Modeling Terms in the λ-Calculus with letrec

(by Term Graphs and Finite-State Automata)

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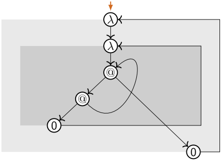

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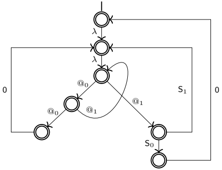

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Explain graph representations for (abstracted) functional programs (λ-terms with recursive bindings) that:

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- facilitate use of standard methods for term graphs and DFAs,
- stay close to the term notation:

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  - use scope sharing,

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Results from the interdisciplinary research project
ROS (Realising Optimal Sharing, Utrecht University, 2009–2014/16),
which brought together:

- term rewriters and logicians (philosophy department, UU)
  - Vincent van Oostrom, CG
- Haskell implementors (CS department, UU)
  - Doaitse Swierstra, Atze Dijkstra, Jan Rochel

Overview

- λ-calculus with letrec ($\boldsymbol{\lambda}_{\text{letrec}}$)
- Expressibility of $\boldsymbol{\lambda}_{\text{letrec}}$ via unfolding
- Maximal sharing of functional programs in $\boldsymbol{\lambda}_{\text{letrec}}$
- Nested term graphs

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  - Which infinite λ-terms are unfoldings of $\boldsymbol{\lambda}_{\text{letrec}}$-terms?
- Maximal sharing of functional programs in $\boldsymbol{\lambda}_{\text{letrec}}$
  - How can $\boldsymbol{\lambda}_{\text{letrec}}$-terms be compressed maximally while preserving their nested scope-structure?
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  - How to get a general framework for terms with nested scopes?

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    - minimization / readback / efficiency / Haskell implementation
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- Nested term graphs
  - How to get a general framework for terms with nested scopes?
    - term graphs with inbuilt nesting

The λ-Calculus with letrec

$$(\lambda f.\,\mathsf{letrec}\ r = f\,r\ \mathsf{in}\ r)\,M$$

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$$(\lambda f.\, \mathsf{let}\ r = f\, r\ \mathsf{in}\ r)\ M$$

The λ-Calculus

Terms in the λ-calculus (over set Var of variables):

(term)	M	$::=$	x	(variable, $x \in Var$)
		$\mid$	$M_1\, M_2$	(application)
		$\mid$	$\lambda x.\, M$	(abstraction)

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Rewriting in $\boldsymbol{\lambda}$:

$$(\lambda x.\, M)\, N \;\to_\beta\; M[x := N] \qquad (\beta\text{-reduction step})$$

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$\lambda x.\, M$	$\to_\alpha$	$\lambda y.\, M[x := y]$	(α-conversion step)

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Terms in the λ-calculus ($\boldsymbol{\lambda}_{\mathsf{letrec}}$) with letrec (over set Var of variables):

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		$\mid$	$M_1\, M_2$	(application)
		$\mid$	$\lambda x.\, M$	(abstraction)
		$\mid$	letrec B in M	(letrec)

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		|	letrec B in M	(letrec)
(binding group)	B	::=	$f_1 = M_1, \ldots, f_n = M_n$	(bindings, $f_1, \ldots, f_n \in Var$)

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		$\mid$	$\lambda x.\, M$	(abstraction)
		$\mid$	$\text{let } B \text{ in } M$	(letrec)
(binding group)	B	$::=$	$f_1 = M_1, \ldots, f_n = M_n$	(bindings, $f_1, \ldots, f_n \in Var$)

Notation: letrec = let (like in Haskell).

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		$\mid$	$\lambda x.\, M$	(abstraction)
		$\mid$	let B in M	(letrec)
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Notation: letrec = let (like in Haskell).

Rewriting in $\boldsymbol{\lambda}_{\text{letrec}}$:

$(\lambda x.\, M)\, N$	$\to_\beta$	$M[x := N]$	(β-reduction step)
$\lambda x.\, M$	$\to_\alpha$	$\lambda y.\, M[x := y]$	(α-conversion step)
$\text{let } B \text{ in } M$	$\to_\triangledown$	$\ldots$	(unfolding steps)

Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$

For $\quad \mathsf{fix} := \lambda f.\, \mathsf{let}\ r = f\ r\ \mathsf{in}\ r \quad$ we find:

$$\mathsf{fix}$$

Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$ (infinite unfolding)

For $\mathsf{fix} := \lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r$ we find:

$$\mathsf{fix} = \lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r$$

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For $\mathsf{fix} := \lambda f.\, \mathsf{let}\ r = f\, r\ \mathsf{in}\ r$ we find:

$$\begin{aligned} \mathsf{fix} &= \lambda f.\, \mathsf{let}\ r = f\, r\ \mathsf{in}\ r \\ &\to_{\triangledown} \lambda f.\, \mathsf{let}\ r = f\, r\ \mathsf{in}\ f\, r \end{aligned}$$

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\end{aligned}$$

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&\rightarrow_{\triangledown} \lambda f.\ \mathsf{let}\ r = f\,r\ \mathsf{in}\ f\,r \\
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&\to_{\triangledown}\; \lambda f.\,f\,(\boxed{\text{let } r = f\,r \text{ in } r}) \\
&\twoheadrightarrow_{\triangledown}\; \lambda f.\,f\,(f\,(\boxed{\text{let } r = f\,r \text{ in } r}))
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\rightarrow_{\triangledown}\;& \lambda f.\, f\, (\boxed{\mathsf{let}\ r = f\, r\ \mathsf{in}\ r}) \\
\twoheadrightarrow_{\triangledown}\;& \lambda f.\, f\, (f\, (\boxed{\mathsf{let}\ r = f\, r\ \mathsf{in}\ r})) \\
\twoheadrightarrow_{\triangledown}\;& \lambda f.\, f\, (f\, (\ldots f\, (\boxed{\mathsf{let}\ r = f\, r\ \mathsf{in}\ r}))) \\
\twoheadrightarrow\!\!\!\!\rightarrow_{\triangledown}\;& \lambda f.\, f\, (f\, (\ldots f\, (\ldots)))
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Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$ (infinite unfolding)

For $\text{fix} := \lambda f.\,\text{let } r = f\,r \text{ in } r$ we find:

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\end{aligned}
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Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$ (infinite unfolding)

