Bisimilarity and Simulatability of Processes Parameterized by Join Interactions

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Departing from Larsen's concept of parameterized bisimilarity of processes with respect to interaction with environments, we start an exploration of its natural weakening: bisimilarity of unrestricted join interactions with environments. Parameterized bisimilarity relates processes p and q with respect to an environment e if p and q behave bi-similarly while joining—respectively the same—transitions from e. The weakened variant relates processes p and q with respect to environment e if the join-interaction processes $p \,\&\, e$ and $q \,\&\, e$ of p and q with e are bisimilar. (Hereby join interactions $r \,\&\, f$ facilitate a step with label a to $r' \,\&\, f'$ if and only if r and f permit a-steps to r' and f', respectively.)

Join-interaction parameterized (ji-parameterized) bisimilarity coincides with parameterized bisimilarity for deterministic environments, but that it is a coarser equivalence in general. We explain how Larsen's concept can be recovered from ji-parameterized bisimilarity by 'determinizing' interactions. We show that by adaptation to simulatability (simulation preorder) the same concept arises: parameterized simulatability coincides with ji-parameterized simulatability. For the discrimination preorder of (ji-)parameterized simulatability on environments we obtain the same result as Larsen did for parameterized bisimilarity. Also, we give a modal-logic characterization of (ji-)parameterized simulatability. Finally we gather open problems, and provide an outlook on our current related work.

1 Introduction

With the motivation of developing flexible formal methods for proving correctness of software programs incrementally, by showing compositional correctness under the formation of contexts, Larsen in [9, 10] introduced parameterized bisimilarity of processes as a helpful concept. It turned out, more recently, to be useful in an area with a similar motivation: contextual behavioural metrics (see work of Dal Lago and Murgia [4, 8]), which measure differences between programs as distances by means of pseudo-metrics. This is because parameterized bisimilarity provides natural examples of contextual behavioural metrics.

The idea underlying parameterized bisimilarity is that the behaviors of two processes are compared with respect to a third process that represents a common environment, in which both processes are placed, and with which both can interact separately. The environment is able to 'consume' a transition from a process by performing a transition with the same action label, after which both the process and the environment move to the target state of the interaction transition on their side, respectively. Such consumption interactions are intended to continue as long as possible. Yet in case that an environment state permits no transition with the same label as the current process state (this is the case, for example, if the environment or the considered process is in a deadlock state), the consumption process stops. Given this setup, processes p and q are called bisimilar with respect to an environment process e if p and q behave in a bisimilar way (fulfilling forth and back conditions as typical for bisimulations) for any pair of runs of *synchronous* consumption interactions of the environment with the two processes in which the environment takes the *same* transitions on its side.

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It is distinctive for Larsen's concept of parameterized bisimilarity that the forth and back conditions of two processes p and q, and subsequently, of states reached via transitions from p and q, have to be verified *separately* for every run of the environment but while interacting *synchronously* with both processes. Indeed, comparisons of possible further interactions have to be carried out in synchronicity of the interactions, as long as the environment can interact with either of the processes. Thereby a mismatch is detected in the following situation: Suppose that by successful comparisons in a synchronous run derivative processes p' and q' as well as derivative environment e' are reached. Suppose further that e' permits, say, an a-transition that can be joined only with an a-transition from p', but not from q' (in case q' does not permit a-transitions). Then it has been determined that p and q are not bisimilar with respect to e.

Parameterized bisimilarity thus compares the behavior of two processes with respect to *controlled* and *synchronous* interactions with an environment process. For determining whether two processes p and q are bisimilar with respect to an environment process e it is necessary to observe the consumption interactions of p with e and of q with e in a synchronous step-wise manner. It is not sufficient to be merely presented the completed processes that result from the interactions of p with e, and of q with e, respectively, and then to ask whether these results are bisimilar.

There are, however, conceivable practical situations, in which one lacks sufficient control over the environment process in order to perform, or merely to analyze, controlled and synchronous interactions with the considered processes. That is, situations in which a scientist has access only to the data of completed interactions of two processes with a given environment, but in which she lacks sufficient control over the environment in order to perform the two interactions synchronously in a step-by-step manner so that she can compare the behaviors that remain after each step.

Here we define, and start to investigate, the weaker concept of parameterized bisimilarity in which only the completed outcome processes of the possible interactions of two processes p and q with a given environment are compared as to whether they are bisimilar. For this purpose we stipulate that the consumption interaction takes place in the form of a 'join' operation (&) between each process and the environment, which produces transitions with the same action labels as the two interaction transitions. Indeed, only transitions with the same label from a process and the environment can be 'joined' to interact, and produce a resulting transition with again that same label. We call the concept of bisimilarity between the join interactions of each process with the environment 'join-interaction parameterized' (ji-parameterized) bisimilarity. Larsen briefly mentions this concept at the end of the article [10]. He calls it 'perhaps more immediate', but excludes it from further consideration on the basis that it lacks some distinctive properties that he was able to show for parameterized bisimilarity. (For more details, see the paragraph 'Larsen on ji-parameterized bisimilarity ...' in Section 5.) Although that is true, it remains the case that ji-parameterized bisimilarity has a much easier, and appealingly natural definition, and that it may be of practical use in cases in which parameterized bisimilarity cannot be used.

While these considerations may seem abstract, we got interested in studying ji-parameterized bisimilarity when we made the following concrete observations (many of which are explained here later):

- trying to understand the conceptual difference between parameterized bisimilarity and ji-parameterized bisimilarity, also by means of concrete examples (for an overview see Theorem 3.11);
- noticing that, for deterministic environments, parameterized bisimilarity and ji-parameterized bisimilarity coincide (see Proposition 3.10);
- recognizing that also parameterized bisimilarity can be formulated as bisimilarity of a special kind ($\&_{\bullet}$) of join interaction (see Definition 3.4 and Lemma 3.5);
- recognizing that simulation preorder adaptations of the two concepts of parameterized bisimilarity and ji-parameterized bisimilarity do in fact coincide (see Proposition 3.10, (i));

- discovering an easy adaptation of Larsens modal-logical characterization of parameterized bisimilarity for (ji-)parameterized simulatability[1] (see Theorem 4.2 in Section 4);
- finding ideas for a natural modal-logical characterization also for ji-parameterized bisimilarity (see current work item (W1) in Section 5).

In Section 2 we summarize Larsen's definition and main results on parameterized bisimilarity, and we define parameterized simulatability.[1] In Section 3 we define ji-parameterized bisimilarity and simulatability, and develop basic results about their relationships with parameterized bisimilarity and simulatability. We discover that Larsen's theorem about the discrimination preorder induced by parameterized bisimilarity has an analogous version for the discrimination preorder that is induced by (ji-)parameterized simulatability. Then in Section 4 we specialize Larsen's modal-logical characterization of parameterized bisimilarity to (ji-)parameterized simulatability. Finally in Section 5 we give a list that summarizes our results, we report about the literature and our ongoing related work, and we sketch further ideas and plans.

2 Preliminaries on Larsen's parameterized bisimilarity

In this section we summarize definitions and results by Larsen in [9, 10] concerning parameterized bisimilarity, its induced discrimination preorder, and a modal-logical characterization for it. Additonally we define parameterized simulations, which relate to parameterized bisimulations in the same way as how simulations relate to bisimulations. We start with the basic concept of labeled transition system.

Definition 2.1 (LTSs)**.** A *(simple) labeled transition system (LTS)* is a triple $\mathscr{T} = \langle \mathrm{St}, A, \to \rangle$ that consists of a set St of *states*, a set A of *actions*, and a ternary *transition relation* $\to \subseteq \mathrm{St} \times A \times \mathrm{St}$ that represents A-labeled *transitions* on the state set.

For LTSs we will use notation and terminology for basic properties as follows. For their stipulation, we let $\mathscr{T} = \langle \mathrm{St}, A, \to \rangle$ be an LTS. For $\langle s, a, t \rangle \in \to$ we usually write $s \xrightarrow{a} t$, and say that "in state s there is a transition with label a (symbolizing an action called a) to state t". In this case we also say that t is an *a-derivative* of s. For $s \in \mathrm{St}$ and $a \in A$ we write $s \xrightarrow{a}$ if there is an a-transition from s in $\mathscr{T}$, and $s \not\xrightarrow{a}$ if there is no a-transition from s in $\mathscr{T}$.

We call an LTS $\mathscr{T} = \langle \mathrm{St}, A, \to \rangle$ *deterministic* (respectively *image-finite*) if $|\{s' \mid s \xrightarrow{a} s'\}| \leq 1$ (and respectively if $|\{s' \mid s \xrightarrow{a} s'\}| < \infty$) for all states $s \in \mathrm{St}$ and actions $a \in A$, that is, if every state of $\mathscr{T}$ has at most one a-derivative (resp. has only finitely many a-derivatives), for all $a \in A$. We say that a state $s \in \mathrm{St}$ *is deterministic* (resp. *is image-finite*) if every state of $\mathscr{T}$ that is reachable from s via a path of transitions has at most one a-derivative (resp. has only finitely many a-derivatives), for all $a \in A$.

Following well-known intuitions, bi-/simulations on such simple LTSs can be defined as follows.

Definition 2.2 (bisimulation/bisimilar, simulation/simulated by)**.** Let $\mathscr{T} = \langle \mathrm{St}, A, \to \rangle$ be an LTS.

(i) A *bisimulation B on $\mathscr{T}$* is a non-empty binary relation $B \subseteq \mathrm{St} \times \mathrm{St}$ with the following property: If $s\,B\,t$ for $s, t \in \mathrm{St}$, then the following two conditions hold:

(forth) $(\forall s' \in \mathrm{St})\big[s \xrightarrow{a} s' \implies (\exists t' \in \mathrm{St})[t \xrightarrow{a} t' \wedge s'\,B\,t']\big]$,

(back) $(\forall t' \in \mathrm{St})\big[t \xrightarrow{a} t' \implies (\exists s' \in \mathrm{St})[s \xrightarrow{a} s' \wedge s'\,B\,t']\big]$.

For processes $s, t \in \mathrm{St}$, we write $s \sim t$ and say that *s and t are bisimilar* if there is a bisimulation B on $\mathscr{T}$ such that $s\,B\,t$.

