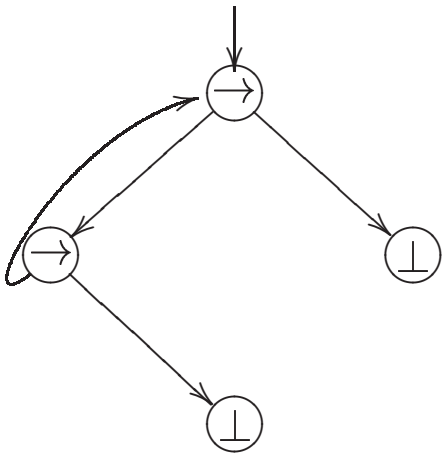
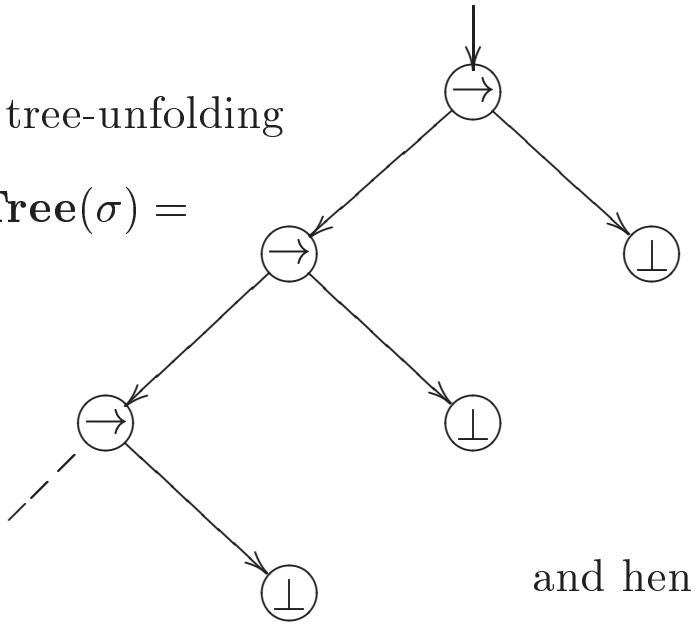


but possess the same tree-unfolding

$\mathbf{Tree}(\tau) = \mathbf{Tree}(\sigma) =$



and hence it holds that $\tau =_\mu \sigma$. ■

Definition 3. (**Tree-Unfolding of Recursive Types in canonical Form,** ***informal version***)

The function **Tree**

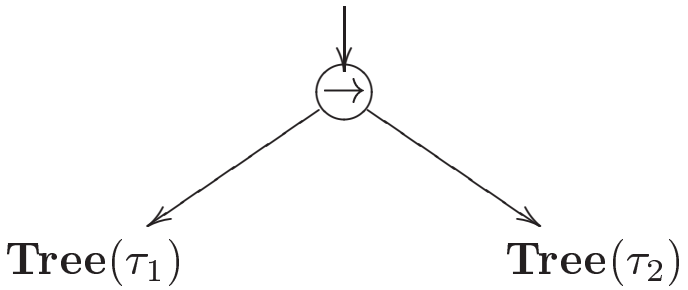
$$\mathbf{Tree}: \ \mathit{can}\text{-}\mu Tp \longrightarrow \text{Labelled Binary Trees}$$
$$\tau \longmapsto \mathbf{Tree}(\tau)$$

which describes the *tree-unfolding* of recursive types in canonical form is defined by:

$\mathbf{Tree}(\bot) :=$

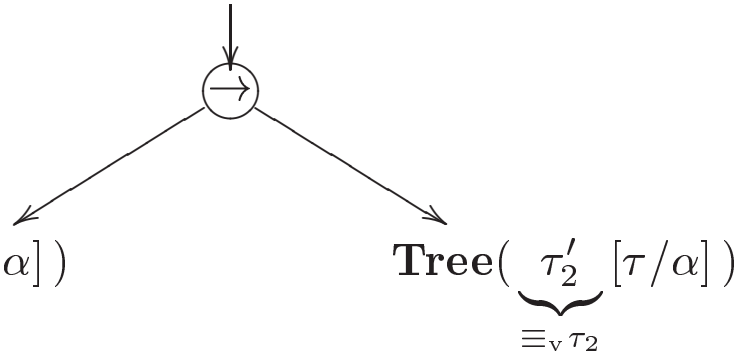
$\mathbf{Tree}(\top) :=$

$\mathbf{Tree}(\alpha) :=$

$\mathbf{Tree}(\tau_1 \to \tau_2) :=$ 

;

$\mathbf{Tree}(\underbrace{\mu\alpha.\,(\tau_1 \to \tau_2)}_{\equiv:\tau}) :=$



(where τ_1' and τ_2' are appropriate variants of τ_1 and τ_2 respectively to make the above two substitutions possible) .

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Definition 4. (Tree-Unfolding of Recursive Types, *formally precise version*)

The function

$$\begin{aligned} \mathbf{Tree} : \mu Tp &\longrightarrow TpTrees \\ \tau &\longmapsto \mathbf{Tree}(\tau) : \{1,2\}^* \to \{\bot, \top, \to\} \cup TVar \\ & \qquad p \longmapsto \mathbf{Tree}(\tau)(p) \end{aligned}$$

which to every recursive type $\tau \in \mu Tp$ assigns its *tree-unfolding* $\mathbf{Tree}(\tau)$ is defined as follows:

For every $\tau \in \mu Tp$ and every (path) $p \in \{1,2\}^*$ the symbol $\mathbf{Tree}(\tau)(p)$ is defined by induction on $|p|$ together with a subinduction on the number of leading μ-bindings in τ according to the following clauses:

(1) If $\tau \equiv \bot$ or $\tau \equiv \top$ or $\tau \equiv \alpha$ (for some $\alpha \in TVar$) , then

$$\mathbf{Tree}(\tau)(\epsilon) := \tau\,, \qquad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(2) If $\tau \equiv \tau_1 \to \tau_2$ and $p = i\,p_0$ (for $i = 1,2$), then

$$\mathbf{Tree}(\tau)(\epsilon) :=\to\,, \qquad \mathbf{Tree}(\tau)(p) := \mathbf{Tree}(\tau_i)(p_0) \text{ (for } i = 1,2).$$

(3) If $\tau \equiv \mu\alpha_1 \ldots \alpha_n.\,\alpha$ with $n \in \mathbb{N}$, $\alpha \not\equiv \alpha_1, \ldots, \alpha_n$, then

$$\mathbf{Tree}(\tau)(\epsilon) := \alpha\,, \qquad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(4) If $\tau \equiv \mu\alpha_1 \ldots \alpha_n.\,\alpha_i$ for $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, then

$$\mathbf{Tree}(\tau)(\epsilon) := \bot\,, \qquad \mathbf{Tree}(\tau)(p) :=\uparrow \text{ (for all } p \neq \epsilon).$$

(5) If $\tau \equiv \mu\alpha_1.\,\tau_0 \equiv \mu\alpha_1 \ldots \alpha_n.\,(\rho_1 \to \rho_2)$ with $n \in \mathbb{N}$, then

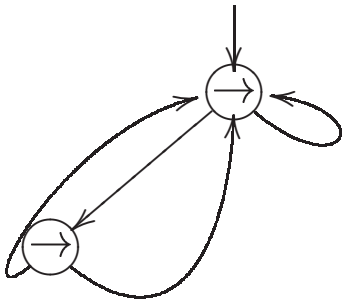
$$\mathbf{Tree}(\tau)(p) := \mathbf{Tree}(\tau_0'[\tau/\alpha_0])(p)$$
(for some variant τ_0' of τ_0). $\boxtimes$

Example 2 (Tree-Unfolding, Strong Recursive Type Equivalence).

The recursive types

$$\tau := \mu\alpha.\,((\alpha \to \alpha) \to \alpha) \qquad \text{and} \qquad \sigma := \mu\alpha.\,(\alpha \to (\alpha \to \alpha))$$

correspond respectively to the different cyclic term-graphs



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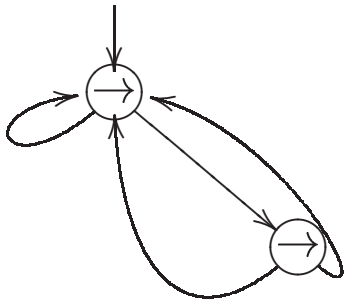
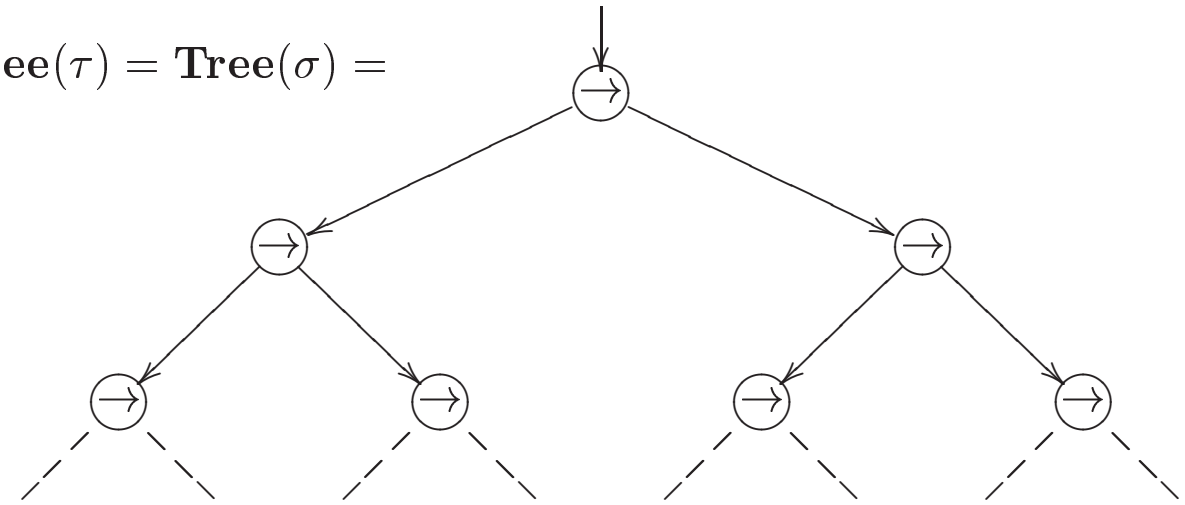


but possess the same tree-unfolding

$\mathbf{Tree}(\tau) = \mathbf{Tree}(\sigma) =$



and hence $\tau =_\mu \sigma$ holds, i.e. recursive type equality holds between τ and σ. ■

Definition 5. (The axiom-system $\mathbf{AC}^{=}$ for recursive type equality $=_\mu$)

Axioms:

$$\text{(REFL)}\ \overline{\tau = \tau} \qquad \text{(VAR)}\ \overline{\tau = \tau'}\ \ (\text{if } \tau \equiv_{\mathrm{v}} \tau')$$

$$(\mu - \bot)\ \overline{\mu\alpha.\,\alpha = \bot} \qquad \text{(FOLD/UNFOLD)}\ \overline{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS} \qquad \frac{\tau = \sigma}{\mu\alpha.\,\tau = \mu\alpha.\,\sigma}\ \mu\text{-COMPAT}$$

$$\frac{\tau_1 = \sigma_1 \qquad \tau_2 = \sigma_2}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \text{ARROW} \qquad \frac{\tau_1 = \tau[\tau_1/\alpha] \qquad \tau_2 = \tau[\tau_2/\alpha]}{\tau_1 = \tau_2}\ \text{UFP}\ \ (\text{if } \alpha \downarrow \tau)$$

(where $\alpha \downarrow \tau$ means that τ is *contractive* w.r.t. α).

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Theorem 1 (R. Amadio, L. Cardelli, '93). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\vdash_{\mathbf{AC}^{=}} \tau = \sigma \iff \tau =_\mu \sigma\ .$$

Definition 6. (The axiom-system HB$^=$ for recursive type equality $=_\mu$)

Axioms/Marked assumptions:

$$\text{(HYP)}\ \ (\tau = \sigma)^x \qquad \text{(REFL)}\ \ \overline{\tau = \tau} \qquad (\mu - \bot)'\ \ \overline{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \bot}$$

$$\text{(VAR)}\ \ \overline{\tau = \tau'} \quad (\text{if } \tau \equiv_{\text{v}} \tau') \qquad \text{(FOLD/UNFOLD)}\ \ \overline{\mu\alpha.\, \tau = \tau[\mu\alpha.\, \tau/\alpha]}$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS}$$

$$\frac{\begin{array}{cc} \langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2 \rangle^x & \langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \tau_1 = \sigma_1 & \tau_2 = \sigma_2 \end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ (\text{ARROW/FIX})_x$$

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Theorem 2 (M. Brandt (and F. Henglein), '98).

