Proof-Theoretic Interconnections between Known Inference- and Axiom-Systems for the Equality and Subtype Relations on Recursive Types

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Recursive Types

Recursive Types μTp:

$$\tau ::= \alpha \mid \bot \mid \top \mid \tau_1 \to \tau_2 \mid \mu\alpha.\, \tau_0 \quad (\alpha \in TVar).$$

Recursive Types in canonical form can-μTp:

$$\tau ::= \alpha \mid \bot \mid \top \mid \tau_1 \to \tau_2 \mid \mu\alpha.\, \underbrace{\tau_1 \to \tau_2}_{\text{where } \alpha \in \mathrm{fv}(\tau_1 \to \tau_2)} \qquad (\alpha \in TVar).$$

Definition 1. (Recursive Type Equality $=_\mu$
(Strong Recursive Type Equivalence))

For all $\tau, \sigma \in \mu Tp$:

$$\tau =_\mu \sigma \;:\Longleftrightarrow\; \mathbf{Tree}(\tau) = \mathbf{Tree}(\sigma)\,.$$

$\boxtimes$

Definition 3. (Type-Trees $TpTrees =$ = Labelled Binary Trees).

A *type-tree* t is a partial function

$$t \,:\, \{0,1\}^* \longrightarrow \{\bot, \top, \to\} \cup TVar$$

with domain $\mathcal{D}(t) := \mathrm{dom}(t)$ s.th.

(i) $\mathcal{D}(t) \neq \emptyset$ and $\mathcal{D}(t)$ is prefix-closed, i.e.

$$\big(\forall\, p, p_1, p_2 \in \{0,1\}^*\big) \\ \big[\, p \equiv p_1 p_2 \;\wedge\; p \in \mathcal{D}(t) \;\Rightarrow\; p_1 \in \mathcal{D}(t) \,\big]\,.$$

(ii) $(\forall\, p \in \{0,1\}^*) \Big[\big(\, t(p) \equiv \to \;\Rightarrow\; p0, p1 \in \mathcal{D}(t) \,\big) \wedge$

$$\wedge \big(\, t(p) \in \{\bot, \top\} \cup TVar \;\Rightarrow\; p0, p1 \notin \mathcal{D}(t) \,\big) \Big]\,.$$

The set of all type-trees is denoted by $TpTrees$.

Definition 6. (The axiom-system $\mathbf{AC}^=$ for recursive type equality $=_\mu$)

Axioms:

(REFL) $\overline{\tau = \tau}$ (VAR) $\overline{\tau = \tau'}$ (if $\tau \equiv_{\mathrm{v}} \tau'$)

$(\mu - \bot)$ $\overline{\mu\alpha.\,\alpha = \bot}$ (FOLD/UNFOLD) $\overline{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau_1 = \sigma_1 \qquad \tau_2 = \sigma_2}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \text{ARROW}$$

$$\frac{\tau = \sigma}{\mu\alpha.\,\tau = \mu\alpha.\,\sigma}\ \mu\text{-COMPAT} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS}$$

$$\frac{\tau_1 = \tau[\tau_1/\alpha] \qquad \tau_2 = \tau[\tau_2/\alpha]}{\tau_1 = \tau_2}\ \text{UFP} \quad (\text{if } \alpha \downarrow \tau)$$

(where $\alpha \downarrow \tau$ means that τ is *contractive* w.r.t. α). ☒

Theorem 2 (R. Amadio, L. Cardelli, '93). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau =_\mu \sigma \iff \vdash_{\mathbf{AC}^=} \tau = \sigma\ .$$

Notation. The notation $\alpha \downarrow \tau$ (α is *contractive* in τ) carries the meaning:

$$\big[\, \alpha \notin \mathrm{fv}(\tau) \ \text{ or } \ \tau \equiv \mu\alpha_1 \ldots \alpha_n.\, (\rho_1 \to \rho_2) \ \text{(for some } n \in \mathbb{N}_0 \text{ and } \rho_1, \rho_2 \in \mu Tp \text{ and variables } \alpha_1 \ldots \alpha_n \in TVar) \,\big] .$$

$\boxtimes$

Lemma 3. *For all $\tau \in \mu Tp$ and variables α:*

$$\alpha \not\downarrow \tau \iff \begin{bmatrix} \tau \equiv \mu\alpha_1 \ldots \alpha_n.\, \alpha \ \textit{ for some } n \in \mathbb{N}_0 \\ \textit{and variables } \alpha_1, \ldots, \alpha_n \textit{ distinct from } \alpha. \end{bmatrix} .$$

Definition 7 (The variant-system $\mathbf{AC}^{=}_{\mathbf{1}}$ of $\mathbf{AC}^{=}$).