For $\text{fix} := \lambda f.\,\text{let } r = f\,r \text{ in } r$ we find:

$$
\begin{aligned}
\text{fix} &= && \lambda f.\,\text{let } r = f\,r \text{ in } r \\
&\to_{\triangledown} && \lambda f.\,\text{let } r = f\,r \text{ in } f\,r \\
&\to_{\triangledown} && \lambda f.\,(\text{let } r = f\,r \text{ in } f)\,(\text{let } r = f\,r \text{ in } r) \\
&\to_{\triangledown} && \lambda f.\,f\,(\text{let } r = f\,r \text{ in } r) \\
&\twoheadrightarrow_{\triangledown} && \lambda f.\,f\,(f\,(\text{let } r = f\,r \text{ in } r)) \\
&\twoheadrightarrow_{\triangledown} && \lambda f.\,f\,(f\,(\ldots f\,(\text{let } r = f\,r \text{ in } r))) \\
&\twoheadrightarrow\!\!\!\!\!\rightarrow_{\triangledown} && \lambda f.\,f\,(f\,(\ldots f\,(\ldots))) \\
&= && [\![\text{fix}]\!]_{\lambda^\infty}
\end{aligned}
$$

Fixed-point combinator in λ_{letrec}

For $\mathsf{fix} := \lambda f.\, \mathsf{let}\ r = f\ r\ \mathsf{in}\ r$ we find:

$$\mathsf{fix}\, M$$

Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$

For $\mathsf{fix} := \lambda f.\, \mathsf{let}\ r = f\, r\ \mathsf{in}\ r$ we find:

$\mathsf{fix}\, M$

$M\, (\mathsf{fix}\, M)$

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For $\text{fix} := \lambda f.\,\text{let}\ r = f\,r\ \text{in}\ r$ we find:

$$\text{fix}\,M \;=\; (\lambda f.\,\text{let}\ r = f\,r\ \text{in}\ r)\,M$$

$$M\,(\text{fix}\,M)$$

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&\to_{\triangledown} \text{let } r = M\,r \text{ in } M\,r \\
&\to_{\triangledown} (\text{let } r = M\,r \text{ in } M)\,(\text{let } r = M\,r \text{ in } r)
\end{aligned}$$

$$M\,(\text{fix}\, M)$$

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$$
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&\to_\beta \text{let } r = M\, r \text{ in } r \\
&\to_\triangledown \text{let } r = M\, r \text{ in } M\, r \\
&\to_\triangledown (\text{let } r = M\, r \text{ in } M)\, (\text{let } r = M\, r \text{ in } r) \\
&\to_\triangledown M\, (\text{let } r = M\, r \text{ in } r)
\end{aligned}
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&\to_{\triangledown}\; (\mathsf{let}\ r = M\,r\ \mathsf{in}\ M)\,(\mathsf{let}\ r = M\,r\ \mathsf{in}\ r) \\
&\to_{\triangledown}\; M\,(\mathsf{let}\ r = M\,r\ \mathsf{in}\ r) \\
&\leftarrow_{\beta}\; M\,((\lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r)\,M) \\
&\phantom{\leftarrow_{\beta}}\; M\,(\mathsf{fix}\,M)
\end{aligned}
$$

Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$

For $\quad \mathsf{fix} := \lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r \quad$ we find:

$$
\begin{aligned}
\mathsf{fix}\,M \;&=\; (\lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r)\,M \\
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&\to_{\triangledown}\; (\mathsf{let}\ r = M\,r\ \mathsf{in}\ M)\,(\mathsf{let}\ r = M\,r\ \mathsf{in}\ r) \\
&\to_{\triangledown}\; M\,(\mathsf{let}\ r = M\,r\ \mathsf{in}\ r) \\
&\leftarrow_{\beta}\; M\,((\lambda f.\,\mathsf{let}\ r = f\,r\ \mathsf{in}\ r)\,M) \\
&=\; M\,(\mathsf{fix}\,M)
\end{aligned}
$$

Fixed-point combinator in $\boldsymbol{\lambda}_{\text{letrec}}$

For $\text{fix} := \lambda f.\,\text{let } r = f\,r \text{ in } r$ we find:

$$
\begin{aligned}
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&\to_{\beta} \text{let } r = M\,r \text{ in } r \\
&\to_{\triangledown} \text{let } r = M\,r \text{ in } M\,r \\
&\to_{\triangledown} (\text{let } r = M\,r \text{ in } M)\,(\text{let } r = M\,r \text{ in } r) \\
&\to_{\triangledown} M\,(\text{let } r = M\,r \text{ in } r) \\
&\leftarrow_{\beta} M\,((\lambda f.\,\text{let } r = f\,r \text{ in } r)\,M) \\
&= M\,(\text{fix}\,M)
\end{aligned}
$$

$$\text{fix}\,M \leftrightarrow^{*}_{\beta\triangledown} M\,(\text{fix}\,M)$$

Fixed-point combinator in λ_{letrec}

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$$
\begin{aligned}
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&\to_{\triangledown} \text{let } r = M\,r \text{ in } M\,r \\
&\to_{\triangledown} (\text{let } r = M\,r \text{ in } M)\,(\text{let } r = M\,r \text{ in } r) \\
&\to_{\triangledown} M\,(\text{let } r = M\,r \text{ in } r) \\
&\leftarrow_{\beta} M\,((\lambda f.\,\text{let } r = f\,r \text{ in } r)\,M) \\
&= M\,(\text{fix}\,M)
\end{aligned}
$$

$$
\begin{aligned}
\text{fix}\,M &\leftrightarrow^{*}_{\beta\triangledown} M\,(\text{fix}\,M) \\
&\leftrightarrow^{*}_{\beta\triangledown} M\,(M\,(\ldots(M\,(\text{fix}\,M))\ldots))
\end{aligned}
$$

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$$
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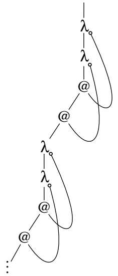
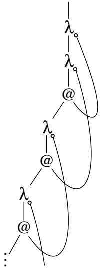
&\to_\beta \text{let } r = M\,r \text{ in } r \\
&\to_\triangledown \text{let } r = M\,r \text{ in } M\,r \\
&\to_\triangledown (\text{let } r = M\,r \text{ in } M)\,(\text{let } r = M\,r \text{ in } r) \\
&\to_\triangledown M\,(\text{let } r = M\,r \text{ in } r) \\
&\leftarrow_\beta M\,((\lambda f.\,\text{let } r = f\,r \text{ in } r)\,M) \\
&= M\,(\text{fix}\,M)
\end{aligned}
$$

$$
\begin{aligned}
\text{fix}\,M &\leftrightarrow^*_{\beta\triangledown} M\,(\text{fix}\,M) \\
&\leftrightarrow^*_{\beta\triangledown} M\,(M\,(\ldots(M\,(\text{fix}\,M))\ldots)) \\
&(\to^+_{\beta\triangledown}\cdot\leftarrow_\beta)^\omega\; M\,(M\,(\ldots(M\,(\ldots))\ldots))\,.
\end{aligned}
$$