For all recursive types $\tau, \sigma \in \mu Tp$:

$$\vdash_{\mathbf{HB}^=} \tau = \sigma \iff \tau =_\mu \sigma\ .$$

Definition 7 (Leading Symbol of a recursive type).

The partial function $\mathcal{L} : \mu Tp \longrightarrow \{\bot, \top, \to\} \cup \{\alpha : \alpha \in TVar\}$,
$\tau \longmapsto \mathcal{L}(\tau)$

which to every recursive type τ assigns its *leading symbol* $\mathcal{L}(\tau)$ (if it exists), is defined by the following clauses:

$$\mathcal{L}(\bot) :\equiv \bot\,, \quad \mathcal{L}(\top) :\equiv \top\,, \quad \mathcal{L}(\alpha) :\equiv \alpha \quad (\text{for } \alpha \in TVar)\,,$$
$$\mathcal{L}(\tau_1 \to \tau_2) :\equiv \to\,,$$
$$\mathcal{L}(\mu\alpha_1 \ldots \alpha_n.\, \alpha) :\equiv \alpha \quad (\text{if } \alpha \not\equiv \alpha_i \text{ for } 1 \leq i \leq n)\,, \quad \mathcal{L}(\mu\alpha_1 \ldots \alpha_n.\, (\rho_1 \to \rho_2)) :\equiv \to\ .$$

Only in the (single remaining) situation, that

$$\tau \equiv \mu\alpha_1 \ldots \alpha_n.\, \alpha_i \quad (\text{for } n \in \mathbb{N} \text{ and } 1 \leq i \leq n)\,,$$

where $n \in \mathbb{N}$, $\mathcal{L}(\tau)$ is undefined. — The *total* function

$$\mathcal{L}' : \mu Tp \longrightarrow \{\bot, \top, \to\} \cup \{\alpha : \alpha \in TVar\}\ ,$$
$$\tau \longmapsto \mathcal{L}'(\tau)$$

which is a slight variant of $\mathcal{L}$ and is defined by the clauses in the definition of $\mathcal{L}$ and by the additional clause

$$\mathcal{L}'(\mu\alpha_1 \ldots \alpha_n.\, \alpha_i) :\equiv \bot \quad (\text{for } 1 \leq i \leq n)\,.$$

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Remark. For all $\tau \in \mu Tp$ it holds: $\mathcal{L}'(\tau) = \mathbf{Tree}(\tau)(\epsilon)$.

Definition 8. (The syntactic-matching inference-system $\mathbf{SM}^{=}$ for consistency-checks concerning recursive type equality $=_\mu$)

Axioms:

$$\text{(REFL)}\ \overline{\tau = \tau} \qquad (\mu - \bot)'\ \overline{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \bot} \qquad \text{(VAR)}\ \overline{\tau = \tau'}\ \ (\text{if } \tau \equiv_{\mathrm{v}} \tau')$$

$$\text{(FOLD/UNFOLD)}\ \overline{\mu\alpha.\, \tau = \tau[\mu\alpha.\, \tau/\alpha]}$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS} \qquad \frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \text{DECOMP}\ (i = 1, 2)$$

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Definition 9. For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau = \sigma \text{ is } \mathbf{SM}^{=}\text{-}\textit{consistent} \ :\iff \text{For all } \chi_1, \chi_2 \in \mu Tp\text{:}\quad \tau = \sigma \vdash_{\mathbf{SM}^{=}} \chi_1 = \chi_2 \implies \mathcal{L}'(\chi_1) \equiv \mathcal{L}'(\chi_2)\,.$$

Hereby for all $\tilde{\tau} \in \mu Tp$ the notation $\mathcal{L}'(\tilde{\tau})$ designates a slight variant of the *leading symbol* $\mathcal{L}(\tilde{\tau})$ of $\tilde{\tau}$. ⊠

Theorem 3 ((∼) Z.M. Ariola, J.W. Klop, '95). *For all recursive types* $\tau, \sigma \in \mu Tp$*:*

$$\tau = \sigma \textit{ is } \mathbf{SM}^{=}\textit{-consistent} \iff \tau =_\mu \sigma\ .$$

Definition 10. ('Normalized type equality rules' à la Brandt, Henglein — The variant-system $\mathbf{HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}^{=}$).

Axioms/Marked Assumptions:

$$\text{(REFL)}\quad \frac{}{\tau = \tau} \qquad\qquad \text{(HYP)}\quad (\tau = \sigma)^x$$

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \text{VAR}$$

$$\frac{\bot = \sigma}{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \sigma}\ (\mu\text{-}\bot)_l^{\bot\text{der}} \qquad \frac{\tau = \bot}{\tau = \mu\beta\beta_1 \ldots \beta_n.\, \beta}\ (\mu\text{-}\bot)_r^{\bot\text{der}}$$

$$\frac{\tau_0[\mu\alpha.\, \tau_0/\alpha] = \sigma}{\mu\alpha.\, \tau_0 = \sigma}\ \text{FOLD}_l \qquad \frac{\tau = \sigma_0[\mu\beta.\, \sigma_0/\beta]}{\tau = \mu\beta.\, \sigma_0}\ \text{FOLD}_r$$

$$\frac{\begin{array}{cc}\langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x & \langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_1 & \mathcal{D}_2 \\ \tau_1 = \sigma_1 & \tau_2 = \sigma_2\end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ (\text{ARROW/FIX})_x$$

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Theorem 4. *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\vdash_{\mathbf{HB}_{\mathbf{0}}^{=}} \tau = \sigma \quad \Longleftrightarrow \quad \tau =_\mu \sigma\ .$$

Definition 11. (**'Normalized syntactic-matching derivation-rules' — — The variant-system $\mathbf{SM}_{\mathbf{0}}^{=}$ of $\mathbf{SM}^{=}$**).

Axioms:

$\mathbf{SM}_{\mathbf{0}}^{=}$ possesses **no** *axioms*.

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR}$$

$$\frac{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \sigma}{\bot = \sigma}\ (\mu\!-\!\bot)_l^{\mathrm{der}\bot} \qquad \frac{\tau = \mu\beta\beta_1 \ldots \beta_n.\, \beta}{\tau = \bot}\ (\mu\!-\!\bot)_r^{\mathrm{der}\bot}$$

$$\frac{\mu\alpha.\, \tau_0 = \sigma}{\tau_0[\mu\alpha.\, \tau_0/\alpha] = \sigma}\ \mathrm{UNFOLD}_l \qquad \frac{\tau = \mu\beta.\, \sigma_0}{\tau = \sigma_0[\mu\beta.\, \sigma_0/\beta]}\ \mathrm{UNFOLD}_r$$

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \mathrm{DECOMP}\quad (i = 1, 2)$$

⊠

Theorem 5. *For all recursive types $\tau, \sigma \in \mu Tp$:*

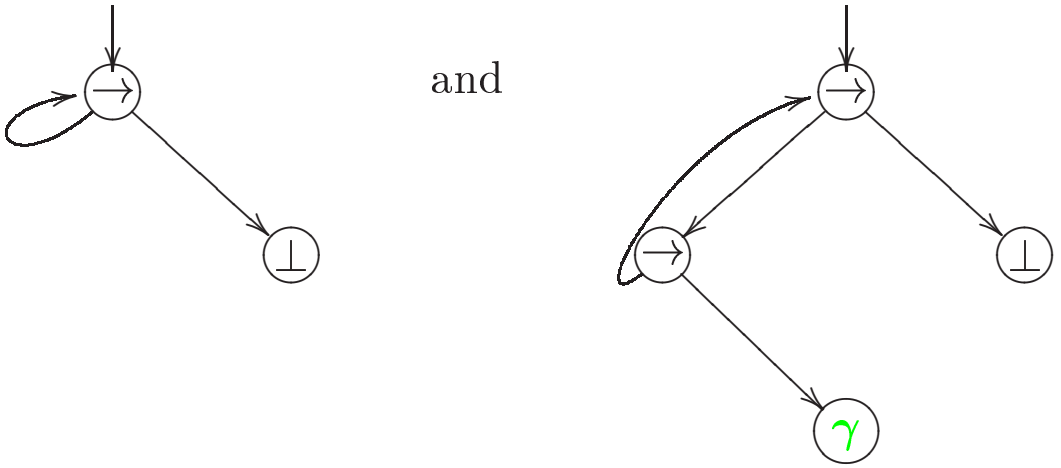
$$\tau = \sigma \text{ is } \mathbf{SM}_{\mathbf{0}}^{=}\text{-consistent} \iff \tau =_\mu \sigma\ .$$

Example 3. ($\mathbf{SM}_{\mathbf{0}}^{=}$-inconsistent equality).

The recursive types

$$\tau := \mu\alpha.\,(\alpha \to \bot) \qquad \text{and} \qquad \sigma := \mu\beta.\,((\beta \to \gamma) \to \bot)$$

correspond respectively to the different cyclic term-graphs



which do clearly not have the same tree-unfolding, hence are not strongly equivalent, i.e. $\tau \not\equiv_{\mu} \sigma$. The equality $\tau = \sigma$ is in fact not $\mathbf{SM}_{\mathbf{0}}^{=}$-consistent:

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Example 4 ("Consistency-Unfolding" in $\mathrm{SM}_{\mathbf{0}}^{=}$).

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Reflection in upwards-direction of this "Consisteny-Unfolding":

$$\dfrac{\dfrac{\dfrac{(\tau \to \tau = \sigma)^{x} \qquad (\tau = \sigma)^{y}}{(\tau \to \tau) \to \tau = \sigma \to \sigma} \qquad (\tau = \sigma)^{y}}{\tau = \sigma \to \sigma} \qquad \dfrac{\dfrac{(\tau = \sigma)^{y} \qquad (\tau = \sigma \to \sigma)^{z}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)} \qquad (\tau = \sigma)^{y}}{\tau \to \tau = \sigma}}{\dfrac{\dfrac{\tau \to \tau = \sigma \to (\sigma \to \sigma)}{(\tau \to \tau = \sigma)^{x}} \qquad \dfrac{(\tau \to \tau) \to \tau = \sigma \to \sigma}{(\tau = \sigma \to \sigma)^{z}}}{(\tau \to \tau) \to \tau = \sigma \to (\sigma \to \sigma)}}{\Big(\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\,\tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\,\sigma}\Big)^{y}}$$

Example 5 (Proof in $\mathrm{HB}_0^{=}$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{x}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}\,\mathrm{y}}{\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\,\tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\,\sigma}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\dots)^{x}}{\tau \to \tau = \sigma} \quad \dfrac{(\dots)^{y}}{\tau = \sigma}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}}{\tau = \sigma \to \sigma}\,\mathrm{x} \quad \dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma} \quad \dfrac{(\dots)^{z}}{\tau = \sigma \to \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{z}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}\,\mathrm{y}$$

$$\dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{x} \quad \dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma} \quad \dfrac{(\dots)^{z}}{\tau = \sigma \to \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{z}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{x} \quad \dfrac{\dfrac{\dfrac{(\dots)^{y}}{\tau = \sigma} \quad \dfrac{(\dots)^{z}}{\tau = \sigma \to \sigma}}{\tau \to \tau = \sigma \to (\sigma \to \sigma)}}{\tau \to \tau = \sigma}\,\mathrm{z}}{(\tau \to \tau) \to \tau = \sigma \to \sigma}\,\mathrm{y}}{\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\,\tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\,\sigma}}$$

■

Approach I:

Formalization with

"Consistency-Unfoldings"

Formalization of this observation [which is due to J.W. Klop] with "Consistency-Unfoldings":

(1) Definition of a (conservative) *extension* $\mathbf{e\text{-}HB}^{=}_{\mathbf{0}}$ of the system $\mathbf{HB}^{=}_{\mathbf{0}}$ with additional rules of the form

$$\frac{\begin{array}{c}\big[\,\mu\alpha.\,\tau_0 = \sigma_0\big]^{x} \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{array}}{\mu\alpha.\,\tau_0 = \sigma_0}\ (\mathrm{FOLD}_l/\mathrm{FIX})_x$$

(if $\mathcal{D}_0$ is *contractive* w.r.t. marked open assms. $(\mu\alpha.\,\tau_0 = \sigma)^x$)

and similar rules $\mathrm{FOLD}_r/\mathrm{FIX}$ and VAR/FIX.