Axioms: All axioms of $\mathbf{AC}^{=}$ $\boxed{\textit{plus}}$

$$(\mu - \bot)' \ \ \overline{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \bot}$$

Rules: All rules of $\mathbf{AC}^{=}$ $\boxed{\textit{except}}$ μ-COMPAT. $\boxtimes$

The axioms $(\mu - \bot)'$ are "more general" than the axioms $(\mu - \bot)$: $\overline{\mu\alpha.\, \alpha = \bot}$ of $\mathbf{AC}^{=}$. In particular: Every equality

$$\mu\alpha.\, \tau_0 = \bot \quad (\text{if } \alpha \not\downarrow \tau_0)$$

(with arbitrary variables α and $\tau_0 \in \mu Tp$) is an axiom of the sort $(\mu - \bot)'$ of $\mathbf{AC}^{=}_{\mathbf{1}}$.

Frequently used derived rules of $\mathbf{AC}^{=}$:

$$\frac{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}{\mu\alpha.\,\tau_0 = \sigma}\ \mathrm{FOLD}_l \qquad \frac{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}{\tau = \mu\beta.\,\sigma_0}\ \mathrm{FOLD}_r$$

$$\frac{\tau = \sigma}{\tau' = \sigma}\ \mathrm{VAR}_l \qquad \frac{\tau = \sigma}{\tau = \sigma'}\ \mathrm{VAR}_r \qquad \frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR}$$

(whenever it holds that $\tau' \equiv_{\mathrm{v}} \tau$ and $\sigma' \equiv_{\mathrm{v}} \sigma$)

$$\frac{\mu\alpha.\,\tau_0 = \sigma}{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}\ \mathrm{UNFOLD}_l$$

$$\frac{\tau = \mu\beta.\,\sigma_0}{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}\ \mathrm{UNFOLD}_r$$

$$\frac{\perp = \sigma}{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}\ (\mu-\perp)_l^{\mathrm{inf}} \qquad + \ (\mu-\perp)_r^{\mathrm{inf}}$$

$$(\mu-\perp)_l^{\mathrm{der}}\ + \qquad \frac{\tau = \mu\beta\beta_1\ldots\beta_n.\,\beta}{\tau = \perp}\ (\mu-\perp)_r^{\mathrm{der}}$$

$$\frac{\tau_1 = \tau[\tau_1/\alpha]}{\tau_1 = \mu\alpha.\,\tau}\ \mathrm{CONTRACT}\ \ (\text{if } \alpha\downarrow\tau)$$

Definition 8. (The axiom-system $\mathbf{HB}^{=}$ for recursive type equality $=_\mu$)

Axioms/Marked assumptions:

$$\text{(REFL)}\ \frac{}{\tau = \tau} \qquad (\mu - \bot)'\ \frac{}{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha = \bot}$$

$$\text{(VAR)}\ \frac{}{\tau = \tau'}\ (\text{if } \tau \equiv_{\mathrm{v}} \tau') \qquad \text{(FOLD/UNFOLD)}\ \frac{}{\mu\alpha.\, \tau = \tau[\mu\alpha.\, \tau/\alpha]}$$

$$\text{(HYP)} \quad (\tau = \sigma)^x$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\,\text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\,\text{TRANS}$$

$$\frac{\begin{array}{c}\langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_1 \\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}\langle \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_2 \\ \tau_2 = \sigma_2\end{array}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\,(\text{ARROW/FIX})_x$$

⊠

Theorem 4 (M. Brandt (and F. Henglein), '98).

For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau =_\mu \sigma \iff \vdash_{\mathbf{HB}^{=}} \tau = \sigma\,.$$

Definition 9. (The syntactic-matching inference-system $\mathbf{SM}^{=}$ for consistency-checks concerning recursive type equality $=_\mu$)

Axioms:

$$\text{(REFL)}\ \frac{}{\tau = \tau} \qquad (\mu - \bot)'\ \frac{}{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \bot}$$

$$\text{(VAR)}\ \frac{}{\tau = \tau'}\ (\text{if } \tau \equiv_{\mathrm{v}} \tau') \qquad \text{(FOLD/UNFOLD)}\ \frac{}{\mu\alpha.\,\tau = \tau[\mu\alpha.\,\tau/\alpha]}$$

Rules:

$$\frac{\sigma = \tau}{\tau = \sigma}\ \text{SYMM} \qquad \frac{\tau = \rho \qquad \rho = \sigma}{\tau = \sigma}\ \text{TRANS}$$

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \text{DECOMP}\ \ (i = 1, 2)$$