Expressibility of λ_{letrec} via unfolding

(joint work with Jan Rochel)

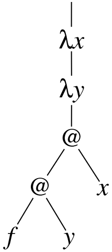




Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

Example

let $f = \lambda x. \lambda y. f\, y\, x$ in f



Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

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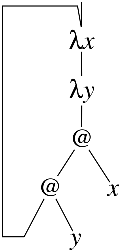
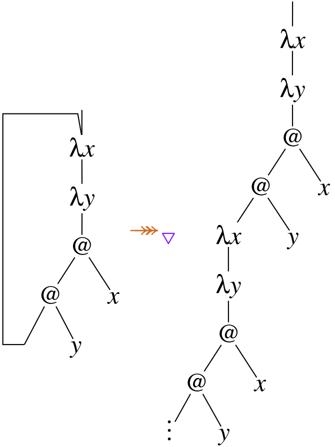
let $f = \lambda x.\, \lambda y.\, f\, y\, x$ in f



Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

Example

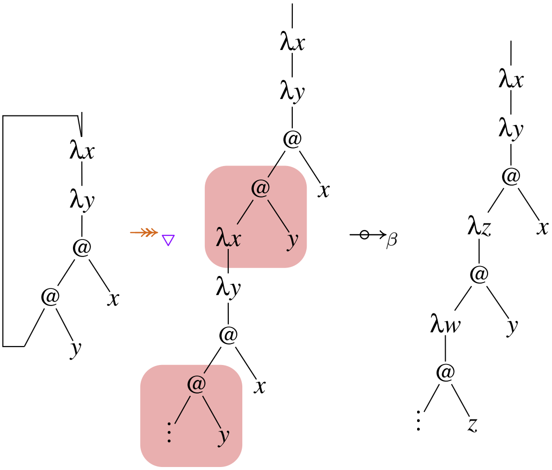
let $f = \lambda x.\, \lambda y.\, f\, y\, x$ in $f \quad \twoheadrightarrow_{\triangledown} \quad \lambda xy.\, (\lambda xy.\, (\lambda xy.\, (\ldots)\, y\, x)\, y\, x)\, y\, x$



Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

Example

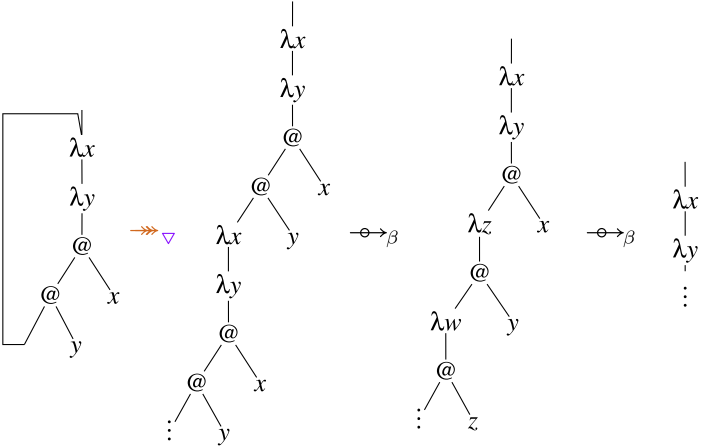
let $f = \lambda x.\, \lambda y.\, f\, y\, x$ in f $\quad \twoheadrightarrow_{\triangledown} \quad \lambda xy.\, (\lambda xy.\, (\lambda xy.\, (\ldots)\, y\, x)\, y\, x)\, y\, x$



Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

Example

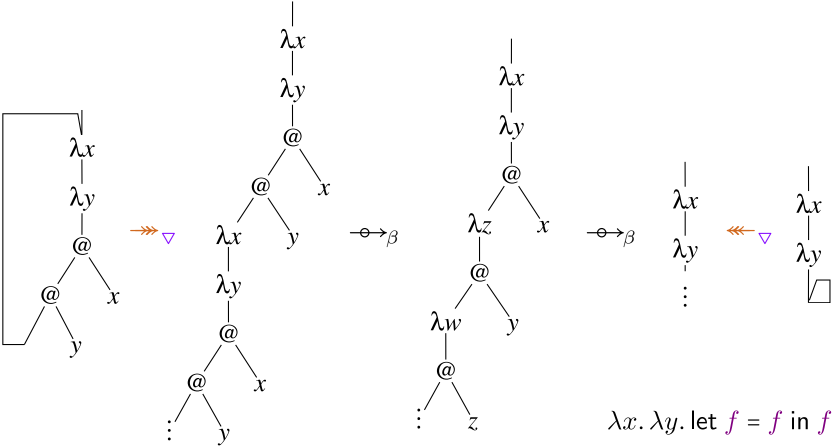
let $f = \lambda x.\, \lambda y.\, f\, y\, x$ in f $\twoheadrightarrow_{\triangledown}$ $\lambda xy.\, (\lambda xy.\, (\lambda xy.\, (\ldots)\, y\, x)\, y\, x)\, y\, x$



Which infinite λ-terms are expressible finitely in $\boldsymbol{\lambda}_{\text{letrec}}$?

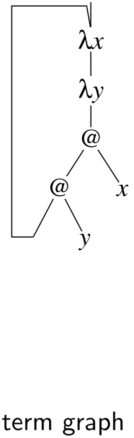
Example

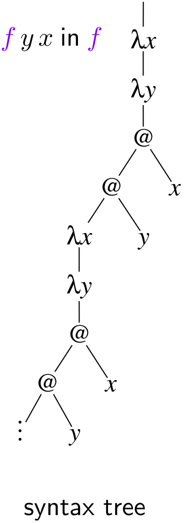
let $f = \lambda x.\, \lambda y.\, f\, y\, x$ in f $\quad \twoheadrightarrow_{\triangledown} \quad$ $\lambda xy.\, (\lambda xy.\, (\lambda xy.\, (\ldots)\, y\, x)\, y\, x)\, y\, x$