[1] We use 'simulatability' instead of 'similarity' for 'simulation preorder' for two reasons: to prevent the impression that a symmetrical relation were meant, and to avoid a possible confusion with 'simulation equivalence' that will also appear here.

(ii) A *simulation* B *on* $\mathscr{T}$ is a non-empty binary relation $S \subseteq \mathrm{St} \times \mathrm{St}$ with the following property: If $s\,B\,t$ for $s,t \in \mathrm{St}$, then the condition (forth) in (i) holds for $B := S$ (but not necessarily the condition (back)). For processes $s,t \in \mathrm{St}$, we write $s \le t$ and say that *s can be simulated by t*, and we write $t \ge s$ and say that *t can simulate s*, if there is a simulation S on $\mathscr{T}$ such that $s\,S\,t$.

Rather than defining simulations as weakened versions of bisimulations as above, bisimulations can also be defined from simulations, as follows. A relation B on an LTS $\mathscr{T}$ is a bisimulation if and only if both B as its converse relation $B^{\smile} := \{\langle t, s\rangle \mid s\,B\,t\}$ are simulations on $\mathscr{T}$.

For modeling processes whose behavior is studied according to how they interact with environments, both processes and environments are formalized as LTSs. However, in order to indicate their intended roles for occurring LTSs, we distinguish in notation, name, and in how they are referenced between *process LTSs* $\mathscr{P} = \langle \mathrm{Pr}, A, \to\rangle$, whose states we call *processes*, and *environment LTSs* $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow\rangle$, whose states we call *environments*. We follow Larsen [9, 10] in this terminology and in most of the notation. Based on this distinction, Larsen defines parameterized bisimulation and bisimilarity as follows.

Definition 2.3 (parameterized bisimulation (Larsen [9, 10]))**.** Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to\rangle$ be a process LTS, and let $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow\rangle$ be an environment LTS. An *$\mathscr{E}$-parameterized bisimulation $\mathscr{B}$ on $\mathscr{P}$* is an Env-indexed family $\mathscr{B} = \{B_f\}_{f \in \mathrm{Env}}$ of non-empty binary relations $B_f \subseteq \mathrm{Pr} \times \mathrm{Pr}$ such that the following holds:[2] If $p\,B_e\,q$ for $e \in \mathrm{Env}$, then if $e \overset{a}{\Rightarrow} e'$ for $a \in A$ the following conditions hold:

$$\text{(forth)}\ \ (\forall p' \in \mathrm{Pr})\big[\,p \overset{a}{\to} p' \implies (\exists q' \in \mathrm{Pr})[\,q \overset{a}{\to} q' \wedge p'\,B_{e'}\,q'\,]\big],$$
$$\text{(back)}\ \ (\forall q' \in \mathrm{Pr})\big[\,q \overset{a}{\to} q' \implies (\exists p' \in \mathrm{Pr})[\,p \overset{a}{\to} p' \wedge p'\,B_{e'}\,q'\,]\big].$$

For processes $p,q \in \mathrm{Pr}$, and environments $e \in \mathrm{Env}$ we write $p \sim_e q$ and say that *p and q are bisimilar with respect to e* if there is an $\mathscr{E}$-parameterized bisimulation $\mathscr{B} = \{B_f\}_{f \in \mathrm{Env}}$ such that $p\,B_e\,q$.

While simulation plays a crucial role in Larsen's main theorem on parameterized bisimulation, see Theorem 2.5 below, it is surprising that he did not also define the simulation version of this concept with only the forth condition from its progression conditions. For the reason that it can be linked directly to the simulation version of the concept of 'ji-parameterized bisimulation' that we will introduce in Section 3 (see Definition 3.3), we also define 'parameterized simulation' here.

Definition 2.4 (parameterized simulation)**.** Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to\rangle$ be a process LTS, and let $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow\rangle$ be an environment LTS. An *$\mathscr{E}$-parameterized simulation $\mathscr{S}$ on $\mathscr{P}$* is an Env-indexed family $\mathscr{S} = \{S_f\}_{f \in \mathrm{Env}}$ of non-empty binary relations $S_f \subseteq \mathrm{Pr} \times \mathrm{Pr}$ such that the following holds: If $p\,S_e\,q$ holds for $e \in \mathrm{Env}$, then for all $a \in A$, if $e \overset{a}{\Rightarrow} e'$ the condition (forth) in Def. 2.3 for $B_{e'} := S_{e'}$ holds:

$$\text{(forth)}\ \ (\forall p' \in \mathrm{Pr})\big[\,p \overset{a}{\to} p' \implies (\exists q' \in \mathrm{Pr})[\,q \overset{a}{\to} q' \wedge p'\,S_{e'}\,q'\,]\big].$$

For processes $p,q \in \mathrm{Pr}$, and environments $e \in \mathrm{Env}$ we write $p \le_e q$ and say that *p can be simulated by q with respect to e*, and $q \ge_e p$ and say that *q can simulate p with respect to e*, if there is an $\mathscr{E}$-parameterized simulation $\mathscr{S} = \{S_f\}_{f \in \mathrm{Env}}$ such that $p\,S_e\,q$.

Also parameterized bisimulations can be defined from parameterized simulations: for process LTS $\mathscr{P}$, and environment LTS $\mathscr{E}$, $\mathscr{S} = \{S_f\}_{f \in \mathrm{Env}}$ is an $\mathscr{E}$-parameterized bisimulation on $\mathscr{P}$ if and only if $\{S_f\}_{f \in \mathrm{Env}}$, and the family $\{S_f^{\smile}\}_{f \in \mathrm{Env}}$ of converse relations of S_f are $\mathscr{E}$-parameterized simulations.

Larsen's main result on parameterized bisimilarity concerns the *discrimination preorder* $\sqsubseteq$ that orders environments according to their power of discriminating between processes. It is defined, for a given

[2] Note the occurrence of e' (instead of e) in $B_{e'}$ in both of the conditions (back) and (forth).

process LTS $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$ and a given environment LTS $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$ (so that, for all $e \in \mathrm{Env}$, the relations $\sim_e$ are then fixed as subsets of $\mathrm{Pr} \times \mathrm{Pr}$), for all $e, f \in \mathrm{Env}$ by:

$$e \sqsubseteq f \;:\Longleftrightarrow\; \nsim_e \subseteq \nsim_f \qquad (\Longleftrightarrow\; \sim_f \subseteq \sim_e)\,. \tag{2.1}$$

Now Larsen's result characterizes $\sqsubseteq$ as coinciding with the simulation preorder $\leq$ on environments. For the 'completeness' direction of this characterization to hold ("$\Leftarrow$" in (2.2)), it is, however, necessary to assume that the underlying process LTS is, as Larsen formulates it, 'sufficiently rich' structurally. The weak natural assumption that he uses for the purpose of guaranteeing sufficient structural richness of any considered process LTS is that its set of processes is closed under action prefixing and finite summation (see Definition 3.2 in Section 3).

Theorem 2.5 (Larsen [9, 10])**.** *The following logical equivalence holds, provided that the underlying process LTS is closed under action prefixing and finite summation, for all image-finite environments e, f:*

$$e \leq f \;\;\Longleftrightarrow\;\; e \sqsubseteq f \;\;(\Longleftrightarrow\;\; \sim_f \;\subseteq\; \sim_e)\,. \tag{2.2}$$

The implication "$\Rightarrow$" holds for all (thus also for not necessarily image-finite) environments e and f.

Specifically for the direction "$\Leftarrow$" in (2.2) Larsen provides an impressive, technical proof, which he found, as he writes, only after an intensive search that took several months.

We now turn to modal-logical characterizations of the relations of being able to be simulated by $\leq$, of bisimilarity $\sim$, and of parameterized bisimilarity $\sim_e$. For expressing properties of LTSs such as the existence of a transition with label a from a given state such that at the target state property ϕ_0 holds, modal formulas should include a diamond modality $\langle a \rangle$ to build formulas like $\langle a \rangle \phi_0$. The set $\mathscr{M}$ of simple modal formulas (and the set $\mathscr{L}$ of positive formulas) are now defined with these diamond modalities and basic propositional connectives (resp. such connectives except negation) as constructors.

Definition 2.6 (modal formulas)**.** For given sets A of actions, we define the following classes of formulas: $\mathscr{L}(A)$ of positive formulas, and $\mathscr{M}(A)$ of (simple modal logic) formulas, via the following grammars:

$$\mathscr{L}(A) \qquad \phi \;::=\; \top \mid \phi \wedge \phi \mid \langle a \rangle \phi \qquad \text{(where } a \in A\text{)}, \tag{2.3}$$

$$\mathscr{M}(A) \qquad \phi \;::=\; \top \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle \phi \qquad \text{(where } a \in A\text{)}. \tag{2.4}$$

As above, we usually will keep the underlying set A of actions implicit, and write $\mathscr{L}$ and $\mathscr{M}$ for $\mathscr{L}(A)$ and $\mathscr{M}(A)$, respectively.

Definition 2.7 (satisfaction relation, sets of satisfied formulas)**.** Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$ be a process LTS. The *satisfaction relation* $\vDash \subseteq \mathrm{Pr} \times \mathscr{M}$ *on* $\mathscr{P}$ is defined by the following clauses:

$$p \vDash \top \;:\Longleftrightarrow\; p \in \mathrm{Pr}\,, \qquad p \vDash \phi_1 \wedge \phi_2 \;:\Longleftrightarrow\; p \vDash \phi_1 \text{ and } p \vDash \phi_2\,,$$

$$p \vDash \neg\phi_0 \;:\Longleftrightarrow\; p \nvDash \phi_0\,, \qquad p \vDash \langle a \rangle \phi_0 \;:\Longleftrightarrow\; \exists p' \in \mathrm{Pr}\,(\,p \xrightarrow{a} p' \text{ and } p' \vDash \phi_0\,)\,.$$

by induction on the structure of formulas in $\mathscr{M}$. For all processes $p \in \mathrm{Pr}$ we define by:

$$\mathscr{L}(p) := \{\phi \in \mathscr{L} \mid p \vDash \phi\}\,, \qquad \mathscr{M}(p) := \{\phi \in \mathscr{M} \mid p \vDash \phi\}\,,$$

the set $\mathscr{L}(p)$ of positive formulas in $\mathscr{L}$ that are satisfied in p, and respectively, the set $\mathscr{M}(p)$ of formulas in $\mathscr{M}$ that are satisfied in p.