(2) Formal definition of *consistency-unfoldings* of $\tau = \sigma$ in $\mathbf{SM}^{=}_{\mathbf{0}}$ as finite downward-growing derivation-trees with backbinding markers [which satisfy a number of properties].

(3) Definition of *reflection-functions* $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$ between [slightly more general objects than] consistency-unfoldings in $\mathbf{SM}^{=}_{\mathbf{0}}$ and derivations in $\mathbf{HB}^{=}_{\mathbf{0}}$.

Definition 12. (The rule FIX).

Let $\mathcal{S}$ be one of the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ or $\mathbf{HB}^{=}$.

Applications of the additional rule FIX are of the form

$$\dfrac{\begin{array}{c}[\tau = \sigma]^x \\ \mathcal{D}_0 \\ \tau = \sigma\end{array}}{\tau = \sigma}\,(\mathrm{FIX})_x \qquad \begin{array}{l}(\text{if } \mathcal{D}_0 \text{ is } \textit{contractive} \text{ w.r.t.} \\ \quad \text{open marked assms. } (\tau = \sigma)^x\,)\end{array} \tag{1}$$

in derivations in $\mathcal{S}$+FIX. Hereby the displayed side-condition has to be understood as:

> For every undischarged marked assumption of the form $(\tau = \sigma)^x$ in $\mathcal{D}_0$ the thread down to the conclusion $\tau = \sigma$ of $\mathcal{D}_0$ *crosses* an application of ARROW or ARROW/FIX *at least once*.

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In the case of $\mathcal{S} := \mathbf{HB}_{\mathbf{0}}^{=}$ the side-condition on applications of the rule FIX in derivations $\mathcal{D}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$+FIX of the form (1) is actually equivalent to the weaker formulation:

> There are *either no* undischarged marked assumptions in $\mathcal{D}_0$ of the form $(\tau = \sigma)^x$ *or* there is *at least one* occurrence of an application of ARROW or ARROW/FIX in $\mathcal{D}_0$.

Lemma 6. *The rule FIX is an admissible rule in* $\mathbf{HB}_{\mathbf{0}}^{=}$*, which can be eliminated effectively from arbitrary derivations in* $\mathbf{HB}_{\mathbf{0}}^{=}$*+FIX .*

More precisely, every derivation $\mathcal{D}$ *in* $\mathbf{HB}_{\mathbf{0}}^{=}$*+FIX can effectively be transformed into a derivation* $\mathcal{D}'$ *in* $\mathbf{HB}_{\mathbf{0}}^{=}$ *with the same conclusion and the same (if any) open assumption-classes.*

Definition 13. (The extension-system e-HB$_0^=$ of the system HB$_0^=$).

The extension **e-HB$_0^=$** of the system **HB$_0^=$** has the same formulas and axioms as **HB$_0^=$**, allows the same marked hypotheses and contains all derivation-rules of **HB$_0^=$**. Additionally, **e-HB$_0^=$** also contains the following 3 rules:

$$\frac{\begin{array}{c}[\tau' = \sigma']^x \\ \mathcal{D}_0 \\ \tau = \sigma\end{array}}{\tau' = \sigma'}\,(\text{VAR/FIX})_x \qquad \begin{array}{l}(\text{where } \tau' \equiv_{\text{v}} \tau \text{ and } \sigma' \equiv_{\text{v}} \sigma \\ \quad \text{and if side-cond.s. } \mathbf{C} \text{ and } \mathbf{I})\end{array}$$

$$\frac{\begin{array}{c}\big[\mu\alpha.\,\tau_0 = \sigma_0\big]^x \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{array}}{\mu\alpha.\,\tau_0 = \sigma_0}\,(\text{FOLD}_l/\text{FIX})_x \qquad (\text{if side-cond.s. } \mathbf{C} \text{ and } \mathbf{I})$$

and FOLD$_r$/FIX, which is of an analogous form as FOLD$_l$/FIX. Hereby:

- *Side-condition* **C**: $\mathcal{D}_0$ is *contractive* w.r.t. the assumptions $(\dots)^x$, which get discharged in the respective rule-application.
- *Side-condition* **I**: The respective class of assumptions $[\dots]^x$ in $\mathcal{D}_0$, which is discharged, is in fact *inhabitated*.

$\boxtimes$

Lemma 7. (Equivalence of the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and e-$\mathbf{HB}_{\mathbf{0}}^{=}$).

The rules VAR/FIX, FOLD_l/FIX and FOLD_r/FIX are admissible rules in $\mathbf{HB}_{\mathbf{0}}^{=}$*, which can effectively be eliminated from derivations in* e-$\mathbf{HB}_{\mathbf{0}}^{=}$*. – Thus the systems* $\mathbf{HB}_{\mathbf{0}}^{=}$ *and* e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *are equivalent (i.e. they possess the same theorems).*

Furthermore every derivation $\mathcal{D}$ in e-$\mathbf{HB}_{\mathbf{0}}^{=}$ *can effectively be transformed into a derivation $\mathcal{D}'$ in* $\mathbf{HB}_{\mathbf{0}}^{=}$ *with the same conclusion and the same (if any) open assumption-classes.*

Proof. As an example we consider the case, in which we want to eliminate the application of a rule FOLD_l/FIX from the bottom of a derivation $\mathcal{D}$ in e-$\mathbf{HB}_{\mathbf{0}}^{=}$ with immediate subderivation $\mathcal{D}_0$ in $\mathbf{HB}_{\mathbf{0}}^{=}$. This can be done as follows:

(1) Replacing the application of $(\mathrm{FOLD}_l/\mathrm{FIX})_x$ by an application of FOLD_l, which is succeeded by an application of $(\mathrm{FIX})_x$:

$$\dfrac{\begin{array}{c}\big[\,\mu\alpha.\,\tau_0 = \sigma_0\big]^{x} \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{array}}{\mu\alpha.\,\tau_0 = \sigma_0}\,(\mathrm{FOLD}_l/\mathrm{FIX})_x \quad\Longmapsto\quad \dfrac{\dfrac{\begin{array}{c}\big[\,\mu\alpha.\,\tau_0 = \sigma_0\big]^{x} \\ \mathcal{D}_0 \\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma_0\end{array}}{\mu\alpha.\,\tau_0 = \sigma_0}\,\mathrm{FOLD}_l}{\mu\alpha.\,\tau_0 = \sigma_0}\,(\mathrm{FIX})_x$$

(2) Eliminating $(\mathrm{FIX})_x$ in an effective way (which is possible due to Lemma 6). □

Definition 14 (Finite Trees of Consequences in $\mathbf{SM}_{\mathbf{0}}^{=}$).

A *finite tree of consequences* (a *f.t.o.c.*) $\mathcal{C}$ of (or from) an equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is an object, which can be formed by a finite number of applications of the following three generation rules. The depth $|\mathcal{C}|$ of $\mathcal{C}$ is defined in parallel:

(i) $\mathcal{C}$ is the equality between recursive types of the form $\boxed{\tau = \sigma}$. – Here $|\mathcal{C}| := 0$.

(ii) $\mathcal{C}$ is of the form $\boxed{\dfrac{\dfrac{\tau = \sigma}{\tau_0 = \sigma_0}\, R}{\mathcal{C}_0}}$, given that $\mathcal{C}_0$ is a f.t.o.c. of $\tau_0 = \sigma_0$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and that R is a rule-application of one of the rules $\text{UNFOLD}_{l/r}$, $(\mu{-}\bot)_{l/r}^{\text{der}\bot}$ or VAR of $\mathbf{SM}_{\mathbf{0}}^{=}$. – Here $|\mathcal{C}| := |\mathcal{C}_0| + 1$.

(iii) $\mathcal{C}$ is of the form $\boxed{\dfrac{\dfrac{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}{\tau_{01} = \sigma_{01} \qquad \tau_{02} = \sigma_{02}}\,\text{DECOMP}}{\mathcal{C}_{01} \qquad\quad \mathcal{C}_{02}}}$, given that $\mathcal{C}_{01}$ is a f.t.o.c. of $\tau_{01} = \sigma_{01}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and that $\mathcal{C}_{02}$ is a f.t.o.c. of $\tau_{02} = \sigma_{02}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$.

Hereby the part $\dfrac{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}{\tau_{01} = \sigma_{01} \qquad \tau_{02} = \sigma_{02}}\,\text{DECOMP}$ is called a *branching*.

– In this case $|\mathcal{C}| := 1 + \max\{|\mathcal{C}_{01}|, |\mathcal{C}_{02}|\}$.

$\boxtimes$

Definition 15. (Finite trees of consequences with marked formulas; back-bound leaf-occurrences of marked formulas).

Let τ and σ be recursive types.

(i) A finite tree of consequences *with marked formulas* (a f.t.o.c. with m.f.) $\mathcal{C}$ of the equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ results from a f.t.o.c. $\mathcal{C}^{(u)}$ of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ by marking some formula-occurrences in $\mathcal{C}^{(u)}$ through attaching subscript-markers to them.

(ii) If a f.t.o.c. with m.f. $\mathcal{C}$ with a single displayed leaf at its bottom of the form

$$\begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ (\rho = \chi)^x \\ \mathcal{C}_1 \\ (\rho = \chi)^x \end{array} \quad . \tag{2}$$

with respect to two f.t.o.c.'s with m.f. $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ from $\tau = \sigma$ and from $\rho = \chi$ is given, such that $|\mathcal{C}_1| \geq 1$ and there is no occurrence of $(\rho = \chi)^x$ in the thread from the root of $\mathcal{C}_1$ to the displayed leaf-occurrence of $(\rho = \chi)^x$, then we say that the occurrence of $(\rho = \chi)^x$ at the considered leaf is *bound back* in $\mathcal{C}$ to the displayed occurrence of $(\rho = \chi)^x$ at the position *inside* $\mathcal{C}$ corresponding to the root of $\mathcal{C}_1$ in the symbolic denotation (2).

$\boxtimes$

Definition 16. (Finite trees of consequences with back-bound leaf-occurrences of marked formulas).

A *finite tree of consequences with back-bound leaf-occurrences of marked formulas* (a f.t.o.c. with b.l.o.m.f.) $\mathcal{C}$ from the equality $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ is a f.t.o.c. with m.f., which satisfies the following two conditions:

(i) Each leaf at the bottom of $\mathcal{C}$ carries either an occurrence of an unmarked formula $\rho = \rho$ (which corresponds to an axiom (REFL) of e.g. the system $\mathbf{SM}^{=}$) or an occurrence of a marked formula $(\rho = \chi)^x$ (which may or may not be bound back in $\mathcal{C}$).

(ii) For each occurrence of a marked formula $(\rho = \chi)^x$ *inside* $\mathcal{C}$, i.e. such that $\mathcal{C}$ with this occurrence displayed can be written as

$$\begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ (\rho = \chi)^x \\ \mathcal{C}_1 \end{array}$$

for some f.t.o.c. with m.f. $\mathcal{C}_0$ from $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ and a f.t.o.c. $\mathcal{C}_1$ from $\rho = \chi$ with depth $|\mathcal{C}_1| \geq 1$, there exists another leaf-occurrence of $(\rho = \chi)^x$ in $\mathcal{C}$ (in a position in $\mathcal{C}$ corresponding to a leaf-occurrence of this marked formula in $\mathcal{C}_1$), which is bound back to the considered occurrence of $(\rho = \chi)^x$ inside $\mathcal{C}$.

$\boxtimes$

Definition 17. ((Partial) Consistency-Unfoldings).

Let τ and σ be recursive types.

(1) A *partial consistency-unfolding* $\mathcal{C}$ of the equation $\tau = \sigma$ in $\mathbf{SM_0^=}$ is a f.t.o.c. $\mathcal{C}$ with b.l.o.m.f., which satisfies the following property:

For all leaf-occurrences $(\rho = \chi)^x$ of marked formulas, which are bound back in $\mathcal{C}$, there is at least one occurrence of a branching DECOMP in the thread in $\mathcal{C}$ from the considered leaf-occurrence of $(\rho = \chi)^x$ up to that occurrence of $(\rho = \chi)^x$ inside $\mathcal{C}$, to which the leaf-occurrence is bound back.