λ_{letrec}-Expressible 'regular' λ^∞-term

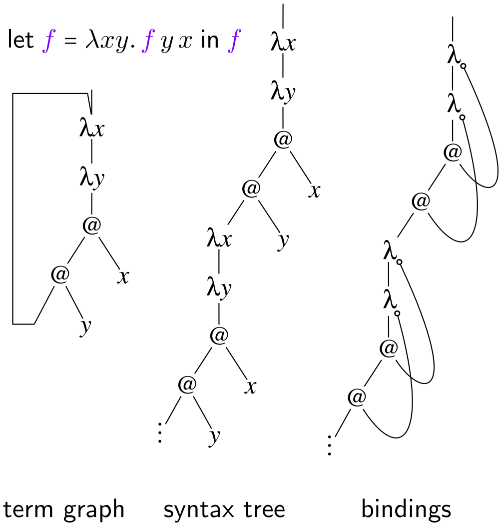
let $f = \lambda xy.\ f\ y\ x$ in f





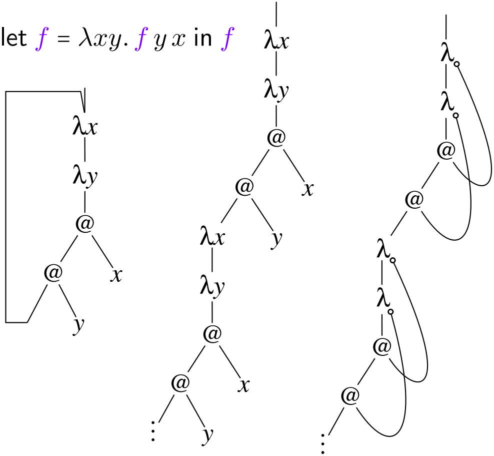
λ_{letrec}-Expressible 'regular' λ^∞-term

let $f = \lambda xy.\, f\, y\, x$ in f



λ_{letrec}-Expressible 'regular' λ^∞-term

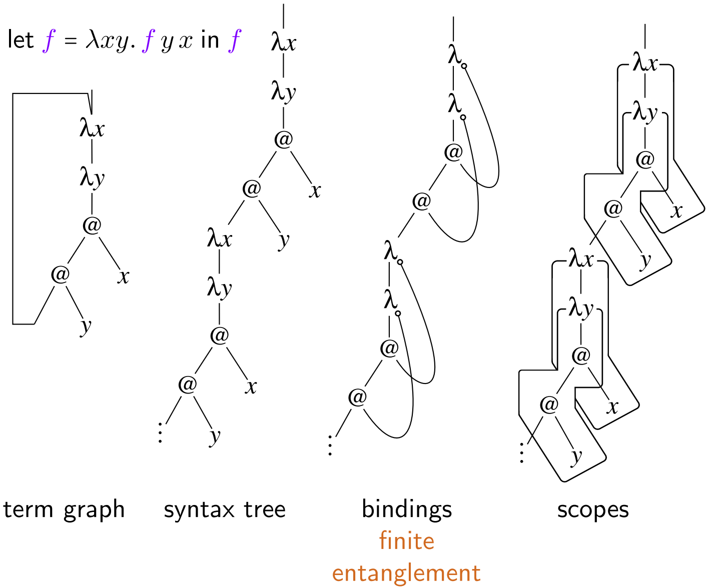
let $f = \lambda xy.\, f\, y\, x$ in f



term graph syntax tree bindings finite entanglement

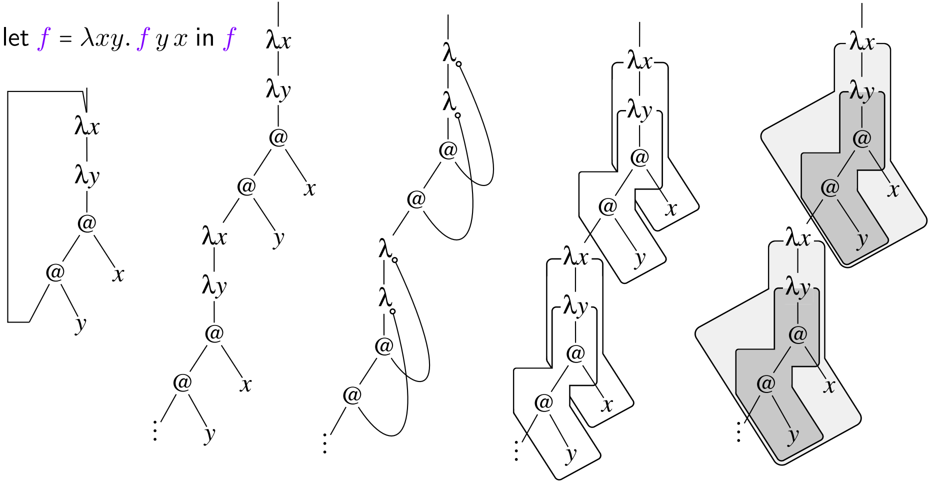
λ_{letrec}-Expressible 'regular' λ^{∞}-term

let $f = \lambda xy.\, f\, y\, x$ in f



λ_{letrec}-Expressible 'regular' λ^∞-term

let $f = \lambda xy.\, f\, y\, x$ in f

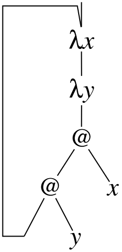


term graph | syntax tree | bindings | scopes | scope⁺s

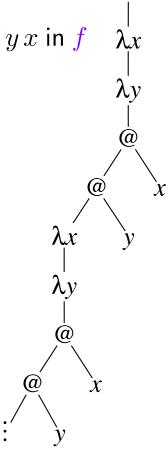
finite entanglement

λ_{letrec}-Expressible 'regular' λ^∞-term

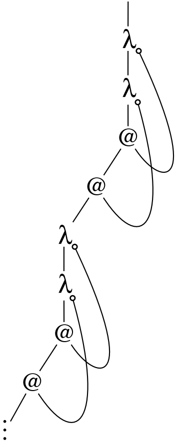
let $f = \lambda xy.\, f\, y\, x$ in f



term graph

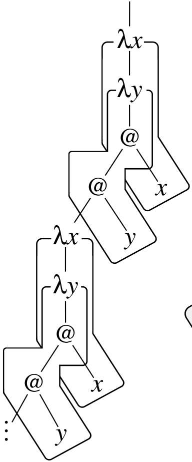


syntax tree

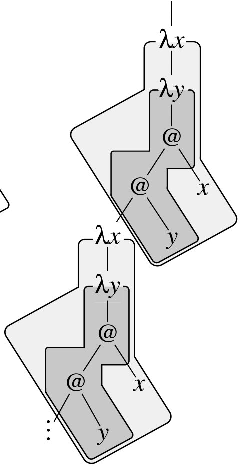


bindings

finite entanglement



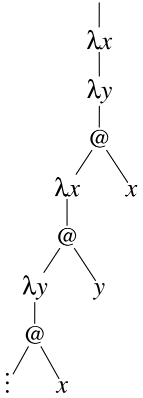
scopes



scope$^+$s

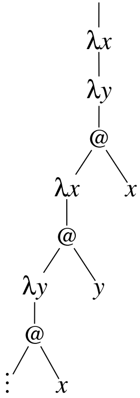
finite nesting

Not λ_{letrec}-expressible 'regular' λ^{∞}-term

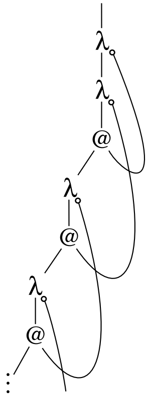


syntax tree

Not $\boldsymbol{\lambda}_{\text{letrec}}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term