$\boxtimes$

Definition 10. For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau = \sigma \text{ is } \mathbf{SM}^{=}\textit{-consistent} \ :\iff$$
$$\iff \text{For all } \chi_1, \chi_2 \in \mu Tp: \quad \tau = \sigma \vdash_{\mathbf{SM}^{=}} \chi_1 = \chi_2 \implies \mathcal{L}'(\chi_1) \equiv \mathcal{L}'(\chi_2)\,.$$

Hereby for all $\tilde{\tau} \in \mu Tp$ the notation $\mathcal{L}'(\tilde{\tau})$ designates a slight variant of the *leading symbol* $\mathcal{L}(\tilde{\tau})$ of $\tilde{\tau}$. $\boxtimes$

Theorem 5 (($\sim$) Z.M. Ariola, J.W. Klop, '95). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau =_\mu \sigma \iff \tau = \sigma \textit{ is } \mathbf{SM}^{=}\textit{-consistent}.$$

Definition 12. (The axiom-system $\mathbf{HB}_0^{=}$ for recursive type equality $=_\mu$)

Axs./M. Assms.: (REFL) $\overline{\tau = \tau}$ (HYP) $(\tau = \sigma)^x$

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR} \qquad \frac{\bot = \sigma}{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}\ (\mu-\bot)_l^{\mathrm{inf}} + (\mu-\bot)_r^{\mathrm{inf}}$$

$$\frac{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}{\mu\alpha.\,\tau_0 = \sigma}\ \mathrm{FOLD}_l \qquad \frac{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}{\tau = \mu\beta.\,\sigma_0}\ \mathrm{FOLD}_r$$

$$\cfrac{\begin{matrix}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_1 \\ \tau_1 = \sigma_1\end{matrix} \qquad \begin{matrix}\langle\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^x \\ \mathcal{D}_2 \\ \tau_2 = \sigma_2\end{matrix}}{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ (\mathrm{ARROW/FIX})_x$$

$\boxtimes$

Definition 13. (The axiom-system $\mathbf{SM}_0^{=}$ for recursive type equality $=_\mu$)

Rules:

$$\frac{\tau = \sigma}{\tau' = \sigma'}\ \mathrm{VAR} \qquad \frac{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}{\bot = \sigma}\ (\mu-\bot)_l^{\mathrm{der}} + (\mu-\bot)_r^{\mathrm{der}}$$

$$\frac{\mu\alpha.\,\tau_0 = \sigma}{\tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}\ \mathrm{UNFOLD}_l \qquad \frac{\tau = \mu\beta.\,\sigma_0}{\tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}\ \mathrm{UNFOLD}_r$$

$$\frac{\tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}{\tau_i = \sigma_i}\ \mathrm{DECOMP}\quad (i = 1, 2)$$

$\boxtimes$

Example 3 ("Consistency Check" in $\mathbf{SM}^{=}_{\mathbf{0}}$).

$$\begin{array}{c}
\left(\overbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}^{\equiv:\,\tau} = \overbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}^{\equiv:\,\sigma}\right)^{y} \\
\hline\hline
(\tau \to \tau) \to \tau = \sigma \to (\sigma \to \sigma) \\
\hdashline
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{c}
(\tau \to \tau = \sigma)^{x} \\ \hline \tau \to \tau = \sigma \to (\sigma \to \sigma) \\ \hdashline
\begin{array}{cc}
(\tau = \sigma)^{y} & \begin{array}{c} \tau = \sigma \to \sigma \\ \hline (\tau \to \tau) \to \tau = \sigma \to \sigma \\ \hdashline (\tau \to \tau = \sigma)^{x} \qquad (\tau = \sigma)^{y} \end{array}
\end{array}
\end{array}
& \qquad &
\begin{array}{c}
(\tau = \sigma \to \sigma)^{z} \\ \hline (\tau \to \tau) \to \tau = \sigma \to \sigma \\ \hdashline
\begin{array}{cc}
\begin{array}{c} \tau \to \tau = \sigma \\ \hline \tau \to \tau = \sigma \to (\sigma \to \sigma) \\ \hdashline (\tau = \sigma)^{y} \qquad (\tau = \sigma \to \sigma)^{z} \end{array} & (\tau = \sigma)^{y}
\end{array}
\end{array}
\end{array}$$

$\blacksquare$

Example 4 (Proof in $\mathbf{HB}_{\mathbf{0}}^{=}$).