(2) A *consistency-unfolding* $\mathcal{C}$ of the equality $\tau = \sigma$ in $\mathbf{SM_0^=}$ is a partial consistency-unfolding $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM_0^=}$,

in which each leaf-occurrence of a marked formula $(\rho = \chi)^x$ at the bottom of $\mathcal{C}$ is bound back in $\mathcal{C}$.

$\boxtimes$

Theorem 8. *For all recursive types* $\tau, \sigma \in \mu Tp$*:*

$$\tau = \sigma \text{ is } \mathbf{SM}_{\mathbf{0}}^{=}\text{-consistent} \iff \text{There exists a consistency-unfolding } \mathcal{C} \text{ of } \tau = \sigma \text{ in } \mathbf{SM}_{\mathbf{0}}^{=} .$$

Proof.

(a) "$\Rightarrow$": By a proof, which is in a way just the "dual" of a proof for the completeness of $\mathbf{HB}_{\mathbf{0}}^{=}$ (such a proof follows easily from the completeness-proof for $\mathbf{HB}^{=}$ in [BrHe98]).

(b) "$\Leftarrow$": By an easy, but rather tedious proof, which generalizes the intuitive argumentation hinted at in the case of the previous example.

□

Notation (Dual Rules in $\mathbf{HB}_{\mathbf{0}}^{=}$ and in $\mathbf{SM}_{\mathbf{0}}^{=}$).

In the two forthcoming definitions we will use the following notation for related rules in the systems $\mathbf{HB}_{\mathbf{0}}^{=}$ and in $\mathbf{SM}_{\mathbf{0}}^{=}$:

Rule $R^{(d)}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$	FOLD_l	FOLD_r	VAR	$(\mu-\bot)_l^{\bot\mathrm{der}}$	$(\mu-\bot)_r^{\bot\mathrm{der}}$
Rule $R^{(cu)}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$	UNFOLD_l	UNFOLD_r	VAR	$(\mu-\bot)_l^{\mathrm{der}\bot}$	$(\mu-\bot)_r^{\mathrm{der}\bot}$

⊠

Remark. For all one-premise-rules $R^{(d)}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$ the following is true:

For all $\tau, \sigma, \tau_0, \sigma_0 \in \mu Tp$ it holds:

$$\left.\begin{array}{l} \dfrac{\tau_0 = \sigma_0}{\tau = \sigma} R^{(d)} \text{ is a rule-application in } \mathbf{HB}_{\mathbf{0}}^{=} \iff \\ \iff \dfrac{\tau = \sigma}{\tau_0 = \sigma_0} R^{(cu)} \text{ is a rule application in } \mathbf{SM}_{\mathbf{0}}^{=}. \end{array}\right\} \quad (3)$$

And respectively (3) holds also for all rules $R^{(cu)}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ *except* the rule DECOMP.

Definition 18. (The Reflection-Operation $\mathcal{D}(\cdot)$ for f.t.o.c.'s with b.l.o.m.f.).

The *reflection* $\mathcal{D}(\mathcal{C})$ of an arbitrary f.t.o.c. with b.l.o.m.f. $\mathcal{C}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ via the *reflection-operation* $\mathcal{D}(\cdot)$ is defined inductively according to the following 5 clauses:

$$\boxed{(\tau = \sigma)^{\boldsymbol{m}}} \overset{\mathcal{D}(\cdot)}{\longmapsto} \boxed{(\tau = \sigma)^{\boldsymbol{m}}}$$

$$\boxed{\dfrac{\tau = \sigma}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)} \quad \mathcal{C}_0} \overset{\mathcal{D}(\cdot)}{\longmapsto} \boxed{\dfrac{\mathcal{D}(\mathcal{C}_0) \quad \tau_0 = \sigma_0}{\tau = \sigma}\, R^{(d)}}$$

(for $R^{(cu)} \in \{\text{UNFOLD}_{l/r}, \text{VAR}, (\mu - \bot)^{\text{der}\bot}_{l/r}\}$)

$$\boxed{\dfrac{(\tau = \sigma)^{x}}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)} \quad \mathcal{C}_0 \quad [\tau = \sigma]^{x}} \overset{\mathcal{D}(\cdot)}{\longmapsto} \boxed{[\tau = \sigma]^{x} \quad \mathcal{D}(\mathcal{C}_0) \quad \dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\, (R^{(d)}/\text{FIX})_{x}}$$

(for $R^{(cu)} \in \{\text{UNFOLD}_{l}, \text{UNFOLD}_{r}, \text{VAR}\}$)

$$\frac{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}{(\tau_{01} = \sigma_{01})^{m_{01}} \quad (\tau_{02} = \sigma_{02})^{m_{02}}}\ \text{DECOMP} \qquad \overset{\mathcal{D}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \frac{\mathcal{D}(\mathcal{C}_{01}) \quad \mathcal{D}(\mathcal{C}_{02}) \quad \tau_{01} = \sigma_{01} \quad \tau_{02} = \sigma_{02}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ \text{ARROW}$$

$$\frac{(\overbrace{\tau_{01} \to \tau_{02}}^{\equiv \tau} = \overbrace{\sigma_{01} \to \sigma_{02}}^{\equiv \sigma})^{x}}{(\tau_{01} = \sigma_{01})^{m_{01}} \quad (\tau_{02} = \sigma_{02})^{m_{02}}}\ \text{DECOMP} \qquad \overset{\mathcal{D}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \frac{\langle \tau = \sigma \rangle^{x} \quad \langle \tau = \sigma \rangle^{x} \quad \mathcal{D}(\mathcal{C}_{01}) \quad \mathcal{D}(\mathcal{C}_{02}) \quad \tau_{01} = \sigma_{01} \quad \tau_{02} = \sigma_{02}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ (\text{ARROW/FIX})_{x}$$

$\boxtimes$

Definition 19. (The Reflection-Operation $\mathcal{C}(\cdot)$ for e-HB$_0^=$-derivations).

The *reflection* $\mathcal{C}(\mathcal{D})$ of an arbitrary derivation $\mathcal{D}$ in **e-HB$_0^=$** with (possibly) open assumption-classes via the *reflection-operation* $\mathcal{C}(\cdot)$ is defined inductively according to the following 5 clauses:

$$\boxed{(\tau = \sigma)^{\boldsymbol{m}}} \overset{\mathcal{C}(\cdot)}{\longmapsto} \boxed{(\tau = \sigma)^{\boldsymbol{m}}}$$

$$\boxed{\dfrac{\mathcal{D}_0}{\dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\, R^{(d)}}} \overset{\mathcal{C}(\cdot)}{\longmapsto} \boxed{\dfrac{\dfrac{\tau = \sigma}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)}}{\mathcal{C}(\mathcal{D}_0)}}$$

$$(\text{for } R^{(d)} \in \{\text{FOLD}_{l/r}, \text{VAR}, (\mu{-}\bot)^{\bot\text{der}}_{l/r}\})$$

$$\boxed{\dfrac{\dfrac{[\tau = \sigma]^x}{\mathcal{D}_0}}{\dfrac{\tau_0 = \sigma_0}{\tau = \sigma}\, (R^{(d)}/\text{FIX})_x}} \overset{\mathcal{C}(\cdot)}{\longmapsto} \boxed{\dfrac{\dfrac{(\tau = \sigma)^x}{(\tau_0 = \sigma_0)^{\boldsymbol{m}_0}}\, R^{(cu)}}{\dfrac{\mathcal{C}(\mathcal{D}_0)}{[\tau = \sigma]^x}}}$$

$$(\text{for } R^{(d)} \in \{\text{FOLD}_l, \text{FOLD}_r, \text{VAR}\})$$

$$\dfrac{\begin{array}{cc}\mathcal{D}_{01} & \mathcal{D}_{02}\\ \tau_{01}=\sigma_{01} & \tau_{02}=\sigma_{02}\end{array}}{\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02}}\ \text{ARROW} \quad \overset{\mathcal{C}(\cdot)}{\longmapsto}$$

$$\longmapsto \quad \dfrac{\begin{array}{cc}\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02} & \\ (\tau_{01}=\sigma_{01})^{m_{01}} & (\tau_{02}=\sigma_{02})^{m_{02}}\\ \mathcal{C}(\mathcal{D}_{01}) & \mathcal{C}(\mathcal{D}_{02})\end{array}}{}\ \text{DECOMP}$$

$$\dfrac{\begin{array}{cc}\langle\tau=\sigma\rangle^{x} & \langle\tau=\sigma\rangle^{x}\\ \mathcal{D}_{01} & \mathcal{D}_{02}\\ \tau_{01}=\sigma_{01} & \tau_{02}=\sigma_{02}\end{array}}{\underbrace{\tau_{01}\to\tau_{02}}_{\equiv\tau}=\underbrace{\sigma_{01}\to\sigma_{02}}_{\equiv\sigma}}\ (\text{ARROW/FIX})_{x} \quad \overset{\mathcal{C}(\cdot)}{\longmapsto}$$

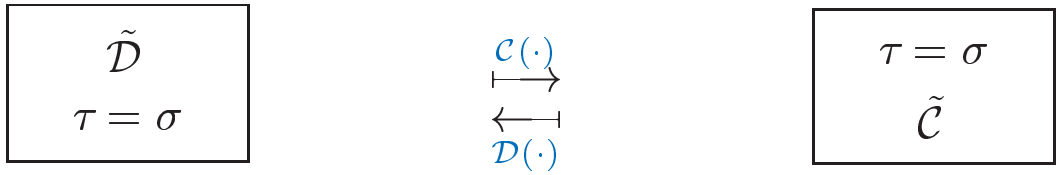
$$\longmapsto \quad \dfrac{\begin{array}{cc}(\tau_{01}\to\tau_{02}=\sigma_{01}\to\sigma_{02})^{x} & \\ (\tau_{01}=\sigma_{01})^{m_{01}} & (\tau_{02}=\sigma_{02})^{m_{02}}\\ \mathcal{C}(\mathcal{D}_{01}) & \mathcal{C}(\mathcal{D}_{02})\\ \langle\tau=\sigma\rangle^{x} & \langle\tau=\sigma\rangle^{x}\end{array}}{}\ \text{DECOMP}$$

$\boxtimes$

Theorem 9. [Main theorem of Approach I] (A Duality between derivations in e-HB$_0^=$ and consistency-unfoldings in SM$_0^=$).

There is a bijective functional relationship *between derivations in* **e-HB$_0^=$** *and consistency-unfoldings in* **SM$_0^=$** *via the reflection-operations* $\mathcal{D}(\cdot)$ *and* $\mathcal{C}(\cdot)$ *in the following sense:*

— *For all* $\tau, \sigma \in \mu Tp$*:*



e-HB$_0^=$-*derivation of* $\tau = \sigma$ *without open assumption-classes*

consistency-unfolding of $\tau = \sigma$ *in* **SM$_0^=$**

— *Thereby the reflection-functions* $\mathcal{C}(\cdot)$ *and* $\mathcal{D}(\cdot)$ *are each other's inverses.*

The very immediate kind of this bijective functional relationship and the possibility to visualize it geometrically is reason to call it a duality.