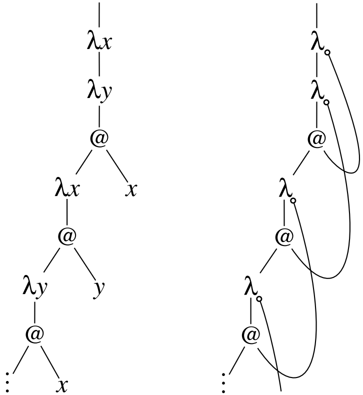


syntax tree



bindings

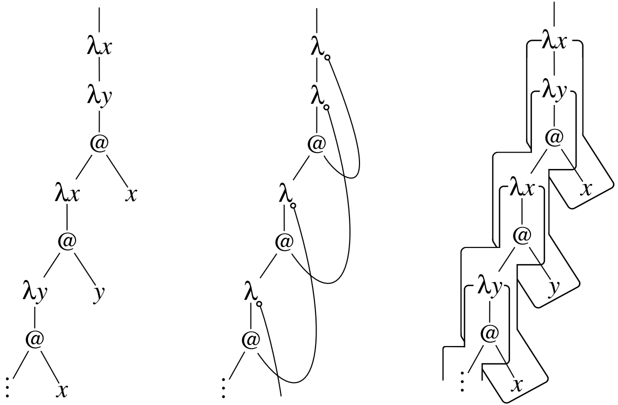
Not $\boldsymbol{\lambda}_{\text{letrec}}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term



syntax tree

bindings
infinitely entangled

Not $\boldsymbol{\lambda}_{\text{letrec}}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term



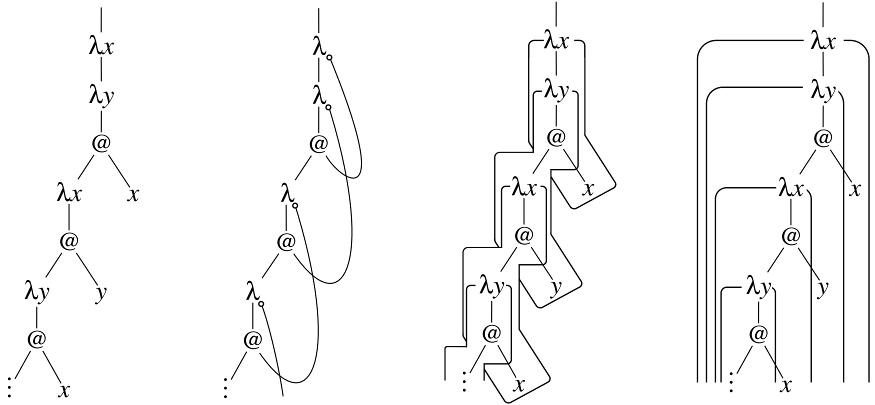
syntax tree

bindings

infinitely entangled

scopes

Not $\boldsymbol{\lambda}_{\text{letrec}}$-expressible 'regular' $\boldsymbol{\lambda}^{\infty}$-term



syntax tree

bindings
infinitely entangled

scopes

scope^{+}s
infinite nesting

Deconstructing/observing λ^∞-terms

$$()\, \lambda x.\, \lambda y.\, x\, x\, y$$

Deconstructing/observing λ^∞-terms

$$( )\, \lambda x.\, \lambda y.\, x\, x\, y \rightarrow_\lambda$$
$$(x)\, \lambda y.\, x\, x\, y$$

$$(x_1 \dots x_n)\, \lambda x_{n+1}.\, M_0 \;\rightarrow_\lambda\; (x_1 \dots x_{n+1})\, M_0$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&()\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda \\
&(x)\,\lambda y.\,x\,x\,y \to_\lambda \\
&(xy)\,x\,x\,y
\end{aligned}
$$

$$(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 \;\to_\lambda\; (x_1 \ldots x_{n+1})\,M_0$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&(\,)\,\lambda x.\,\lambda y.\,x\,x\,y \rightarrow_{\lambda}\\
&(x)\,\lambda y.\,x\,x\,y \rightarrow_{\lambda}\\
&(xy)\,x\,x\,y \rightarrow_{@_0}\\
&(xy)\,x\,x
\end{aligned}
$$

$$
\begin{aligned}
(x_1 \dots x_n)\, M_0\, M_1 \;&\rightarrow_{@_i}\; (x_1 \dots x_n)\, M_i \qquad (i \in \{0,1\})\\
(x_1 \dots x_n)\, \lambda x_{n+1}.\, M_0 \;&\rightarrow_{\lambda}\; (x_1 \dots x_{n+1})\, M_0
\end{aligned}
$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&(\,)\,\lambda x.\,\lambda y.\,x\,x\,y \rightarrow_{\lambda}\\
&(x)\,\lambda y.\,x\,x\,y \rightarrow_{\lambda}\\
&(xy)\,x\,x\,y \rightarrow_{@_0}\\
&(xy)\,x\,x \rightarrow_{\mathsf{S}}\\
&(x)\,x\,x
\end{aligned}
$$

$$
\begin{aligned}
(x_1 \ldots x_n)\,M_0\,M_1 &\rightarrow_{@_i} (x_1 \ldots x_n)\,M_i && (i \in \{0,1\})\\
(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 &\rightarrow_{\lambda} (x_1 \ldots x_{n+1})\,M_0 &&\\
(x_1 \ldots x_n x_{n+1})\,M_0 &\rightarrow_{\mathsf{S}} (x_1 \ldots x_n)\,M_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{aligned}
$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&() \, \lambda x.\, \lambda y.\, x\, x\, y \to_\lambda \\
&(x) \, \lambda y.\, x\, x\, y \to_\lambda \\
&(xy) \, x\, x\, y \to_{@_0} \\
&(xy) \, x\, x \to_{\mathsf{S}} \\
&(x) \, x\, x \to_{@_0} \\
&(x) \, x
\end{aligned}
$$

$$
\begin{aligned}
(x_1 \dots x_n)\, M_0\, M_1 &\to_{@_i} (x_1 \dots x_n)\, M_i && (i \in \{0,1\}) \\
(x_1 \dots x_n)\, \lambda x_{n+1}.\, M_0 &\to_\lambda (x_1 \dots x_{n+1})\, M_0 && \\
(x_1 \dots x_n x_{n+1})\, M_0 &\to_{\mathsf{S}} (x_1 \dots x_n)\, M_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{aligned}
$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&()\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda \\
&(x)\,\lambda y.\,x\,x\,y \to_\lambda \\
&(xy)\,x\,x\,y \to_{@_0} \\
&(xy)\,x\,x \to_{\mathsf{S}} \\
&(x)\,x\,x \to_{@_0} \\
&(x)\,x
\end{aligned}
$$