The classical characterization result via modal-logical formulas of the relations bisimilarity $\sim$, and 'being able to be simulated by' $\leq$ is the following well-known theorem by Hennessy and Milner.

Theorem 2.8 (Hennessy, Milner [6]). *For all image-finite processes p and q the following statements hold:*

$$p \leq q \iff \mathscr{L}(p) \subseteq \mathscr{L}(q)\,, \tag{2.5}$$
$$p \sim q \iff \mathscr{M}(p) = \mathscr{M}(q)\,. \tag{2.6}$$

The implications "⇒" hold for all (thus also for not necessarily image-finite) processes p and q.

For a modal-logical characterization of parameterized bisimilarity, the concept of negation closure of positive formulas will be needed. By departing slightly from Larsen's exposition in [9, 10] we define it via a projection of general formulas to positive formulas.

Definition 2.9 (positive-formula projection, negation closure). The *positive-formula projection* is the function $|\cdot|_+ : \mathscr{M} \longrightarrow \mathscr{L}$ that maps formulas $\phi \in \mathscr{M}$ to positive formulas $|\phi|_+ \in \mathscr{L}$, and that is defined by induction on the structure of ϕ via the following clauses, for all formulas $\phi_0, \phi_1, \phi_2 \in \mathscr{M}$:

$$|\top|_+ := \top\,, \qquad |\neg\phi_0|_+ := |\phi_0|_+\,, \qquad |\phi_1 \wedge \phi_2|_+ := |\phi_1|_+ \wedge |\phi_2|_+\,, \qquad |\langle a\rangle\phi_0|_+ := \langle a\rangle|\phi_0|_+\,.$$

For every positive formula $\phi \in \mathscr{L}$, we define by $\overline{\neg\phi} := \{\psi \in \mathscr{M} \mid |\psi|_+ = \phi\}$ the *negation closure of ϕ in $\mathscr{M}$*. For subclasses $\mathscr{F} \subseteq \mathscr{L}$ of positive formulas, we define the *negation closure of $\mathscr{F}$ in $\mathscr{M}$* by $\overline{\neg\mathscr{F}} := \bigcup\{\overline{\neg\phi} \mid \phi \in \mathscr{F}\} = \{\psi \in \mathscr{M} \mid |\psi|_+ \in \mathscr{F}\}$.

Larsen presents [9, 10] the following modal characterization theorem of parameterized bisimilarity, which he attributes to Colin Stirling. The characterization restricts consideration for possible discriminating formulas to those in the negation-closure of positive formulas that are satisfied by the environment.

Theorem 2.10 (Stirling and Larsen, [9, 10]). *For all image-finite processes p, q, and environments e:*[3]

$$\begin{aligned} p \sim_e q &\overset{(\star)}{\iff} \mathscr{M}(p) \cap \overline{\neg\mathscr{L}(e)} = \mathscr{M}(q) \cap \overline{\neg\mathscr{L}(e)} \\ &\iff \forall \phi_0 \in \mathscr{L}\left[e \vDash \phi_0 \;\Rightarrow\; \forall \phi \in \overline{\neg\phi_0}\left(p \vDash \phi \;\Leftrightarrow\; q \vDash \phi\right)\right]. \end{aligned} \tag{2.7}$$

The implication "⇒" in (⋆) holds for all (thus also for not necessarily image-finite) p, q, and e.

3 Join-Interaction parameterized simulatability and bisimilarity

In this section we first define the weaker versions of parameterized bisimilarity and simulatability (simulation preorder) that are based on a definition of 'join-interaction' of LTSs (Definition 3.4): ji-parameterized simulatability and bisimilarity (Definition 3.3). Then we investigate the basic relationship between the new concepts and parameterized bisimilarity and simulatability (Theorem 3.11), and explain that also parameterized bisimilarity and simulatability can be viewed as bisimilarity and simulatability, resp., with respect to a special kind of join operation (Lemma 3.5). Finally we present a theorem (Theorem 3.14) that characterizes the discrimination preorder of (ji-)parameterized simulatability in analogy with Larsen's characterization of the discrimination preorder of parameterized bisimilarity, see Theorem 2.5.

[3] The condition of being image-finite can be dropped for the environments e. This can be verified by means of a careful analysis of the proof in [9, 10] for this logical characterization.

Definitions of ji-parameterized simulatablity and bisimilarity

In order to prepare for the definition of ji-parameterized bisimilarity we define 'join-interaction LTSs' by using an operation of processes that Larsen calls 'join' [9, p. 43,44]. Later we also need the subsequent stipulation of when a single LTS is closed under the 'join' operation.

Definition 3.1 (join-interaction of LTSs)**.** Let $\mathscr{T}_1 = \langle \mathrm{Pr}_1, A, \to_1 \rangle$ and $\mathscr{T}_2 = \langle \mathrm{Pr}_2, A, \to_2 \rangle$ two LTSs. By the *join-interaction of* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ we mean the LTS $\mathscr{T}_1 \,\&\, \mathscr{T}_2 = \langle \mathrm{Pr}_1 \,\&\, \mathrm{Pr}_2, A, \to \rangle$ where $\to \subseteq (\mathrm{Pr}_1 \,\&\, \mathrm{Pr}_2) \times A \times (\mathrm{Pr}_1 \,\&\, \mathrm{Pr}_2)$ with $\mathrm{Pr}_1 \,\&\, \mathrm{Pr}_2 := \{ p_1 \,\&\, p_2 \mid p_1 \in \mathrm{Pr}_1,\, p_2 \in \mathrm{Pr}_2 \}$ is defined via the transition system rule:

$$\frac{p_1 \xrightarrow{a}_1 p_1' \qquad p_2 \xrightarrow{a}_2 p_2'}{p_1 \,\&\, p_2 \xrightarrow{a} p_1' \,\&\, p_2'}$$

The symbol "&" in processes $p_1 \,\&\, p_2$ of $\mathscr{T}_1 \,\&\, \mathscr{T}_2$ is to be understood as a term constructor that from any two processes q_1 in Pr_1 and q_2 in Pr_2 constructs a formal join-interaction process $p_1 \,\&\, p_2$ in $\mathrm{Pr}_1 \,\&\, \mathrm{Pr}_2$.

Definition 3.2 (closure of an LTS under action prefixing, sum, and join)**.** Let $\mathscr{T} = \langle \mathrm{St}, A, \to \rangle$ be a labeled transition system. We say that $\mathscr{T}$ *is closed under action prefixing*, resp. *under sum*, and resp. *under join* if for every states $s, s_1, s_2 \in \mathrm{St}$ there exists a state $a.s \in \mathrm{St}$ for all $a \in A$, resp. there exists a state $s_1 + s_2 \in \mathrm{St}$, and resp. there exists a state $s_1 \,\&\, s_2 \in \mathrm{St}$ such that the respective transition rule below is satisfied:

$$\frac{}{a.s \xrightarrow{a} s} \qquad\qquad \frac{s_i \xrightarrow{a} s_i'}{s_1 + s_2 \xrightarrow{a} s_i'}\ (\text{where } i \in \{1,2\}) \qquad\qquad \frac{s_1 \xrightarrow{a} s_1' \qquad s_2 \xrightarrow{a} s_2'}{s_1 \,\&\, s_2 \xrightarrow{a} s_1' \,\&\, s_2'}$$

We now proceed to defining join-interaction versions of parameterized simulatability, simulation equivalence, and bisimilarity as simulatability, simulation equivalence, and bisimilarity, respectively, of join-interactions between two processes and an environment.

Definition 3.3. Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$ be a process LTS, and let $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$ be an environment LTS.

For all environments $e \in \mathrm{Env}$, we define three binary relations on Pr: *ji-parameterized simulatability* $\leq_{\&e}$, *ji-parameterized bisimilarity* $\sim_{\&e}$, and finally, *ji-parameterized simulation equivalence* $(\leq\geq)_{\&e}$ where $\leq_{\&e}, \sim_{\&e}, (\leq\geq)_{\&e} \subseteq \mathrm{Pr} \times \mathrm{Pr}$, are defined by the following clauses, for all processes $p, q \in \mathrm{Pr}$:

(*p can be simulated by q with respect to join-interaction with e*)
$$p \leq_{\&e} q \;:\Longleftrightarrow\; p \,\&\, e \;\leq\; q \,\&\, e\,, \tag{3.1}$$
(*p is bisimilar to q with respect to join-interaction with e*)
$$p \sim_{\&e} q \;:\Longleftrightarrow\; p \,\&\, e \;\sim\; q \,\&\, e\,, \tag{3.2}$$
(*p and q are simulation equivalent with respect to join-interaction with e*)
$$p\ (\leq\geq)_{\&e}\ q \;:\Longleftrightarrow\; p \leq_{\&e} q \;\wedge\; q \leq_{\&e} p\,, \tag{3.3}$$

where $p \,\&\, e$ and $q \,\&\, e$ on the right in (3.1) and in (3.2) are processes from the join-interaction LTS $\mathscr{P} \,\&\, \mathscr{E}$. By $\geq_{\&e}$ we denote the converse of $\leq_e$, and express $q \geq_{\&e} p$ verbally by saying that *q can simulate p with respect to join-interaction with e*.

Relationship of ji-parameterized bisimilarity with parameterized bisimilarity

In order to recognize Larsen's parameterized bisimilarity as bisimilarity with respect to a specific form of join-interaction, we introduce a 'right-determinizing' variant $\&_{\bullet}$ of the join operation &. For interactions of a process p with an environment e this operation yields the process $p \,\&_{\bullet}\, e$ from which transitions are labeled by pairs $\langle a, e' \rangle$ that result from joining an a-transition from p with an a-transition from e to target e'. In this way different environment steps that originally have the same action label are distinguished from

$\&_\bullet$-joins. Indeed, by making different targets of environment transitions visible as different transitions from $p \&_\bullet e$ for processes p and q and an environment e, a correspondence arises between bisimulations that link $p \&_\bullet e$ and $q \&_\bullet e$ and parameterized bisimulations that link p and q with respect to e.