Proof. By induction on $|\tilde{\mathcal{D}}|$ of arbitrary $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivations $\tilde{\mathcal{D}}$ with n open assumption classes and respectively by induction on $|\tilde{\mathcal{C}}|$ of arbitrary partial consistency-unfoldings $\tilde{\mathcal{C}}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with n classes of unbound leaf-occurrences of marked formulas (where $n \in \mathbb{N}_0$):

$$\boxed{\begin{array}{c}\{\,[\tau_i = \sigma_i]^{x_i}\,\}_{i=1,\ldots,n} \\ \tilde{\mathcal{D}} \\ \tau = \sigma\end{array}} \quad \begin{array}{c}\overset{\mathcal{C}(\cdot)}{\longmapsto} \\ \overset{\longmapsto}{\underset{\mathcal{D}(\cdot)}{\longleftarrow}}\end{array} \quad \boxed{\begin{array}{c}(\tau = \sigma)^{\boldsymbol{m}} \\ \tilde{\mathcal{C}} \\ \{\,[\tau_i = \sigma_i]^{x_i}\,\}_{i=1,\ldots,n}\end{array}}$$

$\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivation of $\tau = \sigma$ with n open assumption-classes

partial consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with n classes of leaf-occurrences of unbound marked formulas

□

Example 6. (Duality between a derivation in e-HB$_0^=$ and a consistency-unfolding in SM$_0^=$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\tau = \sigma)^{x}}{\tau \to \bot = \sigma \to \bot}\,\text{ARROW}}{\tau = \sigma \to \bot} \qquad \dfrac{\bot = \bot}{\bot = \bot}\,\text{ARROW}}{\tau \to \bot = (\sigma \to \bot) \to \bot}\,\text{FOLD}_r}{\tau \to \bot = \sigma}\,(\text{FOLD}_l/\text{FIX})_{x}}{\mu\alpha.\,(\alpha \to \bot) = \mu\beta.\,((\beta \to \bot) \to \bot)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overbrace{(\mu\alpha.\,(\alpha \to \bot)}^{\equiv:\,\tau} = \overbrace{\mu\beta.\,((\beta \to \bot) \to \bot))}^{\equiv:\,\sigma}{}^{x}}{\tau \to \bot = \sigma}\,\text{UNFOLD}_l}{\tau \to \bot = (\sigma \to \bot) \to \bot}\,\text{UNFOLD}_r \qquad \dfrac{}{\bot = \bot}}{\tau = \sigma \to \bot}\,\text{DECOMP}}{\tau \to \bot = \sigma \to \bot}\,\text{DECOMP}}{(\tau = \sigma)^{x} \qquad \bot = \bot}$$

■

Definition 20. (The property D for finite trees of consequences with back-binding markers for leaves).

Let τ and σ be recursive types.

A finite tree of consequences $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM_0^=}$ with back-bound leaf-occurrences of marked formulas satisfies the *property* **D** iff all occurrences of marked formulas $(\rho = \chi)^x$ in leaves at the bottom of $\mathcal{C}$ (with arbitrary $\rho, \chi \in \mu Tp$ and markers x), which are bound back in $\mathcal{C}$, are exclusively bound back to upper premises of branchings DECOMP in $\mathcal{C}$.

$\boxtimes$

This means, that a f.t.o.c. with b.l.o.m.f. $\mathcal{C}$ of $\tau = \sigma$ in $\mathbf{SM_0^=}$, which satisfies the property **D**, can be written w.r.t. an arbitrary back-bound leaf-occurrence of a m.f. $(\rho = \chi)^x$ in $\mathcal{C}$ as e.g.

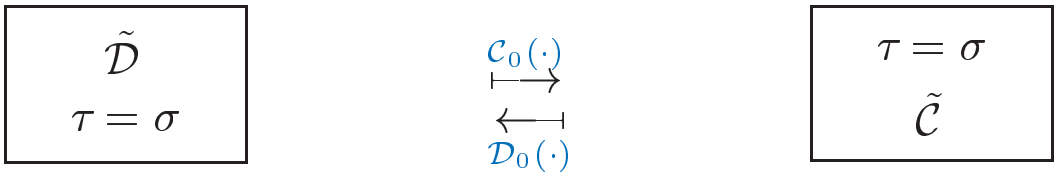
$$\text{E.g.} \quad \begin{array}{c} \tau = \sigma \\ \mathcal{C} \\ (\rho = \chi)^x \end{array} \quad \equiv \quad \cfrac{\begin{array}{c} \tau = \sigma \\ \mathcal{C}_0 \\ (\rho_1 \to \rho_2 = \chi_1 \to \chi_2)^x \end{array}}{\begin{array}{cc} \rho_1 = \chi_1 & \rho_2 = \chi_2 \\ \mathcal{C}_1 & \mathcal{C}_2 \\ (\rho_1 \to \rho_2 = \chi_1 \to \chi_2)^x \end{array}} \text{ DECOMP}$$

with $\rho_1, \rho_2, \chi_1, \chi_2 \in \mu Tp$ s.th. $\rho \equiv \rho_1 \to \rho_2$ and $\chi \equiv \chi_1 \to \chi_2$ and with some f.t.o.c.'s with m.f. $\mathcal{C}_0$ from $\tau = \sigma$, $\mathcal{C}_1$ from $\rho_1 = \chi_1$ and $\mathcal{C}_2$ from $\rho_2 = \chi_2$ in $\mathbf{SM_0^=}$.

Theorem 10. (Fine grained picture of the duality: A Duality between derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property D).

The reflection-operations $\mathcal{D}(\cdot)$ and $\mathcal{C}(\cdot)$ also yield a bijective functional relationship *between derivations in* $\mathbf{HB}_{\mathbf{0}}^{=}$ *and consistency-unfoldings in* $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property **D** *in the following sense:*

— *For all $\tau, \sigma \in \mu Tp$:*



$\mathbf{HB}_{\mathbf{0}}^{=}$*-derivation of $\tau = \sigma$ without open assumption-classes*

consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ *with the property* **D**

where $\mathcal{C}_0(\cdot) := \mathcal{C}|_A$ is the restriction of the reflection-operation $\mathcal{C}(\cdot)$ to the set A of derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ *without open assumption-classes and $\mathcal{D}_0(\cdot) := \mathcal{C}|_B$ is the restriction of the reflection-operation $\mathcal{D}(\cdot)$ to the set B of all consistency-unfoldings in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *with the property* **D**.

— *With this it follows from Theorem 9, that also the reflection-operations $\mathcal{C}_0$ and $\mathcal{D}_0$* are each other's inverses.

Proof. By induction on $|\tilde{\mathcal{D}}|$ of arbitrary $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations $\tilde{\mathcal{D}}$ with (possibly) open assumption-classes and respectively by induction on $|\tilde{\mathcal{C}}|$ of arbitrary partial consistency-unfoldings $\tilde{\mathcal{C}}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$:

$\{ [\tau_i = \sigma_i]^{x_i} \}_{i=1,\dots,n}$ $\tilde{\mathcal{D}}$ $\tau = \sigma$	$\overset{\mathcal{C}(\cdot)}{\longmapsto}$ $\underset{\mathcal{D}(\cdot)}{\longleftarrow}$	$(\tau = \sigma)^{\boldsymbol{m}}$ $\tilde{\mathcal{C}}$ $\{ [\tau_i = \sigma_i]^{x_i} \}_{i=1,\dots,n}$
$\mathbf{HB}_{\mathbf{0}}^{=}$-derivation of $\tau = \sigma$ with (possibly) open assumption-classes		partial consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$

□

Example 7. (Duality between a derivation in $\mathrm{HB}_0^{=}$ and a consistency-check in $\mathrm{SM}_0^{=}$ with the property D).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\underbrace{\mu\alpha.\,(\alpha\to\bot)}_{\equiv:\,\tau}=\underbrace{\mu\beta.\,((\beta\to\bot)\to\bot)}_{\equiv:\,\sigma}}\,\mathrm{FOLD}_{l/r}}{\tau\to\bot=(\sigma\to\bot)\to\bot}\,(\mathrm{ARR./FIX})_x}{\tau=\sigma\to\bot}\qquad\dfrac{}{\bot=\bot}}{\tau\to\bot=\sigma\to\bot}\,\mathrm{ARROW}}{\tau=\sigma}\qquad\dfrac{}{\bot=\bot}}{(\tau\to\bot=(\sigma\to\bot)\to\bot)^x}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mu\alpha.\,(\alpha\to\bot)=\mu\beta.\,((\beta\to\bot)\to\bot)}\,\mathrm{UNFOLD}_{l/r}}{(\tau\to\bot=(\sigma\to\bot)\to\bot)^x}}{\tau=\sigma\to\bot}\,\mathrm{DECOMP}\qquad\dfrac{}{\bot=\bot}}{\tau\to\bot=\sigma\to\bot}}{\tau=\sigma}\,\mathrm{DECOMP}\qquad\dfrac{}{\bot=\bot}}{(\tau\to\bot=(\sigma\to\bot)\to\bot)^x}$$

■

Approach II:

Formalization with

"Syntactic-Matching Tableaux"

Example 8. (Closed analytic tableau as introduced by R. Smullyan for a valid formula of the propositional calculus).

With respect to the rules $\dfrac{F\, p \to q}{\begin{array}{c} T\, p \\ F\, q \end{array}}$ and $\dfrac{T\, p \to q}{F\, p \mid T\, q}$ for *signed formulas* the following is a *closed analytic tableau* for $(p \to q) \to \big((q \to r) \to (p \to r)\big)$, a valid formula of propositional calculus:

<table>
<tr><td colspan="3">$F\,(p \to q) \to \big((q \to r) \to (p \to r)\big)$
$T\, p \to q$
$F\,(q \to r) \to (p \to r)$
$T\, q \to r$
$F\, p \to r$
• $T\, p$
• $F\, r$</td></tr>
<tr><td>• $F\, p$
X</td><td colspan="2">• $T\, q$</td></tr>
<tr><td></td><td>• $F\, q$
X</td><td>• $T\, r$
X</td></tr>
</table>

■

The *subformula-principle*: "Analysing a formula into its successive components".

Example 9. (Extraction of a boolean valuation from a not-closed analytic tableau for a not valid propositional formula).

Let Y be the *not valid* formula $\big((p \to q) \to (q \to r)\big) \to (p \to r)$ of propositional calculus. The tableau for Y of the form

$F\,\big((p \to q) \to (q \to r)\big) \to (p \to r)$ $T\,(p \to q) \to (q \to r)$ $F\,(p \to r)$ $T\,p$ $\times\ F\,r$	
$F\,p \to q$ $T\,p$ $F\,q$	$T\,q \to r$
	$F\,q$ | $\times\ T\,r$ X

is analytic, but not closed. It is possible to extract from it the (atomic) interpretation $v_0 := \{p \mapsto t,\, q \mapsto f,\, r \mapsto f\}$ with the property that $v(Y) = f$, i.e. Y is false under v.

■

Example 10. (Extraction of information from a failed consistency-check in $\mathbf{SM}_0^{=}$).

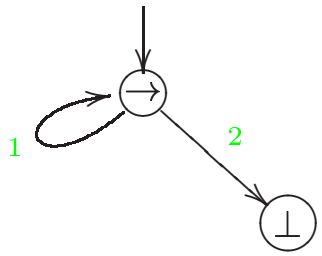
By annotating the "failed consistency-check"

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\left(\varepsilon : \overbrace{\mu\alpha.\,(\alpha \to \bot)}^{\equiv:\,\tau} = \overbrace{\mu\beta.\,((\beta \to \boldsymbol{\gamma}) \to \bot)}^{\equiv:\,\sigma}\right)^{x}}{\varepsilon : \tau \to \bot = (\sigma \to \boldsymbol{\gamma}) \to \bot}\ \text{UNFOLD}_{l/r}}{1 : \tau = \sigma \to \boldsymbol{\gamma}}\ \text{DECOMP}}{1 : \tau \to \bot = \sigma \to \boldsymbol{\gamma}}\ \text{UNFOLD}_{l} \qquad 2 : \bot = \bot}{\left(11 : \tau = \sigma\right)^{x} \qquad 12 : \bot = \boldsymbol{\gamma}}\ \text{DECOMP}$$

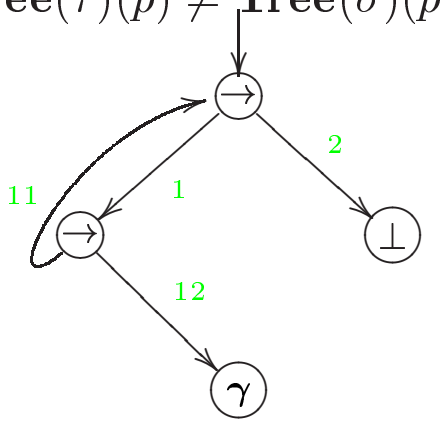
Contradiction!

with position-labels (displayed in green) it is actually possible to find a position $p := 12$ in $\mathrm{dom}(\mathbf{Tree}(\tau)) \cap \mathrm{dom}(\mathbf{Tree}(\sigma))$ such that $\mathbf{Tree}(\tau)(p) \neq \mathbf{Tree}(\sigma)(p)$:

Compare the corresponding cyclic term-graphs



and



■

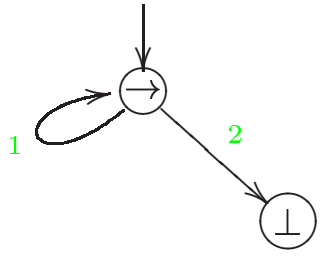
Example 11. (Extraction of information from a *not closed* "syntactic-matching tableau").