$\to_{\text{reg}^+}$-generated subterms of $\lambda x.\,\lambda y.\,x\,x\,y$ w.r.t. rewrite relation $\to_{\text{reg}^+}$:

$$
\begin{aligned}
(x_1 \dots x_n)\,M_0\,M_1 &\to_{@_i} (x_1 \dots x_n)\,M_i && (i \in \{0,1\}) \\
(x_1 \dots x_n)\,\lambda x_{n+1}.\,M_0 &\to_\lambda (x_1 \dots x_{n+1})\,M_0 \\
(x_1 \dots x_n x_{n+1})\,M_0 &\to_{\mathsf{S}} (x_1 \dots x_n)\,M_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{aligned}
$$

Deconstructing/observing λ^∞-terms

$$
\begin{aligned}
&(\,)\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda\\
&(x)\,\lambda y.\,x\,x\,y \to_\lambda\\
&(xy)\,x\,x\,y \to_{@_0}\\
&(xy)\,x\,x \to_{\mathsf{S}}\\
&(x)\,x\,x \to_{@_0}\\
&(x)\,x
\end{aligned}
$$

$\to_{\text{reg}^+}$-generated subterms of $\lambda x.\,\lambda y.\,x\,x\,y$ w.r.t. rewrite relation $\to_{\text{reg}^+}$:

$$
\begin{aligned}
(x_1 \ldots x_n)\,M_0\,M_1 &\to_{@_i} (x_1 \ldots x_n)\,M_i && (i \in \{0,1\})\\
(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 &\to_\lambda (x_1 \ldots x_{n+1})\,M_0\\
(x_1 \ldots x_n x_{n+1})\,M_0 &\to_{\mathsf{S}} (x_1 \ldots x_n)\,M_0 && (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{aligned}
$$

formalized as a CRS, e.g. rule:

$$\mathsf{pre}_n([x_1 \ldots x_n]\mathsf{abs}([x_{n+1}]Z(\vec{x}))) \to \mathsf{pre}_{n+1}([x_1 \ldots x_{n+1}]Z(\vec{x}))$$

Deconstructing/observing λ^∞-terms

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda \\
(x)\,\lambda y.\,x\,x\,y \to_\lambda \\
(xy)\,x\,x\,y \to_{@_1} \\
(xy)\,y
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda \\
(x)\,\lambda y.\,x\,x\,y \to_\lambda \\
(xy)\,x\,x\,y \to_{@_0} \\
(xy)\,x\,x \to_{\mathsf{S}} \\
(x)\,x\,x \to_{@_0} \\
(x)\,x
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_\lambda \\
(x)\,\lambda y.\,x\,x\,y \to_\lambda \\
(xy)\,x\,x\,y \to_{@_0} \\
(xy)\,x\,x \to_{\mathsf{S}} \\
(x)\,x\,x \to_{@_1} \\
(x)\,x
\end{array}
$$

$\to_{\mathsf{reg}^+}$-generated subterms of $\lambda x.\,\lambda y.\,x\,x\,y$ w.r.t. rewrite relation $\to_{\mathsf{reg}^+}$:

$$
\begin{array}{rcll}
(x_1 \ldots x_n)\,M_0\,M_1 & \to_{@_i} & (x_1 \ldots x_n)\,M_i & (i \in \{0,1\}) \\
(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 & \to_\lambda & (x_1 \ldots x_{n+1})\,M_0 & \\
(x_1 \ldots x_n x_{n+1})\,M_0 & \to_{\mathsf{S}} & (x_1 \ldots x_n)\,M_0 & (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{array}
$$

formalized as a CRS, e.g. rule:

$$
\mathsf{pre}_n([x_1 \ldots x_n]\,\mathsf{abs}([x_{n+1}]\,Z(\vec{x}))) \to \mathsf{pre}_{n+1}([x_1 \ldots x_{n+1}]\,Z(\vec{x}))
$$

Generated subterms

$( )\, \lambda x.\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(x)\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(xy)\, x\, x\, y \rightarrow_{@_1}$
$(xy)\, y$

$( )\, \lambda x.\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(x)\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(xy)\, x\, x\, y \rightarrow_{@_0}$
$(xy)\, x\, x \rightarrow_{\mathsf{S}}$
$(x)\, x\, x \rightarrow_{@_0}$
$(x)\, x$

$( )\, \lambda x.\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(x)\, \lambda y.\, x\, x\, y \rightarrow_{\lambda}$
$(xy)\, x\, x\, y \rightarrow_{@_0}$
$(xy)\, x\, x \rightarrow_{\mathsf{S}}$
$(x)\, x\, x \rightarrow_{@_1}$
$(x)\, x$

$$(x_1 \ldots x_n)\, M_0\, M_1 \rightarrow_{@_i} (x_1 \ldots x_n)\, M_i \qquad (i \in \{0,1\})$$
$$(x_1 \ldots x_n)\, \lambda x_{n+1}.\, M_0 \rightarrow_{\lambda} (x_1 \ldots x_{n+1})\, M_0$$
$$(x_1 \ldots x_n x_{n+1})\, M_0 \rightarrow_{\mathsf{S}} (x_1 \ldots x_n)\, M_0 \qquad (\text{if } \lambda x_{n+1} \text{ is vacuous})$$

$\rightarrow_{\text{reg}}$-generated subterms w.r.t. rewrite relation $\rightarrow_{\text{reg}}$, rules above plus:

$$(x_1 \ldots x_i \ldots x_{n+1})\, M_0 \rightarrow_{\text{del}} (x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\, M_0 \qquad (\text{if } \lambda x_i \text{ is vacuous})$$

Generated subterms

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(xy)\,x\,x\,y \to_{@_1} \\
(xy)\,y
\end{array}
$$

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(xy)\,x\,x\,y \to_{@_0} \\
(xy)\,x\,x \to_{\mathsf{S}} \\
(x)\,x\,x \to_{@_0} \\
(x)\,x
\end{array}
$$