Definition 3.4 (right-determinizing join-interaction with environment LTSs)**.** Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$ be a process LTS, and $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$ be an environment LTS. By the *right-determinizing join-interaction of $\mathscr{P}$ and $\mathscr{E}$* we understand the LTS of the form $\mathscr{P} \&_\bullet \mathscr{E} = \langle \mathrm{Pr} \&_\bullet \mathrm{Env}, A \times \mathrm{Env}, \to \rangle$ with $\mathrm{Pr} \&_\bullet \mathrm{Env} := \{p \&_\bullet e \mid p \in \mathrm{Pr},\, e \in \mathrm{Env}\}$ and where $\to \subseteq (\mathrm{Pr} \&_\bullet \mathrm{Env}) \times (A \times \mathrm{Env}) \times (\mathrm{Pr} \&_\bullet \mathrm{Env})$ is defined by as transitions that are generated by the following rules:

$$\frac{p \xrightarrow{a} p' \qquad e \overset{a}{\Rightarrow} e'}{p \&_\bullet e \xrightarrow{\langle a, e' \rangle} p' \&_\bullet e'}$$

Hereby "$\&_\bullet$" in processes $p \&_\bullet e$ of $\mathscr{P} \&_\bullet \mathrm{Env}$ has to be understood as a term constructor that from any process $p \in \mathrm{Pr}$ and environment $e \in \mathrm{Env}$ constructs a formal join-interaction process $p \&_\bullet e$ in $\mathrm{Pr} \&_\bullet \mathrm{Env}$.

Now this variant "$\&_\bullet$" of the join operation "$\&$" facilitates characterizations of parameterized simulatability and bisimilarity that are analogous in kind to the definitions of ji-parameterized simulatability and bisimilarity in Definition 3.3. As stated by logical equivalences in the following lemma, parameterized simulatability, and parameterized bisimilarity correspond to simulatability, and respectively to bisimilarity, of $\&_\bullet$-interactions between two processes and an environment. From this we obtain inclusions of parameterized simulatability and bisimilarity in ji-parameterized simulatability and bisimilarity.

Lemma 3.5. *For all processes p and q, and environments e the following two chains of statements hold:*

$$\begin{aligned} p \le_e q &\iff (p \&_\bullet e) \le (q \&_\bullet e) & \qquad p \sim_e q &\iff (p \&_\bullet e) \sim (q \&_\bullet e) & (3.4)\\ &\implies (p \& e) \le (q \& e) & &\implies (p \& e) \sim (q \& e) & (3.5)\\ &\iff p \le_{\& e} q\,, & &\iff p \sim_{\& e} q\,, & \end{aligned}$$

where $p \&_\bullet e$ and $q \&_\bullet e$ are processes from the right-determinizing join-interaction LTS $\mathscr{P} \&_\bullet \mathscr{E}$, and $p \& e$ and $q \& e$ are processes from the join-interaction LTS $\mathscr{P} \& \mathscr{E}$.

Proof (Sketch). We consider a process LTS $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$, and an environment LTS $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$.

We only argue for the chain of equivalences and implications on the right for $\sim_e$, $\sim$, and $\sim_{\& e}$, since the chain of statements on the left for $\le_e$, $\le$, and $\le_{\& e}$ can be demonstrated analogously.

We first consider statement (3.4). For showing "$\Rightarrow$" it suffices to demonstrate that if $\mathscr{B} = \{B_e\}_{e \in \mathrm{Env}}$ is an $\mathscr{E}$-parameterized bisimulation on $\mathscr{P}$, then $B := \{\langle p \&_\bullet e, q \&_\bullet e \rangle \mid p\, B_e\, q\}$ is a bisimulation on $\mathscr{P} \&_\bullet \mathscr{E}$. For "$\Leftarrow$" it suffices to show that if B is a bisimulation on $\mathscr{P} \&_\bullet \mathscr{E}$, then $\mathscr{B} = \{B_e\}_{e \in \mathrm{Env}}$ with the defining clause $B_e := \{\langle p, q \rangle \mid \langle p \&_\bullet e, q \&_\bullet e \rangle \in B\}$ for $e \in \mathrm{Env}$ is an $\mathscr{E}$-parameterized bisimulation on $\mathscr{P}$. Both auxiliary statements can be shown by using the conditions (forth) and (back) from the assumed (parameterized) bisimulation in order to demonstrate the conditions (forth) and (back) of the (parameterized) bisimulation in the conclusion of the implication.

The implication in (3.5) can be easily verified similarly: by showing that whenever $B_\bullet$ is a bisimulation on $\mathscr{P} \&_\bullet \mathscr{E}$, then $B := \{\langle p \& e, q \& e \rangle \mid \langle p \&_\bullet e, q \&_\bullet e \rangle \in B_\bullet\}$ is a bisimulation on $\mathscr{P} \& \mathscr{E}$. ☐

In Figure 1 we illustrate the characterization (3.4) of parameterized bisimilarity $\sim_e$ as bisimilarity of $\&_\bullet$-interactions by an example. By using Lemma 3.5 we can now show that with respect to deterministic environments no difference arises between parameterized bisimilarity and ji-parameterized bisimilarity.

Proposition 3.6. $\sim_e \;=\; \sim_{\& e}$ *holds for all deterministic environments e.*

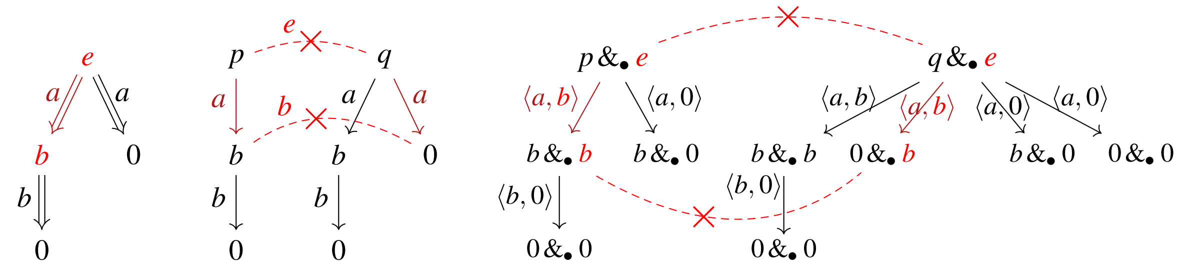

Figure 1: Example that witnesses the correspondence (3.4) in Lemma 3.5: For the environment $e := a.b + a$ and the processes $p := a.b$ and $q := e$, it holds that $p \not\sim_e q$ (indicated by the mismatches $\times$ when building a parameterized bisimulation on the left), and also $p\,\&_\bullet\, e \not\sim q\,\&_\bullet\, e$ (indicated by the mismatches $\times$ when building a bisimulation on the right). Note that in contrast $p \sim_{\&e} q$ holds p, q, e, see Fig. 2 later.

Proof. Let $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$ be a process LTS, and $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$ be an environment LTS. The Proposition follows from Lemma 3.5, once the converse implication "$\Leftarrow$" in (3.5), which is the only implication that is missing there for $\sim_e$ to coincide with $\sim_{\&e}$, is shown to hold for deterministic environments e:

$$e \text{ is deterministic} \implies \big[\,(p\,\&_\bullet\, e) \sim (q\,\&_\bullet\, e) \iff (p\,\&\, e) \sim (q\,\&\, e)\,\big]\,. \tag{3.6}$$

For this, it suffices to show, that whenever B is a bisimulation on $\mathscr{P}\,\&\,\mathscr{E}$ in which all environments that occur in joins in pairs in B are deterministic, then $B_\bullet := \{\langle p\,\&_\bullet\, e, q\,\&_\bullet\, e\rangle \mid \langle p\,\&\, e, q\,\&\, e\rangle \in B\}$ is a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$. The assumption that only deterministic environments occur in B is not too restrictive, because derivatives of deterministic environments are deterministic again.

We let B be a bisimulation on $\mathscr{P}\,\&\,\mathscr{E}$ as described, and let $B_\bullet$ be defined as above. We have to show that $B_\bullet$ is a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$.

For showing the condition (forth) for $B_\bullet$ to be a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$, we let $\langle p\,\&_\bullet\, e, q\,\&_\bullet\, e\rangle \in B_\bullet$, and a transition $p\,\&_\bullet\, e \xrightarrow{l} r$ be arbitrary, where $r \in \mathrm{Pr}\,\&_\bullet\,\mathrm{Env}$, l some label in $A \times \mathrm{Pr}$. Due to operational semantics of $\mathscr{P}\,\&_\bullet\,\mathscr{E}$, this transition must actually be of the form $p\,\&_\bullet\, e \xrightarrow{\langle a,e'\rangle} p'\,\&_\bullet\, e'$, for $p' \in \mathrm{Pr}$ and $e' \in \mathrm{Env}$. We have to show that there is $s \in \mathrm{Pr}\,\&_\bullet\,\mathrm{Env}$ such that $q\,\&_\bullet\, e \xrightarrow{\langle a,e'\rangle} s$ and $\langle r, s\rangle = \langle p'\,\&_\bullet\, e', s\rangle \in B_\bullet$.