The consistency-check of the previous example can be rewritten in the form of a *not closed* "syntactic-matching tableau" of the form

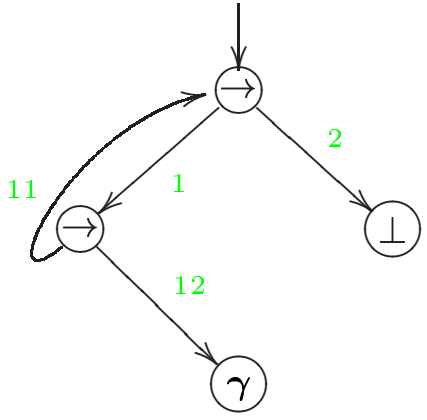
<table>
<tr><td colspan="3">$\bullet\ \varepsilon : \overbrace{\mu\alpha.\,\alpha \to \bot}^{\equiv:\,\tau} = \overbrace{\mu\beta.\,((\beta \to \boldsymbol{\gamma}) \to \bot)}^{\equiv:\,\sigma}$
$\varepsilon : \tau \to \bot = (\sigma \to \boldsymbol{\gamma}) \to \bot$</td></tr>
<tr><td colspan="2">$1 : \tau = \sigma \to \boldsymbol{\gamma}$
$1 : \tau \to \bot = \sigma \to \boldsymbol{\gamma}$</td><td>$2 : \bot = \bot$
X</td></tr>
<tr><td>$\bullet\ 11 : \tau = \sigma$
X</td><td>$12 : \bot = \boldsymbol{\gamma}$</td><td></td></tr>
</table>

and it is actually again possible to find a position $p := 12$ in $\mathrm{dom}(\mathbf{Tree}(\tau)) \cap \mathrm{dom}(\mathbf{Tree}(\sigma))$ such that $\mathbf{Tree}(\tau)(p) \neq \mathbf{Tree}(\sigma)(p)$.

Compare the corresponding cyclic term-graphs



and



■

Definition 21. (The annotated tableau-system ann-t-SM$_0^=$).

Formulas: All annotated equalities $p : \tau = \sigma$ for some $p \in \{1,2\}^*$.

Axioms: **ann-t-SM$_0^=$** possesses **no** *axioms*.

Rules A of the form $\dfrac{\boldsymbol{\alpha}}{\boldsymbol{\alpha_1}}$:

$$\frac{p : \tau = \sigma}{p : \tau' = \sigma'}\ \text{VAR}$$

$$\frac{p : \mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \sigma}{p : \bot = \sigma}\ (\mu\text{-}\bot)_l^{\text{der}\bot} \qquad \frac{p : \tau = \mu\beta\beta_1 \ldots \beta_n.\, \beta}{p : \tau = \bot}\ (\mu\text{-}\bot)_r^{\text{der}\bot}$$

$$\frac{p : \mu\alpha.\, \tau_0 = \sigma}{p : \tau_0[\mu\alpha.\, \tau_0/\alpha] = \sigma}\ \text{UNFOLD}_l \qquad \frac{p : \tau = \mu\beta.\, \sigma_0}{p : \tau = \sigma_0[\mu\beta.\, \sigma_0/\beta]}\ \text{UNFOLD}_r$$

Rules B of the form $\dfrac{\boldsymbol{\beta}}{\boldsymbol{\beta_1} \mid \boldsymbol{\beta_2}}$:

$$\frac{p : \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{p1 : \tau_1 = \sigma_1 \mid p2 : \tau_2 = \sigma_2}\ \text{DECOMP}$$

$\boxtimes$

Definition 22. (Syntactic-Matching Tableaux in ann-t-SM$_0^=$).

Inductive Clauses:

(i) $\boxed{\varepsilon : \tau = \sigma}$ is a syntactic-matching tableau for $\varepsilon : \tau = \sigma$ in **ann-t-SM$_0^=$**.

(ii) $\boxed{\begin{array}{c}\varepsilon : \tau = \sigma \\ \mathcal{T}_0 \\ (\boldsymbol{\alpha}) \\ \boldsymbol{\alpha_1}\end{array}}$ is a s.-m. tableau, given that $\boxed{\begin{array}{c}\varepsilon : \tau = \sigma \\ \mathcal{T}_0 \\ (\boldsymbol{\alpha})\end{array}}$ is a s.-m. tableau for

$\varepsilon : \tau = \sigma$ in **ann-t-SM$_0^=$** and that $\dfrac{\boldsymbol{\alpha}}{\boldsymbol{\alpha_1}}$ is an application of a rule A.

(iii) $\boxed{\begin{array}{c}\varepsilon : \tau = \sigma \\ \mathcal{T}_0 \\ (\boldsymbol{\beta}) \\ \hline \boldsymbol{\beta_1} \mid \boldsymbol{\beta_2}\end{array}}$ is a s.-m. tableau, given that $\boxed{\begin{array}{c}\varepsilon : \tau = \sigma \\ \mathcal{T}_0 \\ (\boldsymbol{\beta})\end{array}}$ is a s.-m. tableau for

$\varepsilon : \tau = \sigma$ in **ann-t-SM$_0^=$** and that $\dfrac{\boldsymbol{\beta}}{\boldsymbol{\beta_1} \mid \boldsymbol{\beta_2}}$ is an application of a rule B. $\boxtimes$

Definition 23. (Closed and reduced tableaux in ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$).

- A branch Θ in a tableau $\mathcal{T}$ in **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$** is called *closed* iff Θ contains an occurrence o of a formula A such that
  - A is a formula of the form $\rho = \rho$ for some $\rho \in \mu Tp$, or
  - A is of the form $p_1 : \chi_1 = \chi_2$ for some $p_1 \in \{0,1\}^*$ and $\chi_1, \chi_2 \in \mu Tp$, and there is also an occurrence in Θ of the (different) formula $p_2 : \chi_1 = \chi_2$ for some $p_2 \in \{0,1\}^*$ with $p_2 \neq p_1$.

  holds (in this case we say that *closure happens for the occurrence o of A in Θ and in $\mathcal{T}$*).
- A tableau $\mathcal{T}$ in **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$** is called *closed* iff every branch in $\mathcal{T}$ is closed.
- A tableau $\mathcal{T}$ in **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$** is called *reduced* iff for each closed branch Θ closure happens for the formula at the bottom of Θ.

$\boxtimes$

Properties of the tableau-system ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$:

Lemma 11. *Let $\mathcal{T}$ be a tableau for $\tau = \sigma$ in* **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$**, *that contains a formula $p : \chi_1 = \chi_2$ for some $p \in \{0,1\}^*$ and $\chi_1, \chi_2 \in \mu Tp$. Then it holds that*

$$\mathbf{Tree}(\tau)|_p = \mathbf{Tree}(\chi_1) \quad \textit{and} \quad \mathbf{Tree}(\sigma)|_p = \mathbf{Tree}(\chi_2) \; .$$

Lemma 12. *Let $\mathcal{T}$ be a closed tableau for $\tau = \sigma$ in* **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$**, *that contains a formula $p : \chi_1 = \chi_2$ for some $p \in \{0,1\}^*$ and $\chi_1, \chi_2 \in \mu Tp$.*

Then it holds that χ_1 and χ_2 have the same leading symbol, i.e.

$$\mathcal{L}'(\chi_1) = \mathcal{L}'(\chi_2) \; .$$

Theorem 13. (Soundness and Completeness of ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$ with respect to recursive type equality $=_\mu$).

For all $\tau, \sigma \in \mu Tp$ it holds that:

$$(\exists\, \mathcal{T}) \left[\begin{array}{l} \mathcal{T} \textit{ is a closed syntactic-matching} \\ \textit{tableau for } \varepsilon : \tau = \sigma \textit{ in } \mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=} \end{array} \right] \iff$$

$$(\exists\, \mathcal{T}) \left[\mathcal{T} \textit{ is a closed tableau for } \varepsilon : \tau = \sigma \textit{ in } \mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=} \right] \iff \tau =_\mu \sigma \; .$$

Definition 24. (The Reflection-Operation $\mathcal{D}(\cdot)$ for syntactic-matching tableaux in ann-t-SM$_0^{=}$).

The *reflection* $\mathcal{D}(\mathcal{C})$ of an arbitrary syntactic-matching tableau in **ann-t-SM$_0^{=}$**, *in which closure has been made explicit,* via the *reflection-operation* $\mathcal{D}(\cdot)$ is defined inductively according to the following 5 clauses:

$$\boxed{\tau = \sigma} \quad \overset{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{(\tau = \sigma)^{\boldsymbol{m}}}$$

$$\boxed{\begin{array}{c} p:\ \tau = \sigma \\ p:\ \tau_0 = \sigma_0 \\ \mathcal{T}_0 \end{array}} \quad \overset{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} \mathcal{D}(\mathcal{T}_0) \\ \tau_0 = \sigma_0 \\ \hline \tau = \sigma \end{array} R^{(d)}}$$

(for some $R^{(d)} \in \{\text{UNFOLD}_{l/r},\ \text{VAR},\ (\mu{-}\bot)^{\text{der}\bot}_{l/r}\}$)

$$\boxed{\begin{array}{c} \bullet\ p:\ \tau = \sigma \\ p:\ \tau_0 = \sigma_0 \\ \mathcal{T}_0 \\ [(\bullet\ p\,p_i:\ \tau = \sigma)_i] \end{array}} \quad \overset{\mathcal{D}(\cdot)}{\longmapsto} \quad \boxed{\begin{array}{c} [\tau = \sigma]^x \\ \mathcal{D}(\mathcal{T}_0) \\ \tau_0 = \sigma_0 \\ \hline \tau = \sigma \end{array} (R^{(d)}/\text{FIX})_x}$$

(for some $R^{(cu)} \in \{\text{UNFOLD}_l,\ \text{UNFOLD}_r,\ \text{VAR}\}$ and a new assumption-marker x)

$p : \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2$	
$p1 : \tau_1 = \sigma_1$ $\mathcal{T}_{01}$	$p2 : \tau_2 = \sigma_2$ $\mathcal{T}_{02}$

$\overset{\mathcal{D}(\cdot)}{\longmapsto}$

$$\frac{\mathcal{D}(\mathcal{T}_{01}) \quad \mathcal{D}(\mathcal{T}_{02}) \quad \tau_1 = \sigma_1 \quad \tau_2 = \sigma_2}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \text{ARROW}$$

$\bullet\ p : \overbrace{\tau_1 \to \tau_2}^{\equiv\tau} = \overbrace{\sigma_1 \to \sigma_2}^{\equiv\sigma}$	
$p1 : \tau_1 = \sigma_1$ $\mathcal{T}_{01}$ $\langle (\bullet\ pp_i : \tau = \sigma)_i \rangle$	$p2 : \tau_2 = \sigma_2$ $\mathcal{T}_{02}$ $\langle (\bullet\ pp_j : \tau = \sigma)_j \rangle$

$\overset{\mathcal{D}(\cdot)}{\longmapsto}$

$\longmapsto$

$$\frac{\langle \tau = \sigma \rangle^x \quad \mathcal{D}(\mathcal{T}_{01}) \quad \tau_{01} = \sigma_{01} \quad \langle \tau = \sigma \rangle^x \quad \mathcal{D}(\mathcal{T}_{02}) \quad \tau_{02} = \sigma_{02}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ (\text{ARROW/FIX})_x$$

(for some new assumption-marker x)

$\boxtimes$

Definition 25. (The Reflection-Operation $\mathcal{T}(\cdot)$ for syntactic-matching tableaux in ann-t-SM$_0^=$).