$$
\begin{prooftree}
\AxiomC{$(\ldots)^y$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\AxiomC{$(\ldots)^x$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\ldots)^y$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to \sigma$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\RightLabel{x}
\BinaryInfC{$\tau \to \tau = \sigma \to (\sigma \to \sigma)$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\ldots)^y$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\AxiomC{$(\ldots)^z$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\BinaryInfC{$\tau \to \tau = \sigma \to (\sigma \to \sigma)$}
\UnaryInfC{$\tau \to \tau = \sigma$}
\AxiomC{$(\ldots)^y$}
\doubleLine
\UnaryInfC{$\tau = \sigma$}
\RightLabel{z}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to \sigma$}
\UnaryInfC{$\tau = \sigma \to \sigma$}
\RightLabel{y}
\BinaryInfC{$(\tau \to \tau) \to \tau = \sigma \to (\sigma \to \sigma)$}
\doubleLine
\UnaryInfC{$\underbrace{\mu\alpha.\,((\alpha \to \alpha) \to \alpha)}_{\equiv:\,\tau} = \underbrace{\mu\alpha.\,(\alpha \to (\alpha \to \alpha))}_{\equiv:\,\sigma}$}
\end{prooftree}
$$

$\blacksquare$

Definition 17. (The annotated version ann-$\mathbf{HB}_{\mathbf{0}}^{=}$ of the system $\mathbf{HB}_{\mathbf{0}}^{=}$).

Axioms/Marked Assumptions:

$$\text{(REFL)}\ \frac{}{\tau:\ \tau = \tau} \qquad\qquad \text{(HYP)}\ \big(\alpha:\ \tau = \sigma\big)^{\alpha}$$

Rules:

$$\frac{\chi:\ \tau = \sigma}{\chi:\ \tau' = \sigma'}\ \text{VAR} \quad (\text{where } \tau' \equiv_{\mathrm{v}} \tau \text{ and } \sigma' \equiv_{\mathrm{v}} \sigma)$$

$$\frac{\chi:\ \bot = \sigma}{\chi:\ \mu\alpha\alpha_1\ldots\alpha_n.\,\alpha = \sigma}\ (\mu-\bot)_l^{\mathrm{inf}} \qquad \frac{\chi:\ \tau = \bot}{\chi:\ \tau = \mu\beta\beta_1\ldots\beta_n.\,\beta}\ (\mu-\bot)_r^{\mathrm{inf}}$$

$$\frac{\chi:\ \tau_0[\mu\alpha.\,\tau_0/\alpha] = \sigma}{\chi:\ \mu\alpha.\,\tau_0 = \sigma}\ \text{FOLD}_l \qquad \frac{\chi:\ \tau = \sigma_0[\mu\beta.\,\sigma_0/\beta]}{\chi:\ \tau = \mu\beta.\,\sigma_0}\ \text{FOLD}_r$$

$$\begin{array}{l}\text{(if side-}\\ \text{condition } \mathbf{S_1})\end{array}\ \frac{\begin{array}{c}\mathcal{D}_1\\ \chi_1:\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \chi_2:\ \tau_2 = \sigma_2\end{array}}{\chi_1 \to \chi_2:\ \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ \text{ARROW}$$

$$\begin{array}{l}\text{(if side-}\\ \text{condition } \mathbf{S_2})\end{array}\ \frac{\begin{array}{c}\langle\alpha:\ \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^{\alpha}\\ \mathcal{D}_1\\ \chi_1:\ \tau_1 = \sigma_1\end{array} \qquad \begin{array}{c}\langle\alpha:\ \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2\rangle^{\alpha}\\ \mathcal{D}_2\\ \chi_2:\ \tau_2 = \sigma_2\end{array}}{\mu\alpha.\,\chi_1 \to \chi_2:\ \tau_1 \to \tau_2 = \sigma_1 \to \sigma_2}\ (\text{AR./F.})_{\alpha}$$

A *type-tree* $t \in TpTrees$ is a partial function

$$t \,:\, \{0,1\}^* \longrightarrow \{\bot, \top, \rightarrow\} \cup TVar$$

with domain $\mathcal{D}(t) := \mathrm{dom}(t)$ s.th. … [cf. Definition 3].

Definition 18 (Partial order $\leq_T$ on Type-Trees).

The partial order $\leq_T$ on $TpTrees$ is for all type-trees t and s defined by

$$t \leq_T s \;\; :\Longleftrightarrow \;\; \big(\forall\, p \in \mathcal{D}(t) \cap \mathcal{D}(s)\big)\,[\, t(p) \leq_{\pi(p)} s(p)\,] \;\; ,$$

where the function π is defined for all $p \in \{0,1\}^*$ as

$$\pi(p) := (\#\{0\,\text{'s occuring in } p\}) \;\bmod 2\,,$$

and the partial orders $\leq_0$ and $\leq_1$ on $\{\bot, \top, \rightarrow\} \cup TVar$ are given by the respective transitive closure of the below two inequalities (for all $\alpha \in TVar$):

$$\bot \leq_0 \left\{ \begin{matrix} \rightarrow \\ \alpha \end{matrix} \right\} \leq_0 \top \quad\quad \text{and} \quad\quad \top \leq_1 \left\{ \begin{matrix} \rightarrow \\ \alpha \end{matrix} \right\} \leq_1 \bot \;.$$

⊠

Definition 19. [Kozen, Palsberg, Schwartzbach] (Subtyping relation $\leq_\mu$ on recursive types).