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda} \\
(xy)\,x\,x\,y \to_{@_0} \\
(xy)\,x\,x \to_{\mathsf{S}} \\
(x)\,x\,x \to_{@_1} \\
(x)\,x
\end{array}
$$

$$
\begin{array}{rcll}
(x_1 \ldots x_n)\,M_0\,M_1 & \to_{@_i} & (x_1 \ldots x_n)\,M_i & (i \in \{0,1\}) \\
(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 & \to_{\lambda} & (x_1 \ldots x_{n+1})\,M_0 &
\end{array}
$$

$\to_{\text{reg}}$-generated subterms w.r.t. rewrite relation $\to_{\text{reg}}$, rules above plus:

$$(x_1 \ldots x_i \ldots x_{n+1})\,M_0 \to_{\text{del}} (x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\,M_0 \quad \text{(if } \lambda x_i \text{ is vacuous)}$$

Generated subterms

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(xy)\,x\,x\,y \to_{@_1}\\
(xy)\,y \to_{\mathsf{del}}\\
(y)\,y
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(xy)\,x\,x\,y \to_{@_0}\\
(xy)\,x\,x \to_{\mathsf{S}}\\
(x)\,x\,x \to_{@_0}\\
(x)\,x
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(x)\,\lambda y.\,x\,x\,y \to_{\lambda}\\
(xy)\,x\,x\,y \to_{@_0}\\
(xy)\,x\,x \to_{\mathsf{S}}\\
(x)\,x\,x \to_{@_1}\\
(x)\,x
\end{array}
$$

$$
\begin{array}{rll}
(x_1 \ldots x_n)\, M_0\, M_1 & \to_{@_i} & (x_1 \ldots x_n)\, M_i \qquad (i \in \{0,1\})\\
(x_1 \ldots x_n)\, \lambda x_{n+1}.\, M_0 & \to_{\lambda} & (x_1 \ldots x_{n+1})\, M_0
\end{array}
$$

$\to_{\mathsf{reg}}$-generated subterms w.r.t. rewrite relation $\to_{\mathsf{reg}}$, rules above plus:

$$(x_1 \ldots x_i \ldots x_{n+1})\, M_0 \to_{\mathsf{del}} (x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\, M_0 \quad (\text{if } \lambda x_i \text{ is vacuous})$$

Generated subterms

$() \lambda x. \lambda y. x\, x\, y \to_{\lambda}$
$(x) \lambda y. x\, x\, y \to_{\lambda}$
$(xy)\, x\, x\, y \to_{@_1}$
$(xy)\, y \to_{\mathsf{del}}$
$(y)\, y$

$() \lambda x. \lambda y. x\, x\, y \to_{\lambda}$
$(x) \lambda y. x\, x\, y \to_{\lambda}$
$(xy)\, x\, x\, y \to_{@_0}$
$(xy)\, x\, x \to_{\mathsf{S}}$
$(x)\, x\, x \to_{@_0}$
$(x)\, x$

$() \lambda x. \lambda y. x\, x\, y \to_{\lambda}$
$(x) \lambda y. x\, x\, y \to_{\lambda}$
$(xy)\, x\, x\, y \to_{@_0}$
$(xy)\, x\, x \to_{\mathsf{S}}$
$(x)\, x\, x \to_{@_1}$
$(x)\, x$

$\to_{\mathsf{reg}^+}$-generated subterms of $\lambda x. \lambda y. x\, x\, y$ w.r.t. rewrite relation $\to_{\mathsf{reg}^+}$:

$$(x_1 \ldots x_n)\, M_0\, M_1 \to_{@_i} (x_1 \ldots x_n)\, M_i \qquad (i \in \{0,1\})$$
$$(x_1 \ldots x_n)\, \lambda x_{n+1}. M_0 \to_{\lambda} (x_1 \ldots x_{n+1})\, M_0$$
$$(x_1 \ldots x_n x_{n+1})\, M_0 \to_{\mathsf{S}} (x_1 \ldots x_n)\, M_0 \qquad (\text{if } \lambda x_{n+1} \text{ is vacuous})$$

$\to_{\mathsf{reg}}$-generated subterms w.r.t. rewrite relation $\to_{\mathsf{reg}}$, rules above plus:

$$(x_1 \ldots x_i \ldots x_{n+1})\, M_0 \to_{\mathsf{del}} (x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\, M_0 \qquad (\text{if } \lambda x_i \text{ is vacuous})$$

Generated subterms

$$
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(xy)\,x\,x\,y \rightarrow_{@_1} \\
(xy)\,y \rightarrow_{\mathsf{del}} \\
(y)\,y
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(xy)\,x\,x\,y \rightarrow_{@_0} \\
(xy)\,x\,x \rightarrow_{\mathsf{S}} \\
(x)\,x\,x \rightarrow_{@_0} \\
(x)\,x
\end{array}
\qquad
\begin{array}{l}
()\,\lambda x.\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(x)\,\lambda y.\,x\,x\,y \rightarrow_{\lambda} \\
(xy)\,x\,x\,y \rightarrow_{@_0} \\
(xy)\,x\,x \rightarrow_{\mathsf{S}} \\
(x)\,x\,x \rightarrow_{@_1} \\
(x)\,x
\end{array}
$$

$\rightarrow_{\mathsf{reg}^+}$-generated subterms of $\lambda x.\,\lambda y.\,x\,x\,y$ w.r.t. rewrite relation $\rightarrow_{\mathsf{reg}^+}$:

$$
\begin{array}{rcll}
(x_1 \ldots x_n)\,M_0\,M_1 & \rightarrow_{@_i} & (x_1 \ldots x_n)\,M_i & (i \in \{0,1\}) \\
(x_1 \ldots x_n)\,\lambda x_{n+1}.\,M_0 & \rightarrow_{\lambda} & (x_1 \ldots x_{n+1})\,M_0 & \\
(x_1 \ldots x_n x_{n+1})\,M_0 & \rightarrow_{\mathsf{S}} & (x_1 \ldots x_n)\,M_0 & (\text{if } \lambda x_{n+1} \text{ is vacuous})
\end{array}
$$

$\rightarrow_{\mathsf{reg}}$-generated subterms w.r.t. rewrite relation $\rightarrow_{\mathsf{reg}}$, rules above plus:

$$(x_1 \ldots x_i \ldots x_{n+1})\,M_0 \rightarrow_{\mathsf{del}} (x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1})\,M_0 \quad (\text{if } \lambda x_i \text{ is vacuous})$$

We use eager application of scope-closure rules for $\rightarrow_{\mathsf{reg}^+}$ and $\rightarrow_{\mathsf{reg}}$.