The *reflection* $\mathcal{T}(\mathcal{D})$ of an arbitrary derivation $\mathcal{D}$ in **e-HB$_0^=$** with (possibly) open assumption-classes is defined as $\mathcal{T}(\mathcal{D}) := \mathcal{T}(\mathcal{D}, \varepsilon)$ via the *reflection-operation* $\mathcal{T}(\cdot, p)$ (for arbitrary $p \in \{1,2\}^*$), which in its turn is defined inductively according to the following 5 clauses:

$$(\tau = \sigma)^{m} \quad \overset{\mathcal{T}(\cdot,p)}{\longmapsto} \quad \boxed{\tau = \sigma}$$

$$\dfrac{\mathcal{D}_0}{\tau_0 = \sigma_0}\, R^{(d)}(/\mathrm{FIX}) \quad \overset{\mathcal{T}(\cdot,p)}{\longmapsto} \quad \boxed{\begin{array}{c} p : \tau = \sigma \\ p : \tau_0 = \sigma_0 \\ \mathcal{T}(\mathcal{D}_0, p) \end{array}}$$

(for $R^{(d)} \in \{\mathrm{FOLD}_{l/r}, \mathrm{VAR}, (\mu{-}\bot)^{\bot\mathrm{der}}_{l/r}\}$)

$$\dfrac{\begin{array}{c}[\tau = \sigma]^{x} \\ \mathcal{D}_0 \\ \tau_0 = \sigma_0 \end{array}}{\tau = \sigma}\, (R^{(d)}/\mathrm{FIX})_x \quad \overset{\mathcal{T}(\cdot,p)}{\longmapsto} \quad \boxed{\begin{array}{c} \bullet\ p : \tau = \sigma \\ p : \tau_0 = \sigma_0 \\ \mathcal{T}(\mathcal{D}_0, p) \\ [(\bullet\ p\, p_i : \tau = \sigma)_i] \end{array}}$$

(for $R^{(d)} \in \{\mathrm{FOLD}_l, \mathrm{FOLD}_r, \mathrm{VAR}\}$)

$$\frac{\mathcal{D}_{01} \quad \mathcal{D}_{02}}{\tau_{01} = \sigma_{01} \qquad \tau_{02} = \sigma_{02}}{\tau_{01} \to \tau_{02} = \sigma_{01} \to \sigma_{02}}\ \text{ARROW}$$

$\overset{\mathcal{T}(\cdot,p)}{\longmapsto}$

$p: \quad \tau_1 \to \tau_2 \quad = \quad \sigma_1 \to \sigma_2$	
$p1: \ \tau_1 = \sigma_1$	$p2: \ \tau_2 = \sigma_2$
$\mathcal{T}(\mathcal{D}_{01}, p1)$	$\mathcal{T}(\mathcal{D}_{02}, p2)$

$$\frac{\langle \tau = \sigma \rangle^x \quad \langle \tau = \sigma \rangle^x \quad \mathcal{D}_{01} \quad \mathcal{D}_{02}}{\tau_{01} = \sigma_{01} \qquad \tau_{02} = \sigma_{02}}{\underbrace{\tau_{01} \to \tau_{02}}_{\equiv\tau} = \underbrace{\sigma_{01} \to \sigma_{02}}_{\equiv\sigma}}\ (\text{ARROW/FIX})_x$$

$\overset{\mathcal{T}(\cdot,p)}{\longmapsto}$

$\longmapsto$

$\bullet \ p: \overbrace{\tau_1 \to \tau_2}^{\equiv\tau} = \overbrace{\sigma_1 \to \sigma_2}^{\equiv\sigma}$	
$p1: \ \tau_1 = \sigma_1$	$p2: \ \tau_2 = \sigma_2$
$\mathcal{T}(\mathcal{D}_{01}, p1)$	$\mathcal{T}(\mathcal{D}_{02}, p2)$
$\langle (\bullet \ p\,p_i: \ \tau = \sigma)_i \rangle$	$\langle (\bullet \ p\,p_j: \ \tau = \sigma)_j \rangle$

$\boxtimes$

Theorem 14. [Main theorem in Approach II]
(Duality between derivations in e-$\mathbf{HB}_{\mathbf{0}}^{=}$ and syntactic-matching tableaux in ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$).

(i) *For every derivation* $\tilde{\mathcal{D}}$ *in* **e-$\mathbf{HB}_{\mathbf{0}}^{=}$** *with conclusion* $\tau = \sigma$ *its reflection* $\mathcal{T}(\tilde{\mathcal{D}})$ *is a closed and reduced syntactic-matching tableau for* $\tau = \sigma$ *in* **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$**.

(ii) *For every closed and reduced syntactic-matching tableau* $\tilde{\mathcal{T}}$ *for* $\varepsilon : \tau = \sigma$ *in* **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$** *its reflection* $\mathcal{D}(\tilde{\mathcal{T}})$ *is a derivation in* **e-$\mathbf{HB}_{\mathbf{0}}^{=}$** *with conclusion* $\tau = \sigma$ *and without open assumption-classes.*

(iii) *For all syntactic-matching tableaux* $\tilde{\mathcal{T}}$ *in* **ann-t-$\mathbf{SM}_{\mathbf{0}}^{=}$**

$$\mathcal{T} \circ \mathcal{D}\,(\tilde{\mathcal{T}}) = \tilde{\mathcal{T}}\,.$$

holds. And for all derivations $\tilde{\mathcal{D}}$ *in* **e-$\mathbf{HB}_{\mathbf{0}}^{=}$** *without open assumption-classes*

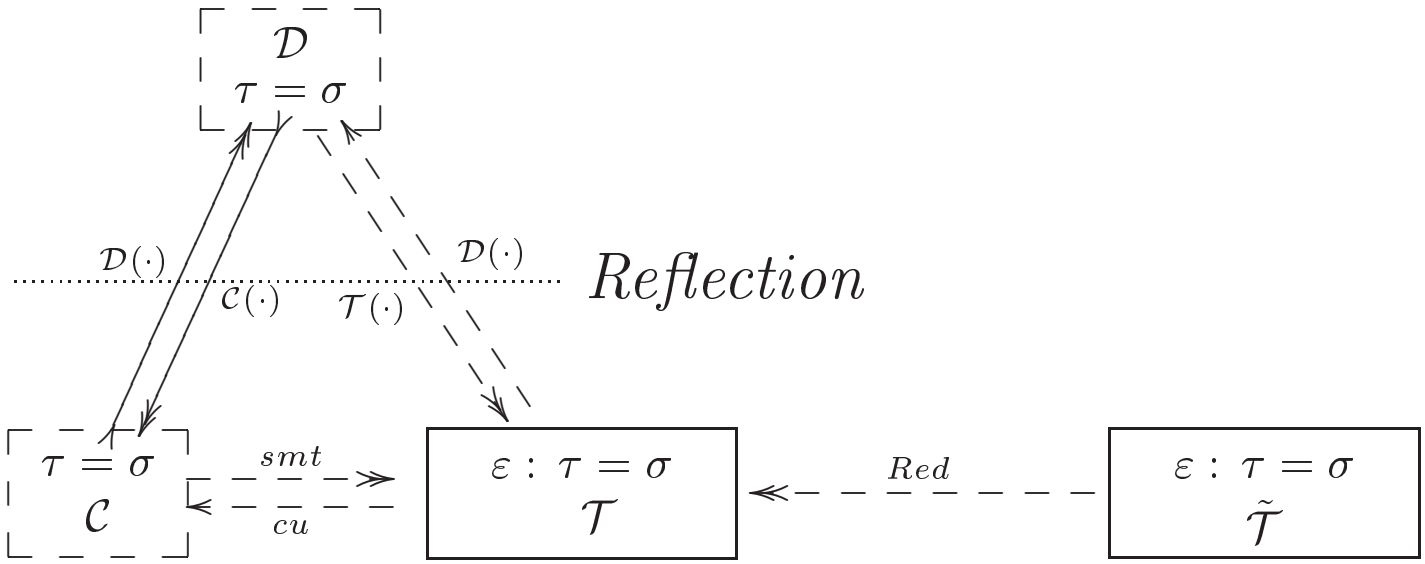
$$\mathcal{D} \circ \mathcal{T}\,(\tilde{\mathcal{D}}) \;\sim\; \tilde{\mathcal{D}}$$

[hereby "$\sim$" means:
"equality up to rearranging (redirecting) the assumption-bindings" in these **e-$\mathbf{HB}_{\mathbf{0}}^{=}$**-derivations].

is the case.

Duality between derivations in e-HB$_0^=$ with consistency-unfoldings in SM$_0^=$ as well as with tableau-proofs in ann-t-SM$_0^=$

derivation of $\tau = \sigma$ in **e-HB$_0^=$** without open assumption-classes



consistency-unfolding of $\tau = \sigma$ in **SM$_0^=$**

closed, reduced syntactic-matching tableau for $\tau = \sigma$ in **ann-t-SM$_0^=$**

(general) tableau-proof for $\tau = \sigma$ in **ann-t-SM$_0^=$**

Example 12. (Duality between a syntactic-matching tableau in ann-t-SM$_0^=$ and a derivation in e-HB$_0^=$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\tau = \sigma)^{\,x} \qquad \dfrac{}{\bot = \bot}\,(\text{REFL})}{\tau \to \bot = \sigma \to \bot}\,\text{ARROW}}{\tau = \sigma \to \bot} \qquad \dfrac{}{\bot = \bot}\,(\text{REFL})}{\tau \to \bot = (\sigma \to \bot) \to \bot}\,\text{ARROW}}{\tau \to \bot = \sigma}\,\text{FOLD}_r}{\mu\alpha.\,(\alpha \to \bot) = \mu\beta.\,((\beta \to \bot) \to \bot)}\,(\text{FOLD}_l/\text{FIX})_x$$

• $\varepsilon : \overbrace{\mu\alpha.\,(\alpha \to \bot)}^{\equiv:\,\tau} = \overbrace{\mu\beta.\,((\beta \to \bot) \to \bot)}^{\equiv:\,\sigma}$ $\varepsilon : \tau \to \bot = (\sigma \to \bot) \to \bot$	
$1 : \tau = \sigma \to \bot$ $1 : \tau \to \bot = \sigma \to \bot$	$2 : \bot = \bot$ X
• $11 : \tau = \sigma$ X	$12 : \bot = \bot$ X

■

Definition 26. (The not-annotated tableau-system t-SM$_0^=$).

Formulas: All equalities between recursive types.

Axioms: **t-SM$_0^=$** possesses **no** *axioms*.

Rules A of the form $\dfrac{\boldsymbol{\alpha}}{\boldsymbol{\alpha_1}}$ *:*

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \text{VAR}$$

$$\frac{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \sigma}{\bot = \sigma}\ (\mu\text{-}\bot)_l^{\text{der}\bot} \qquad \frac{\tau = \mu\beta\beta_1 \ldots \beta_n.\, \beta}{\tau = \bot}\ (\mu\text{-}\bot)_r^{\text{der}\bot}$$

$$\frac{\mu\alpha.\, \tau_0 = \sigma}{\tau_0[\mu\alpha.\, \tau_0/\alpha] = \sigma}\ \text{UNFOLD}_l \qquad \frac{\tau = \mu\beta.\, \sigma_0}{\tau = \sigma_0[\mu\beta.\, \sigma_0/\beta]}\ \text{UNFOLD}_r$$

Rules B of the form $\dfrac{\boldsymbol{\beta}}{\boldsymbol{\beta_1} \mid \boldsymbol{\beta_2}}$ *:*

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_1 = \sigma_1 \mid \tau_2 = \sigma_2}\ \text{DECOMP}$$

⊠

Definition 27. (Closure in t-SM$_0^{=}$-tableaux).

A branch Θ in a tableau $\mathcal{T}$ in **t-SM$_0^{=}$** is called *closed* iff Θ contains an occurrence o of a formula A such that

- A is a formula of the form $\rho = \rho$ for some $\rho \in \mu Tp$, or
- A is of the form $\rho_1 \to \rho_2 = \chi_1 \to \chi_2$ for some $\rho_1, \rho_2, \chi_1, \chi_2 \in \mu Tp$ and if Θ contains another occurrence o' of the same formula above or below o.

⊠

Definition 28. (The property D for s.-m. tableaux in ann-t-SM$_0^{=}$).

A syntactic-matching tableaux $\mathcal{T}$ in **ann-t-SM$_0^{=}$** *fulfills the property* **D** if and only if

all occurrences of formulas at the bottom of $\mathcal{T}$, for which closure happens in $\mathcal{T}$, are either axioms (REFL) or are of the form $\rho_1 \to \rho_2 = \chi_1 \to \chi_2$ for some $\rho_1, \rho_2, \chi_1, \chi_2 \in \mu Tp$.

⊠

Theorem 15. (Specializing the new duality to $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations).