From $p\,\&_\bullet\, e \xrightarrow{\langle a,e'\rangle} p'\,\&_\bullet\, e'$ it follows that $p \xrightarrow{a} p'$ and $e \overset{a}{\Rightarrow} e'$. By the definition of $\mathscr{P}\,\&\,\mathscr{E}$ it follows that there is also the transition $p\,\&\, e \xrightarrow{a} p'\,\&\, e'$ in $\mathscr{P}\,\&\,\mathscr{E}$. From $\langle p\,\&_\bullet\, e, q\,\&_\bullet\, e\rangle \in B_\bullet$ it follows by the definition of $B_\bullet$ that $\langle p\,\&\, e, q\,\&\, e\rangle \in B$, and by the assumption on B also that e is deterministic. Then it follows from the condition (forth) of B as a bisimulation on $\mathscr{P}\,\&\,\mathscr{E}$ that there is some $s \in \mathrm{Pr}\,\&\,\mathrm{Env}$ such that $q\,\&\, e \xrightarrow{a} s$ and $\langle p'\,\&\, e', s\rangle \in B$. By the definition of $\mathscr{P}\,\&\,\mathscr{E}$ we find that $s = q'\,\&\, e''$ and $\langle p'\,\&\, e', q'\,\&\, e''\rangle \in B$ for some $q' \in \mathrm{Pr}$ and $e' \in \mathrm{Env}$ with $q \xrightarrow{a} q'$ and $e \overset{a}{\Rightarrow} e''$. But now from $e \overset{a}{\Rightarrow} e'$ and $e \overset{a}{\Rightarrow} e''$ we can conclude, because e is deterministic, that $e'' = e'$. From this we obtain $\langle p'\,\&\, e', q'\,\&\, e'\rangle \in B$, which entails $\langle p'\,\&_\bullet\, e', q'\,\&_\bullet\, e'\rangle \in B_\bullet$. From $e \overset{a}{\Rightarrow} e'$ and $e \overset{a}{\Rightarrow} e'$ we also obtain $q\,\&_\bullet\, e \xrightarrow{\langle a,e'\rangle} q'\,\&_\bullet\, e'$. Therefore we have found in $s := q'\,\&_\bullet\, e'$ the desired $s \in \mathrm{Pr}\,\&_\bullet\,\mathrm{Env}$ with $q\,\&_\bullet\, e \xrightarrow{\langle a,e'\rangle} s$ and $\langle r, s\rangle = \langle p'\,\&_\bullet\, e', s\rangle \in B_\bullet$.

In this way we have established the condition (forth) for $B_\bullet$ to be a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$. Since the condition (back) can be verified analogously, we conclude that $B_\bullet$ is indeed a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$.

By having shown that $B_\bullet$ is a bisimulation on $\mathscr{P}\,\&_\bullet\,\mathscr{E}$ under the assumption that B is a bisimulation on $\mathscr{P}\,\&\,\mathscr{E}$ in which only deterministic environments occur, we have established (3.6), from which the proposition follows from Lemma 3.5 as argued above. $\square$

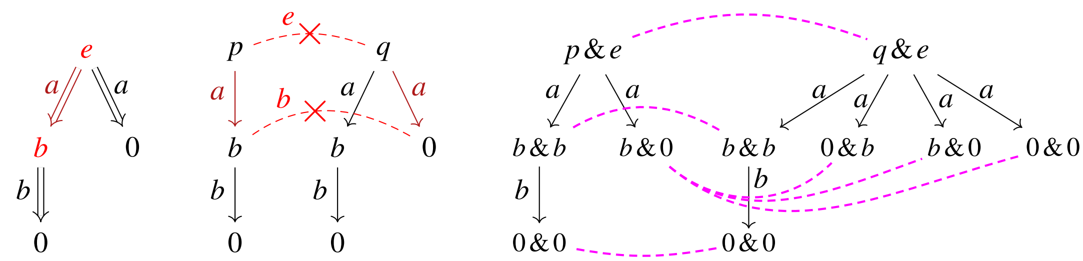

Figure 2: Example for witnessing $\sim_e \;\neq\; \sim_{\&e}$: For $e := a.b + a$, $p := a.b$, and $q := e$ it holds that $p \not\sim_e q$ (indicated by the mismatches $\times$ when building a parameterized bisimulation), but $p \,\&\, e \sim q \,\&\, e$ (indicated by the bisimulation links) and hence $p \sim_{\&e} q$.

The proposition below clarifies which inclusions hold in general between $\sim_e$, $\sim_{\&e}$, and $(\leq\geq)_{\&e}$.

Proposition 3.7. *The following set-theoretical relationships hold between parameterized bisimilarity* $\sim_e$*, ji-parameterized bisimilarity* $\sim_{\&e}$*, and ji-parameterized simulation equivalence* $(\leq\geq)_{\&e}$*:*

(i) $\sim_e \subseteq \sim_{\&e}$ *for all environments e.*

(ii) $\sim_e \neq \sim_{\&e}$ *for some environments e, for which then* $\sim_e \subsetneqq \sim_{\&e}$ *holds due to (i).*

(iii) $\sim_{\&e} \subseteq (\leq\geq)_{\&e}$ *for all environments e.*

(iv) $\sim_{\&e} \neq (\leq\geq)_{\&e}$ *for some environments e, for which then* $\sim_{\&e} \subsetneqq (\leq\geq)_{\&e}$ *holds due to (iii).*

The counterexample statements (ii) and (iv) hold under the proviso that environments are included among processes, they permit at least two actions, and are closed under action prefixing and sums. This can be weakened to merely require that $a.b + a$ *and* $a.b$ *are contained among environments and processes.*

Proof. Statement (i) follows directly from the chain of implications as guaranteed by Lemma 3.5.

A counterexample for (ii) is in Figure 2: We have that $p \sim_{\&e} q$ holds due to $p \,\&\, e = (a.b + a) \,\&\, (a.b) \simeq a.b + a \sim a.b + a + a + a \simeq (a.b + a) \,\&\, (a.b + a) = q \,\&\, e$, where $\simeq$ denotes being isomorphic, which shows $p \,\&\, e \sim q \,\&\, e$. However, $p \not\sim_e q$ holds for the following reason: Suppose that $p \sim_e q$ holds. Then due to $e = a.b + a \overset{a}{\Rightarrow} b$, and the condition (back) of an underlying parameterized bisimilarity the transition $q \overset{a}{\to} 0$ must be matched by the transition $p \overset{a}{\to} b$ so that $b \sim_b 0$ holds. However the latter is false, because $b \not\sim_b 0$ holds, as the environment b and the process b can make a b-step, but 0 cannot. Statement (iii) follows from the fact that bisimilarity is symmetric and it is a simulation [18]. For (iv), let p and q be as in Figure 2, and let $e = p$. Then $p \; (\leq\geq)_{\&e} \; q$ holds due to $p \,\&\, e = (a.b) \,\&\, (a.b) \simeq a.b \leq\geq a.b + a \simeq (a.b + a) \,\&\, (a.b) = q \,\&\, e$. However, $p \not\sim_{\&e} q$: indeed, $q \,\&\, e \overset{a}{\to} b \,\&\, 0 \sim 0$, to which $p \,\&\, e$ can only answer by reducing to $b \,\&\, b \sim b$, and clearly $0 \not\sim b$. □

Parameterized simulatability coincides with ji-parameterized simulatability

While parameterized bisimilarity and ji-parameterized bisimilarity are two different relations in general by Proposition 3.7, (ii), it turns out that this does not hold for the corresponding two concepts of parameterized simulatability. For us it was surprising to find the proof of the first of the following two lemmas, which together show that parameterized simulatability and ji-parameterized simulatability coincide.

Lemma 3.8. $\leq_{\&e} \subseteq \leq_e$ *holds for all environments e.*

Proof. We fix a process LTS $\mathscr{P} = \langle \mathrm{Pr}, A, \to \rangle$, and an environment LTS $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$.

As the crucial stepping stone, we show that $\mathscr{S} = \{S_e\}_{e\in\mathrm{Env}}$ as defined by, for all $e \in \mathrm{Env}$:

$$S_e := \big\{ \langle p,q\rangle \in \mathrm{Pr} \;\big|\; \exists e_2 \in \mathrm{Env}\big[p\,\&\,e \;\le\; q\,\&\,e_2\big]\big\} \subseteq \mathrm{Pr}\times\mathrm{Pr} \tag{3.7}$$

is an $\mathscr{E}$-parameterized simulation on $\mathscr{P}$. For this, we let $e \in \mathrm{Env}$, and $\langle p,q\rangle \in S_e$ be arbitrary. We assume that $e \overset{a}{\Rightarrow} e'$, and $p \overset{a}{\to} p'$ for some $a \in A$, $e' \in \mathrm{Env}$, and $p' \in \mathrm{Pr}$. We have to show that there exists $q' \in \mathrm{Pr}$ with $q \overset{a}{\to} q'$ such that $\langle p',q'\rangle \in S_{e'}$.

From $e \overset{a}{\Rightarrow} e'$ and $p \overset{a}{\to} p'$ we find that $p\,\&\,e \overset{a}{\to} p'\,\&\,e'$ holds. Due to $\langle p,q\rangle \in S_e$ we can pick $e_2 \in \mathrm{Env}$ with $p\,\&\,e \le q\,\&\,e_2$. It follows, by the forward-property (forth) of the (largest) simulation $\le$ applied to $p\,\&\,e \le q\,\&\,e_2$ and $p\,\&\,e \overset{a}{\to} p'\,\&\,e'$, and by the operational semantics of the join operation, that there are $q' \in \mathrm{Pr}$ and $e_2' \in \mathrm{Env}$ such that $q\,\&\,e_2 \overset{a}{\to} q'\,\&\,e_2'$, as well as $q \overset{a}{\to} q'$ and $e_2 \overset{a}{\Rightarrow} e_2'$ and with $p'\,\&\,e' \le q'\,\&\,e_2'$. The latter shows that $\langle p',q'\rangle \in S_{e'}$, and thus we have found $q \overset{a}{\to} q'$ such that $\langle p',q'\rangle \in S_{e'}$. In this way we have verified that $\mathscr{S} = \{S_e\}_{e\in\mathrm{Env}}$ as defined in (3.7) is an $\mathscr{E}$-parameterized simulation on $\mathscr{P}$.

For showing $\le_{\&e} \subseteq \le_e$, suppose now that $p \le_{\&e} q$ holds, for some $p,q \in \mathrm{Pr}$ and $e \in \mathrm{Env}$. By the definition of $\le_{\&e}$, this means that $p\,\&\,e \le q\,\&\,e$ holds. That, however, implies $\langle p,q\rangle \in S_e$ due to (3.7). But since we have recognized $\mathscr{S}$ as an $\mathscr{E}$-parameterized simulation, we conclude that $p \le_e q$ holds. □

Lemma 3.9. $\le_e \subseteq \le_{\&e}$ *holds for all environments e.*

Proof. The inclusion as stated by the lemma follows from the chain of implications displayed on the left in Lemma 3.5, which as stated in its proof can be proved analogously as the implications on the right there. But since we dropped the argument there, we also provide the sketch of a direct proof here.