For all $\tau, \sigma \in \mu Tp$:

$$\tau \leq_\mu \sigma \;\;:\Longleftrightarrow\;\; \mathbf{Tree}(\tau) \leq_T \mathbf{Tree}(\sigma) \;.$$

⊠

Definition 20. (The axiom-system $\mathbf{AC}^{\leq}$ for the subtyping relation $\leq_\mu$ between recursive types)

Axioms/Marked Assumptions:

$$\{\,\text{Axioms of } \mathbf{AC}^{=}\,\} +$$

$$(\text{BOT})\ \overline{\bot \leq \sigma} \qquad (\text{TOP})\ \overline{\tau \leq \top} \qquad (\text{HYP})\ (\alpha \leq \beta)^x$$

Rules:

$$\{\,\text{Rules of } \mathbf{AC}^{=}\,\} +$$

$$\frac{\tau \leq \rho \qquad \rho \leq \sigma}{\tau \leq \sigma}\ \text{TRANS} \qquad \frac{\sigma_1 \leq \tau_1 \qquad \tau_2 \leq \sigma_2}{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}\ \text{ARROW}$$

$$\frac{\tau = \sigma}{\tau \leq \sigma}\ \text{EQUAL} \qquad \frac{\begin{array}{c}[\alpha \leq \beta]^x \\ \mathcal{D}_0 \\ \tau \leq \sigma\end{array}}{\mu\alpha.\,\tau \leq \mu\beta.\,\sigma}\ (\mu\text{-INTRO})_x \ \begin{array}{l}\text{(if side-}\\ \text{condition S)}\end{array}$$

where the *side-condition* S to the rule μ-INTRO is defined as follows:

> $\alpha \notin \text{fv}(\sigma)$, $\beta \notin \text{fv}(\tau)$ and α and β do not occur in any undischarged marked assumption $(\gamma \leq \delta)^y$ of $\mathcal{D}_0$, which does not belong to the assumption-class $[\alpha \leq \beta]^x$ of $\mathcal{D}_0$.

⊠

Theorem 18 (R. Amadio, L. Cardelli, '93). *For all recursive types $\tau, \sigma \in \mu Tp$:*

$$\tau \leq_\mu \sigma \iff \vdash_{\mathbf{AC}^{\leq}} \tau \leq \sigma\ .$$

Definition 21. (The axiom-system $\mathbf{HB}^{\leq}$ for the subtyping relation $\leq_\mu$ between recursive types)

Axioms/Marked Assumptions:

(BOT) $\overline{\bot \leq \sigma}$ (VAR) $\overline{\tau \leq \tau'}$ (if $\tau \equiv_{\mathrm{v}} \tau'$)

(TOP) $\overline{\tau \leq \top}$ $(\mu - \bot)'$ $\overline{\mu\alpha\alpha_1 \ldots \alpha_n.\, \alpha \leq \bot}$ $(n \in \mathbb{N}_0)$

(REFL) $\overline{\tau \leq \tau}$ $(\text{FOLD/UNFOLD})_1$ $\overline{\mu\alpha.\, \tau_0 \leq \tau_0[\mu\alpha.\, \tau_0/\alpha]}$

(HYP) $(\tau \leq \sigma)^x$ $(\text{FOLD/UNFOLD})_2$ $\overline{\tau_0[\mu\alpha.\, \tau_0/\alpha] \leq \mu\alpha.\, \tau_0}$

Rules:

$$\frac{\tau \leq \rho \qquad \rho \leq \sigma}{\tau \leq \sigma}\,\text{TRANS} \qquad \frac{\sigma_1 \leq \tau_1 \qquad \tau_2 \leq \sigma_2}{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}\,\text{ARROW}$$

$$\frac{\begin{array}{c}\langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_1 \\ \sigma_1 \leq \tau_1\end{array} \qquad \begin{array}{c}\langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_2 \\ \tau_2 \leq \sigma_2\end{array}}{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}\,(\text{ARROW/FIX})_x$$

⊠

Theorem 19. (M. Brandt and F. Henglein, '97)
For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau \leq_\mu \sigma \iff \vdash_{\mathbf{HB}^{\leq}} \tau \leq \sigma\,.$$

Definition 22. (The syntactic-matching inference-system $\mathbf{SM}^{\leq}$ for consistency-checks concerning the subtyping relation $\leq_\mu$ between recursive types)