It holds for arbitrary recursive types $\tau, \sigma \in \mu Tp$:

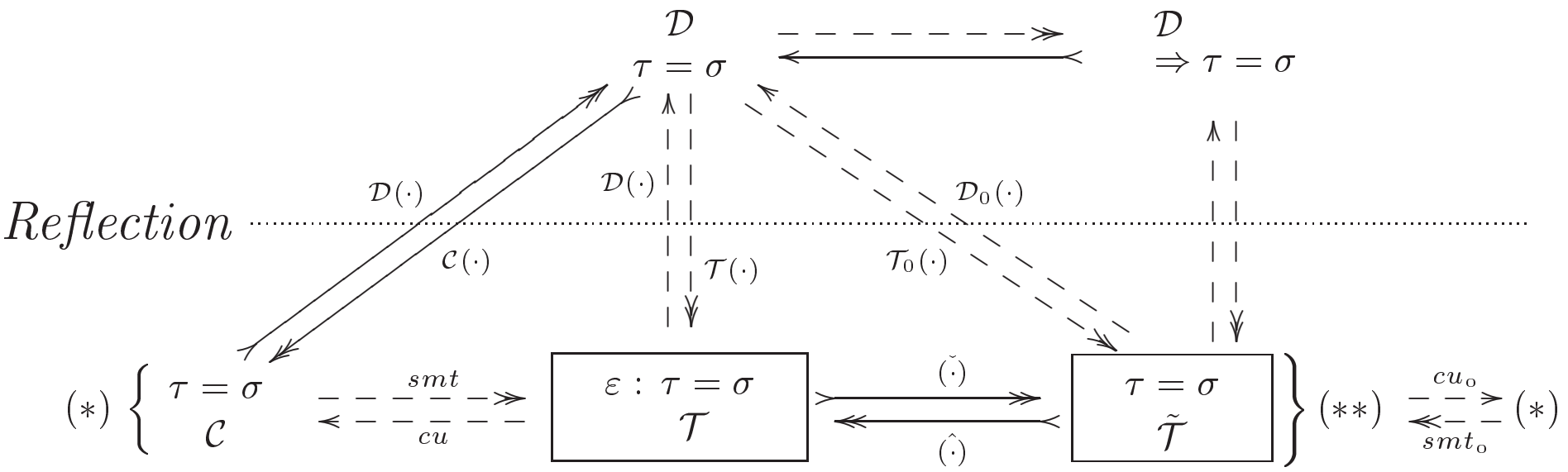
(i) For every derivation $\tilde{\mathcal{D}}$ in $\mathbf{HB}_{\mathbf{0}}^{=}$ with conclusion $\tau = \sigma$ and without open assumption-classes its reflection $\mathcal{T}(\tilde{\mathcal{D}})$ is a closed and reduced syntactic-matching tableau for $\tau = \sigma$ in $\mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$.

(ii) For every closed and reduced syntactic-matching tableau $\tilde{\mathcal{T}}$ for $\tau = \sigma$ in $\mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=}$ with the property $\mathbf{D}$ its reflection $\mathcal{T}(\tilde{\mathcal{D}})$ is a derivation in $\mathbf{HB}_{\mathbf{0}}^{=}$ with conclusion $\tau = \sigma$ and with no open assumption-classes.

(iii) Closed and reduced syntactic-matching tableaux for $\tau = \sigma$ in $\mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=}$ correspond uniquely to closed and reduced syntactic-matching tableaux for $\tau = \sigma$ in the not annotated system $\mathbf{t\text{-}SM}_{\mathbf{0}}^{=}$ via bijective un-annotation- and annotation-functions $\check{(\cdot)}$ *and* $\hat{(\cdot)}$.

derivation of $\tau = \sigma$ in $\mathbf{HB_0^=}$ without open assumption-classes

derivation of $\Rightarrow \tau = \sigma$ in the $\mathbf{G3\text{-}HB_0^=}$-Gentzen-system



consistency-unfolding of $\tau = \sigma$ in $\mathbf{SM_0^=}$ with the property **D**

closed, reduced syntactic-matching tableau for $\tau = \sigma$ in **ann-t-SM**$_\mathbf{0}^=$ with the property **D**

syntactic-matching tableau for $\tau = \sigma$ in **t-SM**$_\mathbf{0}^=$

Example 13. (Duality between a syntactic-matching tableaux in t-SM$_0^=$ and a derivation in HB$_0^=$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\tau \to \bot = (\sigma \to \bot) \to \bot}\,x}{\tau = \sigma}\qquad\dfrac{}{\bot = \bot}\,(\text{REFL})}{\tau \to \bot = \sigma \to \bot}\,\text{ARROW}}{\tau = \sigma \to \bot}\qquad\dfrac{}{\bot = \bot}\,(\text{REFL})}{\tau \to \bot = (\sigma \to \bot) \to \bot}\,(\text{ARR./FIX})_x}{\underbrace{\mu\alpha.\,(\alpha \to \bot)}_{\equiv:\,\tau} = \underbrace{\mu\beta.\,((\beta \to \bot) \to \bot)}_{\equiv:\,\sigma}}\,\text{FOLD}_{l/r}$$

$\mu\alpha.\,(\alpha \to \bot) = \mu\beta.\,((\beta \to \bot) \to \bot)$ $\tau \to \bot = \sigma$ • $\tau \to \bot = (\sigma \to \bot) \to \bot$		
$\tau = \sigma \to \bot$ $\tau \to \bot = \sigma \to \bot$		$\bot = \bot$ X
$\tau = \sigma$ $\tau \to \bot = \sigma$ • $\tau \to \bot = (\sigma \to \bot) \to \bot$ X	$\bot = \bot$ X	

■

What do the *dualities* show/imply?

- They tie the concepts *"consistency in* $\mathbf{SM}_{\mathbf{0}}^{=}$*"* and *"derivability in* $\mathbf{HB}_{\mathbf{0}}^{=}$*"* closely together. – It is an immediate consequence of the *dualities* that proving soundness and completeness for $\mathbf{HB}_{\mathbf{0}}^{=}$ is essentially the same as proving:

  For all $\tau, \sigma \in \mu Tp$:

$$\tau = \sigma \text{ is } \mathbf{SM}_{\mathbf{0}}^{=}\text{-consistent} \iff (\exists\, \mathcal{C})\,(\,\mathcal{C} \text{ is a consistency-unfolding of } \tau = \sigma \text{ in } \mathbf{SM}_{\mathbf{0}}^{=}\,). \tag{4}$$

$$(\iff (\exists\, \mathcal{T})\,[\,\mathcal{T} \text{ is a syntactic-matching tableau for } \varepsilon :\ \tau = \sigma \text{ in } \mathbf{ann\text{-}t\text{-}SM}_{\mathbf{0}}^{=}\,.\,])$$

- The easily graspable (but rather tedious) proof of "$\Leftarrow$" in (4) therefore leads directly to a soundness-proof for $\mathbf{HB}_{\mathbf{0}}^{=}$ independent from the one in [BrHe98].
- This soundness-proof for $\mathbf{HB}_{\mathbf{0}}^{=}$ can be extended to a soundness-proof for $\mathbf{HB}^{=}$. (But this requires me to use concepts from the completeness-proof in [BrHe98].)
- *"Non-Result"*: Showing "$\Rightarrow$" in (4) is essentially the same problem as proving the completeness of $\mathbf{HB}_{\mathbf{0}}^{=}$ w.r.t. $=_{\mu}$. Hence the *duality* does not give rise to a completeness-proof for $\mathbf{HB}_{\mathbf{0}}^{=}$ or for $\mathbf{HB}^{=}$, which were independent from the concepts used in the proof given by [BrHe98] for $\mathbf{HB}^{=}$.

Summary

- **Preliminaries** on recursive types. Recursive Type Equality $=_\mu$.
- The sound and complete axiom-systems $\mathbf{AC}^{=}$ and $\mathbf{HB}^{=}$ for $=_\mu$ and the syntactic-matching system $\mathbf{SM}^{=}$ for consistency-checking with respect to $=_\mu$.
- **Observation by J.W. Klop** of a close connection through "reflection" between a successful consistency-check in $\mathbf{SM}_{\mathbf{0}}^{=}$ and a derivation in $\mathbf{HB}_{\mathbf{0}}^{=}$ in an example.
- **Approach I: Formalization with "Consistency-Unfoldings"**
  - The conservative extension $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}_{\mathbf{0}}^{=}$.
  - Notions: F.t.o.c.'s, f.t.o.c.'s with m.f. and with b.l.o.m.f., (partial) consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$.
  - Reflection-operations $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$.
  - Main Theorem: A *duality* induced by $\mathcal{C}(\cdot)$ and $\mathcal{D}(\cdot)$ between $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$-derivations and consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$.
  - *Finer picture of this duality*: A Duality also between $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations and *such* consistency-unfoldings in $\mathbf{SM}_{\mathbf{0}}^{=}$, that fulfil the property $\mathbf{D}$.

- **Approach II: Formalization with "Syntactic-Matching Tableaux"**
  - 'Analytic' Tableaux as considered by R. Smullyan. Relation to consistency-unfoldings.
  - The annotated tableau-system **ann-t-SM$_0^=$**. Syntactic-Matching Tableaux.
  - Reflection-operations $\mathcal{D}(\cdot)$ and $\mathcal{T}(\cdot)$.
  - New version of the main theorem: A *duality* induced by $\mathcal{D}(\cdot)$ and $\mathcal{T}(\cdot)$ between syntactic-matching tableaux in **ann-t-SM$_0^=$** and derivations in **e-HB$_0^=$**.
  - *Specializing this new duality-statement to* **HB$_0^=$***-derivations*: A Duality between syntactic-matching tableaux in **t-SM$_0^=$** and derivations in **HB$_0^=$**.
- **Consequences of the duality**.

The complete set of the LaTeX-slides presented here is available at:

`http://www.cs.vu.nl/~clemens/PAM.ps.gz`

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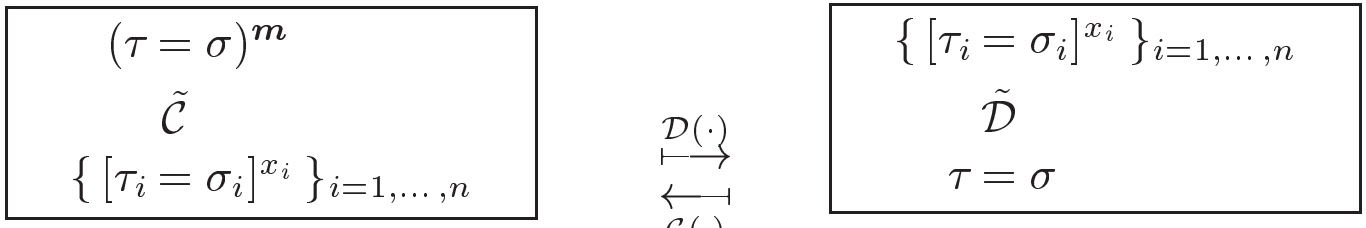
Lemma 16. (Well-Definedness of the reflection-operations $\mathcal{D}(\cdot)$ and $\mathcal{C}(\cdot)$ in Definitions 18 and 19).

(i) For every f.t.o.c. with b.l.o.m.f. $\tilde{\mathcal{C}}$ in $\mathbf{SM}_{\mathbf{0}}^{=}$ *its reflection $\mathcal{D}(\tilde{\mathcal{C}})$ is a "pseudo-derivation" in* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$.

"Pseudo-Derivation" in $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*: Derivation in a variant-system of* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*, which differs from* $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ *only insofar, as that the side-condition* $\mathbf{C}$ *on $R^{(d)}$/FIX-applications for $R^{(d)} \in \{FOLD_{l/r}, VAR\}$ may be violated.*

(ii) For every $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$*-derivation $\tilde{\mathcal{D}}$ with the conclusion $\tau = \sigma$ and with (possibly) open assumption-classes its reflection $\mathcal{C}(\tilde{\mathcal{D}})$ is a f.t.o.c. of $\tau = \sigma$ in* $\mathbf{SM}_{\mathbf{0}}^{=}$ *with b.l.o.m.f..*

This is sketched in the picture:



f.t.o.c. with b.l.o.m.f. of $\tau = \sigma$ in $\mathbf{SM}_{\mathbf{0}}^{=}$

"pseudo-derivation" of $\tau = \sigma$ in $\mathbf{e\text{-}HB}_{\mathbf{0}}^{=}$ with open assumption-classes