Let $\mathscr{S} = \{S_e\}_{e\in\mathrm{Env}}$ be an $\mathscr{E}$-parameterized simulation on a process LTS $\mathscr{P} = \langle \mathrm{Pr}, A, \to\rangle$ with respect to an environment LTS $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow\rangle$. Then it is easy to verify that:

$$S := \{\langle p\,\&\,e, q\,\&\,e\rangle \mid p,q \in \mathrm{Pr} \text{ and } e \in \mathrm{Env} \text{ such that } p\; S_e\; q\} \subseteq \mathrm{Pr}\,\&\,\mathrm{Env}$$

is a simulation on $\mathscr{P}\,\&\,\mathscr{E} = \langle \mathrm{Pr}\,\&\,\mathrm{Env}, A, \to\rangle$. This statement implies that if $p \le_e q$, then $p\,\&\,e \le q\,\&\,e$ follows, and hence $p \le_{\&e} q$. □

Proposition 3.10. *For all environments e the following two statements hold:*

(i) $\le_{\&e} = \le_e$.

(ii) $(\le\ge)_{\&e} = (\le\ge)_e$.

Proof. The inclusions "⊆" and "⊇" that make up statement (i) are guaranteed by Lemma 3.8 and by Lemma 3.9, respectively. Then (ii) follows from (i) by: $(\le\ge)_{\&e} = \le_{\&e} \cap \ge_{\&e} = \le_e \cap \ge_e = (\le\ge)_e$. □

The theorem below collects results we have obtained about which inclusions hold in general between parameterized bisimilarity, ji-parameterized bisimilarity, and (ji-)parameterized simulation equivalence.

Theorem 3.11. *The following set-theoretical relationships hold between parameterized bisimilarity, ji-parameterized bisimilarity, and (ji-)parameterized simulation equivalence, for environments e, f, g:*

$$\sim_e \underset{\text{Prop. 3.7,(i)}}{\subseteq} \sim_{\&e} \underset{\text{Prop. 3.7,(iii)}}{\subseteq} (\le\ge)_e \underset{\text{Prop. 3.10,(ii)}}{=} (\le\ge)_{\&e} \qquad (\textit{for } \underline{\textit{all}}\ e)\,,$$

$$\sim_f \underset{\text{Prop. 3.7,(ii)}}{\subsetneq} \sim_{\&f} \underset{\text{Prop. 3.7,(iii)}}{\subseteq} (\le\ge)_f \underset{\text{Prop. 3.10,(ii)}}{=} (\le\ge)_{\&f} \qquad (\textit{for } \underline{\textit{some}}\ f)\,,$$

$$\sim_g \underset{\text{Prop. 3.7,(i)}}{\subseteq} \sim_{\&g} \underset{\text{Prop. 3.7,(iv)}}{\subsetneq} (\le\ge)_g \underset{\text{Prop. 3.10,(ii)}}{=} (\le\ge)_{\&g} \qquad (\textit{for } \underline{\textit{some}}\ g)\,,$$

where the statements that guarantee the relationship in question are indicated.

Discrimination preorder induced by (ji-)parameterized similarity

Larsen noted in [10, p. 209–210]: "Due to the modal characterization [see Theorem 2.10] and the simple characterization of the discrimination ordering presented [see Theorem 2.5], we are confident that the notion of parameterized bisimulation equivalence proposed is indeed a natural one". Indeed Larsen also explains that "the simulation ordering does not characterize the discrimination ordering generated by this alternative parameterized version [namely $\sim_{\&e}$]".

This is witnessed by the following proposition. Indeed it demonstrates that a characterization of the discrimination preorder induced by ji-parameterized bisimilarity $\sim_{\&e}$ cannot, in analogy with Theorem 2.5 for $\sim_e$, be of the form $e \leq f \iff \sim_{\&f} \subseteq \sim_{\&e}$, for all environments e and f.

Proposition 3.12. *There are environments e and f such that:*

$$e \leq f \;\; \wedge \;\; \sim_{\&f} \not\subseteq \sim_{\&e} \,. \tag{3.8}$$

Proof. Let $e = a.b$ and $f = a.b + a$. We have that $e \leq f$. Set $p = e$ and $q = f$. We have that $p \sim_{\&f} q$ but $p \not\sim_{\&e} q$, hence $\sim_{\&f}$ is not contained in $\sim_{\&e}$. □

Unfortunately we have not yet found an appealing characterization of the discrimination preorder that is induced by $\sim_{\&e}$. We formulate this question together with a perhaps also interesting specialization as the open problems (P1) and (P2) in the conclusion.

However, and somewhat surprisingly, we do obtain characterizations analogous to Theorem 2.5 for the discrimination preorder on environments e with respect to (ji-)parameterized similarity $\leq_{\&e}$ and $\leq_e$, and with respect to (ji-)parameterized simulation equivalence $(\leq\geq)_{\&e}$ and $(\leq\geq)_e$.

Similar to the proviso for the 'completeness' direction "⇒" in (2.2) of Theorem 2.5 our characterization of $(\leq\geq)_{\&e}$ requires an assumption that guarantees that the structure of the underlying process LTS is sufficiently rich in relation to the environment LTS. While Larsen assumed closure under the formation of action prefixing and finite sums, we will assume the existence of a 'universal' process, and that the underlying process LTS contains the environment LTS, and is closed under the formation of joins. (The assumption of a universal process simplifies the proof, but can be dropped.)

Let $\mathscr{P} = \langle \mathrm{Pr}, A, \rightarrow \rangle$ be a process LTS. We say that a process $u \in \mathrm{Pr}$ is *universal* if for all $a \in A$ there is a transition $u \xrightarrow{a} u'$ with $u' \in \mathrm{Pr}$ in $\mathscr{P}$ such that $u' \sim u$ (consequently it permits all actions in transitions from any of its reachable states). Note that all universal processes in $\mathscr{P}$ are bisimilar. If $\mathscr{P}$ is additionally closed under the formation &, then $u \,\&\, p \sim p \,\&\, u \sim p$ holds for all $p, u \in \mathrm{Pr}$ where u is universal.

For proving our characterization below, Theorem 3.14 we will use the following lemma.

Lemma 3.13. *For all environments e, f it holds, provided that the environment LTS is closed under joins:*

$$e \leq f \,\&\, e \iff e \leq f \,.$$

Proof. For showing the direction "⇒", and the direction "⇐" in the statement of the lemma, it suffices to prove that the relation $\{\langle e, f \rangle \mid e \leq f \,\&\, e\}$, and respectively, that the relation $\{\langle e, f \,\&\, e \rangle \mid e \leq f\}$ is a simulation. Both of these statements can be verified in a straightforward manner. □

We now are in a position to show characterizations of the discrimination preorders of (ji-)parameterized simulatability and (ji-)parameterized simulation equivalence, as formulated by the theorem below.

Theorem 3.14. *The following logical equivalences hold, for all environments e and f, provided that: the underlying process LTS contains a universal process and the environment LTS, and additionally is closed under the formation of joins* (but note that image-finiteness as in Theorem 2.5 is not required)*:*

$$e \leq f \iff \leq_{\&f} \subseteq \leq_{\&e} \,, \tag{3.9}$$

$$e \le f \iff \ge_{\&f} \subseteq \ge_{\&e} , \qquad (3.10)$$
$$e \le f \iff (\le\ge)_{\&f} \subseteq (\le\ge)_{\&e} . \qquad (3.11)$$

Proof. We let $\mathscr{E} = \langle \mathrm{Env}, A, \Rightarrow \rangle$ be an environment LTS, and we let $\mathscr{P} = \langle \mathrm{Pr}, A, \rightarrow \rangle$ be a process LTS that contains the universal process U.

We first note that (3.10) follows from, and is equivalent, to (3.9), because $\ge_{\&f}$ is the converse relation of $\le_{\&e}$. Furthermore (3.11) follows from (3.9) and (3.10), due to $(\le\ge)_{\&e} = \le_{\&e} \cap \ge_{\&e}$. Therefore it remains to prove (3.10). For that we show the two directions of 3.9, and proceed as follows:

"$\Rightarrow$": We show that $e \le f \implies \le_f \subseteq \le_e$. Then (3.9) follows from Proposition 3.10, (i). So, let $\{S_e\}_{e\in\mathrm{Env}}$ be a $\mathscr{E}$-parameterized family of binary relations $S_e \subseteq \mathrm{Pr} \times \mathrm{Pr}$ that is defined, for all $e \in \mathrm{Env}$ by:

$$p \, S_e \, q \;\; :\iff \;\; \exists f \ge e \left[p \le_f q \right]$$

It suffices to show that $\{S_e\}_{e\in\mathrm{Env}}$ is an $\mathscr{E}$-parameterized simulation. So, suppose $p \, S_e \, q$. Then $p \le_f q$ for some $f \ge e$. Suppose $p \xrightarrow{a} p'$ and $e \overset{a}{\Rightarrow} e'$. Then $f \overset{a}{\Rightarrow} f'$ for some $f' \ge e'$, and hence $q \xrightarrow{a} q'$ for some q' such that $p' \le_{f'} q'$. Then it follows that $p' \, S_{e'} \, q'$ holds, as required.

"$\Leftarrow$": Assume $\le_{\&f} \subseteq \le_{\&e}$. Let u be a universal process in $\mathscr{P}$. Then $u \,\&\, f \sim f$. Moreover, $f \le f \,\&\, f$ (which is easy to show), and then $u \le_{\&f} f$. By the assumption $\le_{\&f} \subseteq \le_{\&e}$ we have that $u \le_{\&e} f$. In other words: $e \sim u \,\&\, e \le f \,\&\, e$. By Lemma 3.13 we get $e \le f$, as required. □

4 Modal characterization of (ji-)parameterized simulatability

In this section we adapt the modal characterization of parameterized bisimilarity $\sim_e$, see Theorem 2.10, for (ji-)parameterized simulatability $\le_e$ and $\le_{\&e}$. The crucial observation for our adaptation is Lemma 4.1 below which states that the set of positive formulas that a join interaction $p_1 \,\&\, p_2$ satisfies is the intersection of the sets of positive formulas satisfied by the constituent processes p_1 and p_2. Finally we explain why a similar line of argument is not possible in order to adapt the modal characterization for $\sim_e$ to obtain one for ji-parameterized bisimilarity $\sim_{\&e}$. In doing so we provide some evidence for Larsen's assessment, that (in view of that $\sim_e \subsetneqq \sim_{\&e}$ holds in general, see Proposition 3.7) "the modal characterization for $\sim_e$ does not hold for $\sim_{\&e}$, and no other modal characterization seems immediate" [10, p. 210]. However, we report about further work of ours on this issue in the final section (see (W1) in Section 5).