Axioms:

$$\text{(BOT)}\ \overline{\bot \leq \tau} \qquad \text{(VAR)}\ \overline{\tau \leq \tau'} \quad (\text{if } \tau \equiv_{\mathrm{v}} \tau')$$

$$\text{(TOP)}\ \overline{\top \leq \sigma} \qquad (\mu-\bot)'\ \overline{\mu\alpha\alpha_1\ldots\alpha_n.\,\alpha \leq \bot} \quad (n \in \mathbb{N}_0)$$

$$\text{(REFL)}\ \overline{\tau \leq \tau} \qquad \text{(FOLD/UNFOLD)}_1\ \overline{\mu\alpha.\,\tau_0 \leq \tau_0[\mu\alpha.\,\tau_0/\alpha]}$$

$$\text{(FOLD/UNFOLD)}_2\ \overline{\tau_0[\mu\alpha.\,\tau_0/\alpha] \leq \mu\alpha.\,\tau_0}$$

Rules:

$$\frac{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}{\sigma_1 \leq \tau_1}\,\mathrm{DECOMP}_l \qquad \frac{\tau \leq \rho \qquad \rho \leq \sigma}{\tau \leq \sigma}\,\mathrm{TRANS}$$

$$\frac{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}{\tau_2 \leq \sigma_2}\,\mathrm{DECOMP}_r$$

⊠

Definition 23. For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau \leq \sigma \text{ is } \mathbf{SM}^{\leq}\text{-}\textit{consistent} \ :\Longleftrightarrow\ \text{For all } \chi_1, \chi_2 \in \mu Tp:$$
$$\Big[\tau \leq \sigma \vdash_{\mathbf{SM}^{\leq}} \chi_1 \leq \chi_2 \ \Rightarrow\ \mathcal{L}'(\chi_1) \leq_0 \mathcal{L}'(\chi_2)\Big].$$

Theorem 20 ($\sim$ Theorem 5 about $\mathbf{SM}^{=}$).

For all recursive types $\tau, \sigma \in \mu Tp$:

$$\tau \leq_\mu \sigma \iff \tau \leq \sigma \textit{ is } \mathbf{SM}^{\leq}\textit{-consistent}.$$

Definition 24. (The axiom-system $\mathbf{HB}_{\mathbf{0}}^{\leq}$ for the subtyping relation $\leq_\mu$ between recursive types)

Axs./M. Assms.: (REFL) $\dfrac{}{\tau \leq \tau}$ (HYP) $(\tau \leq \sigma)^x$

(BOT) $\dfrac{}{\perp \leq \sigma}$ (TOP) $\dfrac{}{\tau \leq \top}$

Rules:

$$\frac{\tau \leq \sigma}{\tau' \leq \sigma'}\,\mathrm{VAR} \qquad \frac{\perp \leq \sigma}{\mu\alpha\alpha_1 \ldots \alpha_n .\, \alpha \leq \sigma}\,(\mu - \perp)_l^{\mathrm{inf}} + (\mu - \perp)_r^{\mathrm{inf}}$$

$$\frac{\tau_0[\mu\alpha.\,\tau_0/\alpha] \leq \sigma}{\mu\alpha.\,\tau_0 \leq \sigma}\,\mathrm{FOLD}_l \qquad \frac{\tau \leq \sigma_0[\mu\beta.\,\sigma_0/\beta]}{\tau \leq \mu\beta.\,\sigma_0}\,\mathrm{FOLD}_r$$

$$\frac{\begin{array}{c}\langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_1 \\ \sigma_1 \leq \tau_1\end{array} \qquad \begin{array}{c}\langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_2 \\ \tau_2 \leq \sigma_2\end{array}}{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}\,(\mathrm{ARROW/FIX})_x$$

$\boxtimes$

Definition 25. (The checking-system $\mathbf{SM}_{\mathbf{0}}^{\leq}$ for the subtyping relation $\leq_\mu$ between recursive types)

Rules:

$$\frac{\tau \leq \sigma}{\tau' \leq \sigma'}\,\mathrm{VAR} \qquad \frac{\mu\alpha\alpha_1 \ldots \alpha_n .\, \alpha \leq \sigma}{\perp \leq \sigma}\,(\mu - \perp)_l^{\mathrm{der}} + (\mu - \perp)_r^{\mathrm{der}}$$

$$\frac{\mu\alpha.\,\tau_0 \leq \sigma}{\tau_0[\mu\alpha.\,\tau_0/\alpha] \leq \sigma}\,\mathrm{UNFOLD}_l \qquad \frac{\tau \leq \mu\beta.\,\sigma_0}{\tau \leq \sigma_0[\mu\beta.\,\sigma_0/\beta]}\,\mathrm{UNFOLD}_r$$

$$\frac{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}{\sigma_1 \leq \tau_1}\,\mathrm{DECOMP}_{l/r}\,\frac{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2}{\tau_2 \leq \sigma_2}$$

$\boxtimes$

Definition 26. (The variant-system $\mathbf{HB}^{\leq}_{\mathbf{0},\mathbf{rst}}$ of $\mathbf{HB}^{\leq}_{\mathbf{0}}$).