Lemma 4.1. *For all processes p and q of a process LTS $\mathscr{P} = \langle Pr, A, \rightarrow \rangle$ it holds:*

$$\mathscr{L}(p \,\&\, q) = \mathscr{L}(p) \cap \mathscr{L}(q) \, . \qquad (4.1)$$

Proof. Statement (4.1) can be established by induction on the structure of positive modal formulas ϕ according to their definition in grammar (2.4) of Definition 2.6.

The base case of (4.1) for $\phi = \top$ is obviously true, because $\top$ is satisfied for any process. It remains to establish the induction step for formulas of the forms $\phi = \phi_1 \wedge \phi_2$ and $\phi = \langle a \rangle \phi_0$. Since in the first case the induction step is easy to demonstrate, we only treat the more interesting case of $\phi = \langle a \rangle \phi_0$. For this we argue as follows:

$$\begin{aligned} \langle a \rangle \phi_0 \in \mathscr{L}(p \,\&\, q) \iff & \; p \,\&\, q \vDash \langle a \rangle \phi_0 \\ \iff & \; (\exists \, p', q' \in \mathrm{Pr}) \left[p \xrightarrow{a} p' \wedge q \xrightarrow{a} q' \wedge p' \,\&\, q' \vDash \phi_0 \right] \\ \iff & \; (\exists \, p', q' \in \mathrm{Pr}) \left[p \xrightarrow{a} p' \wedge q \xrightarrow{a} q' \wedge \phi_0 \in \mathscr{L}(p' \,\&\, q') \right] \end{aligned}$$

$$\begin{aligned}
&\overset{\text{IH}}{\Longleftrightarrow} \ (\exists\, p', q' \in \mathrm{Pr})\left[\, p \xrightarrow{a} p' \,\wedge\, q \xrightarrow{a} q' \,\wedge\, \phi_0 \in \mathscr{L}(p') \cap \mathscr{L}(q') \,\right] \\
&\Longleftrightarrow \ (\exists p' \in \mathrm{Pr})\left[\, p \xrightarrow{a} p' \,\wedge\, \phi_0 \in \mathscr{L}(p') \,\right] \wedge (\exists q' \in \mathrm{Pr})\left[\, q \xrightarrow{a} q' \,\wedge\, \phi_0 \in \mathscr{L}(q') \,\right] \\
&\Longleftrightarrow \ \langle a \rangle \phi_0 \in \mathscr{L}(p) \,\wedge\, \langle a \rangle \phi_0 \in \mathscr{L}(q) \\
&\Longleftrightarrow \ \langle a \rangle \phi_0 \in \mathscr{L}(p) \cap \mathscr{L}(q)\,,
\end{aligned}$$

where we have marked by (IH) the logical equivalence in which the induction hypothesis is used. □

Based on this lemma, a modal characterization of $\leq_{\&e}$ and $\leq_e$ is now an easy consequence of the modal characterization of the simulation preorder $\leq$ on processes, see (2.5) in Theorem 2.8. In this way we obtain, in analogy with Theorem 2.10 , the following modal characterizations of (ji-)parameterized similarity and of (ji-)parameterized simulation equivalence with respect to positive formulas.

Theorem 4.2. *For all image-finite environments e, the following characterizations of $\leq_{\&e}$ and $\leq_e$, as well as of $(\leq\geq)_{\&e}$ and $(\leq\geq)_e$ hold for all image-finite processes p, q:*

$$p \leq_{\&e} q \quad \big(\Longleftrightarrow\ p \leq_e q\big) \quad \Longleftrightarrow \quad \mathscr{L}(p) \cap \mathscr{L}(e) \subseteq \mathscr{L}(q) \cap \mathscr{L}(e)\,, \tag{4.2}$$

$$p\,(\leq\geq)_{\&e}\, q \quad \big(\Longleftrightarrow\ p\,(\leq\geq)_e\, q\big) \quad \Longleftrightarrow \quad \mathscr{L}(p) \cap \mathscr{L}(e) = \mathscr{L}(q) \cap \mathscr{L}(e)\,. \tag{4.3}$$

The implications "⇒" in (4.2) *and* (4.3) *hold also for not necessarily image-finite p, q, and e.*

Proof. For (4.2) we argue as follows for all image-finite processes p and q, and environments e:

$$\begin{aligned}
p \leq_e q \ &\Longleftrightarrow\ p \leq_{\&e} q && \text{(by Prop. 3.10, (i))} \\
&\Longleftrightarrow\ p \,\&\, e \leq q \,\&\, e && \text{(by the definition of } \leq_{\&e}\text{)} \\
&\Longleftrightarrow\ \mathscr{L}(p \,\&\, e) \subseteq \mathscr{L}(q \,\&\, e) && \text{(by (2.5) in Thm. 2.8)} \\
&\Longleftrightarrow\ \mathscr{L}(p) \cap \mathscr{L}(e) \subseteq \mathscr{L}(q) \cap \mathscr{L}(e) && \text{(by using Lem. 4.1).}
\end{aligned}$$

The implication "⇒" in the third equivalence statement holds also for not necessarily image-finite p, q, and e due to the Hennessy–Milner Theorem 2.8. Together with the fact that "⇒" also holds for the other three equivalence statements above, this demonstrates that "⇒" in (4.2) holds for all p, q, and e.

Statement (4.3) for the (ji-)parameterized simulation equivalences $(\leq\geq)_{\&e}$ and $(\leq\geq)_e$ follows from (4.2) due to the definition of $(\leq\geq)_{\&e}$ from $\leq_{\&e}$ in (3.3), and of $(\leq\geq)_e$ from $\leq_e$ in Definition 2.4. □

There is no obvious generalization of Lemma 4.1 that applies to all formulas of $\mathscr{M}$. In particular, $\mathscr{M}(p_1 \,\&\, p_2) = \mathscr{M}(p_1) \cap \mathscr{M}(p_2)$ does not hold, because certainly "⊆" is violated: in case that p_1 and p_2 are such that $p_1 \xrightarrow{a}$ and $p_2 \overset{a}{\nrightarrow}$ holds for some $a \in A$, then $\neg\langle a \rangle \top \in \mathscr{M}(p_1 \,\&\, p_2)$ due to $(p_1 \,\&\, p_2) \overset{a}{\nrightarrow}$, but $\neg\langle a \rangle \top \notin \mathscr{M}(p_1)$, and hence $\neg\langle a \rangle \top \notin \mathscr{M}(p_1) \cap \mathscr{M}(p_2)$. Therefore the proof of the characterization above cannot be extended, at least not in an analogous manner, to obtain a modal characterization of ji-parameterized bisimilarity $\sim_{\&e}$. (But see the report about our current work (W1) in Section 5.)

Yet an interesting specialization of Lemma 4.1 concerns the specialized join operation $\&_\bullet$ introduced in Definition 3.4: $|\mathscr{M}(p \,\&_\bullet\, e)|_{\pi_A} = \mathscr{M}(p) \cap \overline{{}^{\neg}\mathscr{L}(e)}$ holds for all processes p and environments e, where $|\cdot|_{\pi_A}$ projects modalities $\langle\langle a, e \rangle\rangle$ in formulas to their action components $\langle a \rangle$. This observation can be used, together with the characterization of parameterized bisimilarity $\sim_e$ via $\&_\bullet$ in (3.4) of Lemma 3.5, to obtain, for Larsen's characterization of $\sim_e$ in Theorem 2.10, an alternative proof that is similar to the proof of Theorem 4.2 above.

5 Conclusion (summary, literature, current work, open problems, plans)

Here we first summarize our contributions in a list with references to statements in earlier sections. We then explain our path to the definition of ji-parameterized bisimilarity, and Larsen's comments on the shortcomings of this concept. Furthermore we collect some references to the literature concerning work that has been done based on parameterized bisimilarity in the meantime. Subsequently we report about our current work on a modal characterization of (ji-)parameterized bisimilarity, and about generalizations of the modal characterizations here and by Stirling and Larsen. We also describe some open problems of which the solutions have evaded us thus far. Finally we mention our plan to investigate whether ji-parameterized bisimilarity can be used to refine Larsen's results in his thesis [9] on a method to show program correctness under the formation of contexts.

Contribution. Below we provide a summary by listing the concepts that we have defined and the results we have obtained, together with references to the appertaining formal statements:

(C1) We complemented Larsen's parameterized bisimilarity $\sim_e$ with respect to 'synchronous' interaction with environments e by also defining parameterized simulatability, the simulation preorder $\leq_e$, on processes with respect to 'synchronous' interaction with environment e (see Definition 2.4).

(C2) We defined weaker versions $\leq_{\&e}$ of $\leq_e$ and $\sim_{\&e}$ of $\sim_e$ by relaxing the synchronicity condition of environment interaction for $\leq_e$ and $\sim_e$ to require only the existence of simulations, and respectively of bisimulations, between free join interactions (&) with environments e (see Definition 3.3).

(C3) We showed that $\leq_e$ and $\sim_e$ can be characterized similarly to the definitions of $\leq_{\&e}$, and $\sim_{\&e}$ via join interactions (&) as the existence of a simulation, and as bisimilarity, respectively, of free interactions with the specific form $\&_\bullet$ (see Definition 3.4) of join interactions that record targets of environment transitions in action labels (see Lemma 3.5).

(C4) We established that $\sim_e$ and $\sim_{\&e}$ coincide for deterministic environments e (see Proposition 3.6).