The variant-system $\mathbf{HB}^{\leq}_{\mathbf{0},\mathbf{rst}}$ of $\mathbf{HB}^{\leq}_{\mathbf{0}}$ with only <u>r</u>e<u>st</u>ricted applications of ARROW/FIX

$$\text{(if side-condition } \mathbf{S}\text{)} \quad \frac{\begin{matrix} \langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_1 \\ \sigma_1 \leq \tau_1 \end{matrix} \qquad \begin{matrix} \langle \tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2 \rangle^x \\ \mathcal{D}_2 \\ \tau_2 \leq \sigma_2 \end{matrix}}{\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2} \ (\text{ARR./F.})_x$$

has the same axioms and rules as $\mathbf{HB}^{\leq}_{\mathbf{0}}$, but all applications of ARROW/FIX in this system are subject to side-condition S :

(i) For every undischarged assm. $(\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2)^x$ in $\mathcal{D}_1$ the thread downwards to the conclusion $\sigma_1 \leq \tau_1$ of $\mathcal{D}_1$ crosses the *left* premise of an application of ARROW or ARROW/FIX an *odd* number of times.

(ii) For every undischarged assm. $(\tau_1 \to \tau_2 \leq \sigma_1 \to \sigma_2)^x$ in $\mathcal{D}_2$ the thread downwards to the conclusion $\tau_2 \leq \sigma_2$ of $\mathcal{D}_2$ crosses the *left* premise of an application of ARROW or ARROW/FIX an *even* number of times.

$\boxtimes$

Lemma 21. *Every derivation in the system* $\mathbf{HB}^{\leq}_{\mathbf{0},\mathbf{rst}}$ *can be rebuilt effectively into a derivation in* $\mathbf{HB}^{\leq}_{\mathbf{0},\mathbf{rst}}$.

Hence the systems $\mathbf{HB}^{\leq}_{\mathbf{0}}$ *and* $\mathbf{HB}^{\leq}_{\mathbf{0},\mathbf{rst}}$ *are equivalent theories.*

Definition 27. (The annotated version ann-$\mathbf{HB}^{\leq}_{\mathbf{0,rst}}$ of the system $\mathbf{HB}^{\leq}_{\mathbf{0,rst}}$).

Axioms/Marked Assumptions:

$$\text{(BOT)}\ \overline{(\bot,\sigma):\ \bot\leq\sigma} \qquad \text{(TOP)}\ \overline{(\tau,\top):\ \tau\leq\top}$$

$$\text{(REFL)}\ \overline{(\tau,\tau):\ \tau\leq\tau} \qquad \text{(HYP)}\ \big((\alpha,\bar{\alpha}):\ \tau\leq\sigma\big)^{\alpha}$$

(where $\bar{\alpha}\not\equiv\alpha$)

Rules:

$$\frac{(\chi,\bar{\chi}):\ \tau\leq\sigma}{(\chi,\bar{\chi}):\ \tau'\leq\sigma'}\,\text{VAR}\quad(\text{if }\tau'\equiv_{\mathrm{v}}\tau\text{ and }\sigma'\equiv_{\mathrm{v}}\sigma)$$

$$\frac{(\chi,\bar{\chi}):\ \bot\leq\sigma}{(\chi,\bar{\chi}):\ \mu\alpha\alpha_1\ldots\alpha_n.\,\alpha\leq\sigma}\ (\mu-\bot)^{\mathrm{inf}}_{l/r}\ \frac{(\chi,\bar{\chi}):\ \tau\leq\bot}{(\chi,\bar{\chi}):\ \tau\leq\mu\beta\beta_1\ldots\beta_n.\,\beta}$$

$$\frac{(\chi,\bar{\chi}):\ \tau_0[\mu\alpha.\,\tau_0/\alpha]\leq\sigma}{(\chi,\bar{\chi}):\ \mu\alpha.\,\tau_0\leq\sigma}\,\text{FOLD}_l\quad \frac{(\chi,\bar{\chi}):\ \tau\leq\sigma_0[\mu\beta.\,\sigma_0/\beta]}{(\chi,\bar{\chi}):\ \tau\leq\mu\beta.\,\sigma_0}\,\text{FOLD}_r$$