(C5) We settled the relationships between (ji-)parameterized bisimilarity $\sim_e$ and $\sim_{\&e}$, and the (ji-)parameterized simulation equivalences $(\leq\geq)_e$ and $(\leq\geq)_{\&e}$: for all environments $\sim_e$ is contained in $\sim_{\&e}$, and furthermore $\sim_{\&e}$ is contained in both of $(\leq\geq)_e$ and $(\leq\geq)_{\&e}$, which coincide. The two inclusions in this chain are proper in general. (See Theorem 3.11).

(C6) Larsen's main technical result about $\sim_e$ (see Theorem 2.5), that the discrimination preorder induced by $\sim_e$ on environments coincides with the simulation preorder $\leq$ on environments, does not hold analogously for the discrimination preorder induced by $\sim_{\&e}$ (see Proposition 3.12). However, we showed that this coincidence with the simulation preorder $\leq$ on environments *does* hold analogously for the discrimination preorders induced both by (ji-)parameterized similarity $\leq_e = \leq_{\&e}$ and by (ji-)parameterized simulation equivalence $(\leq\geq)_e = (\leq\geq)_{\&e}$ (see Theorem 3.14).

(C7) We adapted Stirling and Larsen's modal characterization of parameterized bisimilarity $\sim_e$ (see Theorem 2.10) to obtain a modal characterization of (ji-)parameterized similarity $\leq_e = \leq_{\&e}$ and also of (ji-)parameterized simulation equivalence $(\leq\geq)_e = (\leq\geq)_{\&e}$ (see Theorem 4.2).

Larsen on ji-parameterized bisimilarity, and our way to its definition. We formulated ji-parameterized bisimilarity and ji-parameterized simulatability while reading Larsen's article [10] from 1987, and trying to improve our intuitive understanding of parameterized bisimilarity. Afterwards we developed, in stages, the results that we report here. Only when diving deeper into the intricate proof of Larsens main result, the characterization of the discrimination preorder induced by parameterized bisimilarity $\sim_e$ as simulatability of environments (see Theorem 2.5), did we find his remarks about an "alternative and perhaps more immediate parameterized version [of bisimulation equivalence]". This passage appears on page 210 in [10], at the end of Section 5 that is devoted to this central result. The version of bisimulation equivalence that Larsen sketches there coincides with ji-parameterized bisimilarity $\sim_{\&e}$.

Larsen refers to ji-parameterized bisimilarity in order to "give further support for the proposed parameterized version of bisimulation equivalence", in addition to the following assessment: "Due to the modal characterization presented [...] and the simple characterization of the discrimination ordering presented [...], we are confident that the notion of parameterized bisimulation equivalence proposed is indeed a natural one." As for the mentioned further evidence Larsen notes that ji-parameterized bisimilarity "lacks many of the properties presented in this paper". Concretely he mentions three properties.

First, that "$\sim_e$ is strictly included in $\sim_{\&e}$ for all environments e" (in general is meant [we use our notation for $\sim_{\&e}$ here]), corresponding to Proposition 3.7, (i) and (ii)). Second, that "thus the modal characterization for $\sim_e$ does not hold for $\sim_{\&e}$, and no other modal characterization seems immediate." This assessment stimulates us to work out (W1).

Finally third, Larsen writes that: "More important though is that the simulation ordering does not characterize the discrimination ordering generated by this alternative parameterized version[.]", in contrast with his impressive and surprising main result in [10], Theorem 2.5 here, which shows that that is the case for parameterized bisimilarity $\sim_e$. For this observation Larsen uses a counterexample that is slightly different from the one we use for Proposition 3.12, the corresponding statement here. Below we formulate the question of a characterization of the discrimination preorder induced by ji-parameterized bisimilarity as the open problem (P1), and a specialization of this question as the open problem (P2).

Literature on parameterized versions of bisimilarity. Parameterized bisimilarity proved to be a very fruitful concept since its inception by Larsen in [9, 10]. His definition has been applied, specialized, and adapted in multiple ways in the meantime. Please see below for a few examples. But to the best of our knowledge this does not hold for the ji-parameterized concepts of simulatability and bisimilarity, apart from the passages in [10] that we cited and described above.

Parameterized bisimilarity in Larsen's definition [9, 10] has later been called 'relative bisimilarity' and 'relativized bisimilarity' in [12] by Larsen and Milner, who used it also for the practical purpose of verifying the Alternating Bit Protocol [12]. As pointed out in [5], it was also the basis for 'modal transition systems' to which a large body of work has been devoted since, see for example [14, 13, 1, 7].

Environment parameterized bisimulations in the sense of Larsens' definition or adapted and specialized variants of it have been used frequently, for example in [12, 11]. [17] introduces a notion of equivalence parameterized with respect to typing information, which, quoting from [17]: "can be seen as a disciplined instance of Larsen's, in which one uses types to express constraints on the behaviors of the observers, rather than explicitly writing all their possible behavior".

Current Work. We investigate modal-logical characterizations of (ji-)parameterized bisimilarity and simulatability, and of refinements of the modal characterizations already obtained by Larsen, and here.

(W1) We are working out a modal-logical characterization for ji-parameterized bisimilarity $\sim_{\&e}$ that is based on a game characterization of $\sim_{\&e}$. However, our characterization will <u>not</u> just be of a simple form comparable to Theorem 2.10 and Theorem 4.2, for all environments e:

$$p \sim_{\&e} q \iff \mathscr{M}(p) \cap \mathscr{F}(e) = \mathscr{M}(q) \cap \mathscr{F}(e) \quad \text{(for all image-finite processes } p, q\text{).}$$

where $\mathscr{F} \subseteq \mathscr{M}$ would be appropriately defined formulas with then $\mathscr{F}(g) := \{\phi \in \mathscr{F} \mid g \vDash \phi\}$ defined for all environments g. It is nevertheless interesting to note that since $\not\sim_{\&e} \subseteq \not\sim_e$ holds (due to $\sim_e \subseteq \sim_{\&e}$ by Proposition 3.7, (i)), that whenever $p \not\sim_{\&e} q$ holds, always $p \not\sim_e q$ follows, and a formula $\phi \in (\mathscr{M}(p) \cap \overline{\neg\mathscr{L}(e)})\, \Delta\, (\mathscr{M}(q) \cap \overline{\neg\mathscr{L}(e)})$ (where Δ denotes symmetric difference) that distinguishes p and q can always be found via Larsen's characterization, Theorem 2.10.

(W2) The restriction to image-finite processes for the modal characterizations of simulatability $\leq$ and bisimilarity $\sim$ by Hennessy and Milner (Theorem 2.8) can be dropped by permitting infinitary formulas with infinite conjunctions. Indeed, Milner has described such an adaptation for infinitary formulas in [16].

We want to obtain similar extensions to not necessarily image-finite processes for Larsen's characterization of $\sim_e$ (Theorem 2.10) and our ones of $\leq_e = \leq_{\&e}$ and $(\leq\geq)_e = (\leq\geq)_{\&e}$ (Theorem 4.2).

Open problems. As problems to which (satisfactory) answers have evaded us so far, we want to mention:

(P1) How can the discrimination order for $\sim_{\&e}$ be characterized? Note that a similar characterization in terms of simulatability $\leq$ as for the discrimination order of $\sim_e$ in Thm. 2.5 by Larsen, and for $\leq_{\&e}$ and $(\leq\geq)_{\&e}$ in Theorem. 3.14, is not possible due to Proposition 3.12.

(P2) Does equality of ji-parameterized bisimilarity with respect to environments e and f coincide with bisimilarity of e and f? Equivalently, does the implication "$\Leftarrow$" hold in the following statement (of which "$\Rightarrow$" is easy to verify), for all environments e and f:

$$e \sim f \quad \overset{?}{\underset{\Longrightarrow}{\Longleftarrow}} \quad \sim_{\&e} \, = \, \sim_{\&f} \ .$$

Future research. As two lines of research for which the concept of ji-parameterized bisimilarity may lead to new insights we mention: a continuation of Larsen's work in his thesis [9] towards flexible formal methods for showing compositionality of program correctness (see (F1)), and consequences for finding interesting contextual behavioural metrics as introduced in [4] (see (F2)):

(F1) An interesting future work is the study of compositionality properties of $\sim_{\&e}$, that is the behavior of $\sim_{\&e}$ up to context. A context is typically defined as a syntactic process C (expressed in some process algebra) with a hole $[]$. Notation $C[p]$ is used for the process obtained upon substitution of p for the hole in C. In general, $\sim_{\&e}$ is not preserved by contexts: Consider processes $p = a+b$, $q = a$, environment $e = a.b$ and context $C = a.[]$. We have that $p \sim_{\&e} q$, but $C[p] = a.a+b \not\sim_{\&e} a.a = C[q]$. Notice that the above example also applies to $\sim_e$. Indeed, including a process in a context intuitively also affects the environment, as shown in a study of the compositionality of $\sim_e$ in [10]. The idea in that work is to introduce parametric environment-transformer[4] T_C which preserves $\sim_e$ in the following sense:

$$p \ \sim_{T_C(e)} \ q \quad \Longrightarrow \quad <C,p> \equiv_e <C,q> \ ,$$

where $<C,p> \equiv_e <C,q>$ intuitively means that "$C[p] \sim_{\&e} C[q]$ with C interacting identically with p and q" [10] (we omit the formal definition for brevity). We speculate that, for $\sim_{\&e}$, the requirement "C interacting identically with p and q" could be removed. If so, the compositionality of $\sim_{\&e}$ could be expressed as follows (for an appropriate environment- transformer T'_C):

$$p \sim_{\&T'_C(e)} q \quad \Longrightarrow \quad C[p] \sim_{\&e} C[q] \, .$$

(F2) The relatively recent work [4] shows that from $\sim_e$ (and quantitative generalizations of it) one can extract a generalized pseudo-metric between processes, where the codomain of the metric is the set of environments (under some closure assumptions). The idea is that the distance $d(p,q)$ between processes p,q is defined as the largest environment e (according to (2.1)) such that $p \sim_e q$. An obvious future work is exploring whether a metric can be extracted for $\sim_{\&e}$. The main challenge is finding the right notion of "largest environment" for $\sim_{\&e}$, which is related to open problem (P1).

[4]We use a different notation than [10]. There, $T_C(e)$ is rendered as $wie_{\mathbb{E}}(C,e)$

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