$$\text{(if side-condition }\mathbf{S_1'})\quad \frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ (\bar{\chi_1},\chi_1):\ \sigma_1\leq\tau_1 & (\chi_2,\bar{\chi_2}):\ \tau_2\leq\sigma_2\end{array}}{(\chi_1\to\chi_2,\ \bar{\chi_1}\to\bar{\chi_2}):\tau_1\to\tau_2\leq\sigma_1\to\sigma_2}\,\text{ARROW}$$

$$\frac{\begin{array}{cc}\langle(\alpha,\bar{\alpha}):\ \tau_1\to\tau_2\leq\sigma_1\to\sigma_2\rangle^{\alpha} & \langle(\alpha,\bar{\alpha}):\ \tau_1\to\tau_2\leq\sigma_1\to\sigma_2\rangle^{\alpha}\\ \mathcal{D}_1 & \mathcal{D}_2\\ (\bar{\chi_1},\chi_1):\ \sigma_1\leq\tau_1 & (\chi_2,\bar{\chi_2}):\ \tau_2\leq\sigma_2\end{array}}{(\mu\alpha.\,\chi_1\to\chi_2,\ \mu\bar{\alpha}.\,\bar{\chi_1}\to\bar{\chi_2}):\ \tau_1\to\tau_2\leq\sigma_1\to\sigma_2}\,\begin{array}{l}(\text{if s.-c. }\mathbf{S_2'})\\ (\text{AR./F.})_{\alpha}\end{array}$$

Gathered Results

A. In the case of Recursive Type Equality $=_\mu$:

1. There is an immediate "duality" between "consistency checks" in $\mathbf{SM}_{\mathbf{0}}^{=}$ and derivations in $\mathbf{HB}_{\mathbf{0}}^{=}$ (*Jan Willem Klop*).
2. μ-COMPAT is a derived rule of $\mathbf{AC}_{\mathbf{1}}^{=}$.
3. An effective translation of $\mathbf{AC}^{=}$-derivations into $\mathbf{HB}^{=}$-derivations.
4. An effective translation of $\mathbf{HB}^{=}$- into $\mathbf{AC}^{=}$-derivations.
   a. A transformation of $\mathbf{HB}^{=}$- into $\mathbf{HB}_{\mathbf{0}}^{=}$-derivations.
   b. An effective transformation of $\mathbf{HB}_{\mathbf{0}}^{=}$- into $\mathbf{AC}^{=}$-derivations.
   (Use of the annotated version **ann-**$\mathbf{HB}_{\mathbf{0}}^{=}$ of $\mathbf{HB}_{\mathbf{0}}^{=}$).
5. a An effective proof for the eliminibility of the FIX-rule

$$\dfrac{\begin{array}{c}[\tau=\sigma]^x\\ \mathcal{D}_0\\ \tau=\sigma\end{array}}{\tau=\sigma}\,\mathrm{FIX}_x \quad (\mathcal{D}_0 \text{ is } \textit{contractive}) \quad .$$

   in $\mathbf{HB}^{=}$-derivations.

   b A semantical soundness-proof of $\mathbf{HB}^{=}$+FIX.

B. In the case of the Subtyping Relation $\leq_\mu$:

1. There is an immediate “duality” between “consistency checks” in $\mathbf{SM}_{\mathbf{0}}^{\leq}$ and derivations in $\mathbf{HB}_{\mathbf{0}}^{\leq}$.
2. An effective transformation of $\mathbf{AC}^{\leq}$-derivations into $\mathbf{HB}^{\leq}$-derivations.
3. An effective translation of $\mathbf{HB}^{\leq}$- into $\mathbf{AC}^{\leq}$-derivations.
   a. An effective Transformation of $\mathbf{HB}^{\leq}$- into $\mathbf{HB}_{\mathbf{0}}^{\leq}$-derivations.
   b. An effective Transformation of $\mathbf{HB}^{\leq}$- into $\mathbf{HB}_{\mathbf{0,rst}}^{\leq}$-derivations.
   c. An effective translation of $\mathbf{HB}_{\mathbf{0,rst}}^{\leq}$- into $\mathbf{AC}^{\leq}$-derivations.
   (Use of the annotated version $\mathbf{ann\text{-}HB}_{\mathbf{0,rst}}^{\leq}$ of $\mathbf{HB}_{\mathbf{0,rst}}^{\leq}$).
4. The FIX-rule is a derived rule of $\mathbf{HB}^{\leq}$, applications of which can be eliminated effectively from $\mathbf{HB}^{\leq}$-derivations.
5. The rule μ-INTRO of $\mathbf{AC}^{\leq}$ is (in $\mathbf{HB}^{\leq}$) equivalent to the rules ARROW/FIX or FIX.

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