Regularity and strong regularity

An infinite first-order term t is regular if:

t has only finitely many subterms.

Definition

1. A λ^{∞}-term M is strongly regular if:

   $()M$ has only finitely many $\rightarrow_{\text{reg}^+}$-generated subterms.

Regularity and strong regularity

An infinite first-order term t is regular if:

t has only finitely many subterms.

Definition

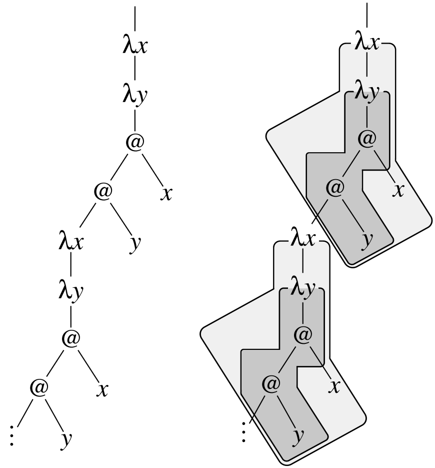
1. A λ^∞-term M is strongly regular if:

   $()M$ has only finitely many $\rightarrow_{\text{reg}^+}$-generated subterms.

2. A λ^∞-term N is regular if:

   $()N$ has only finitely many $\rightarrow_{\text{reg}}$-generated subterms.

Strongly regular λ^∞-term

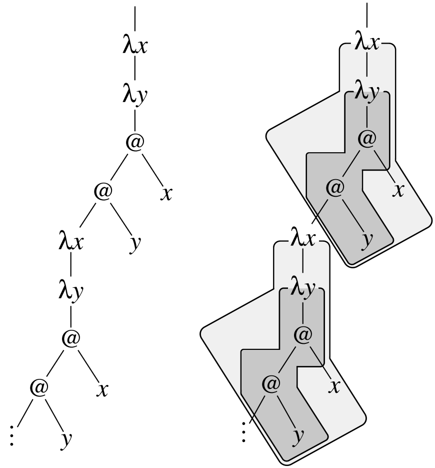


$$()\, M \quad = \quad ()\, \lambda xy.\, M\, y\, x$$

$M = \lambda xy.\, M\, y\, x$

$\rightarrow_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

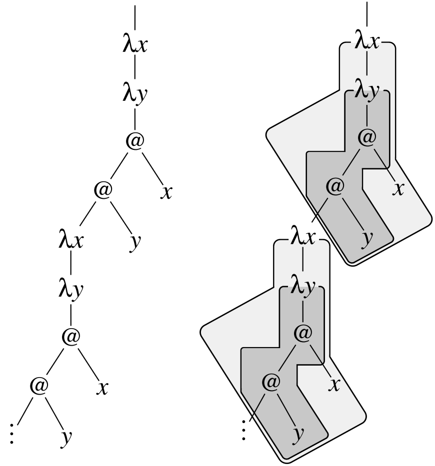


$$
\begin{aligned}
(\,)\,M \quad &= \quad (\,)\,\lambda xy.\, M\, y\, x \\
&\to_\lambda \quad (x)\,\lambda y.\, M\, y\, x
\end{aligned}
$$

$M = \lambda xy.\, M\, y\, x$

$\to_{\text{reg}^+}$-generated subterms

Strongly regular λ^{∞}-term

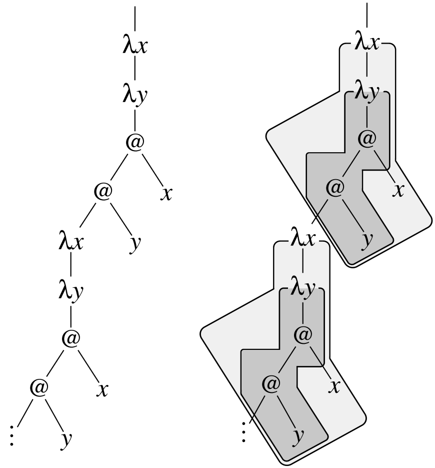


$$\begin{aligned} ()\,M &= ()\,\lambda xy.\,M\,y\,x \\ &\to_{\lambda} (x)\,\lambda y.\,M\,y\,x \\ &\to_{\lambda} (xy)\,M\,y\,x \end{aligned}$$

$M = \lambda xy.\,M\,y\,x$

$\to_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

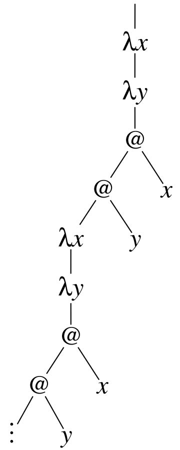


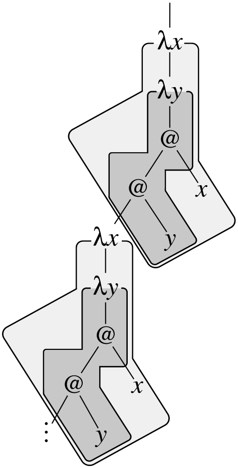
$$
\begin{aligned}
()\,M \quad &= \quad ()\,\lambda xy.\,M\,y\,x\\
&\to_\lambda \quad (x)\,\lambda y.\,M\,y\,x\\
&\to_\lambda \quad (xy)\,M\,y\,x\\
&\to_{@_0} \quad (xy)\,M\,y
\end{aligned}
$$

$M = \lambda xy.\,M\,y\,x$

$\to_{\mathrm{reg}^+}$-generated subterms

Strongly regular λ^∞-term



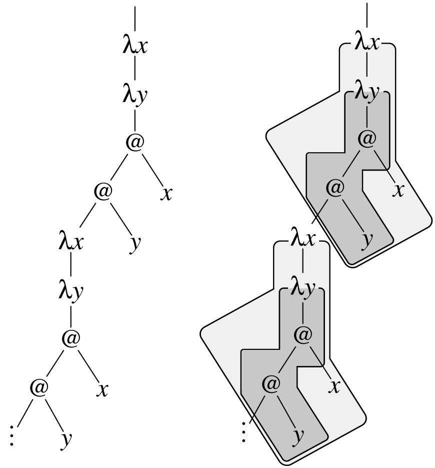


$$\begin{aligned} ()\,M &= ()\,\lambda xy.\,M\,y\,x \\ &\to_\lambda (x)\,\lambda y.\,M\,y\,x \\ &\to_\lambda (xy)\,M\,y\,x \\ &\to_{@_0} (xy)\,M\,y \\ &\to_{@_0} (xy)\,M \end{aligned}$$

$M = \lambda xy.\,M\,y\,x$

$\to_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

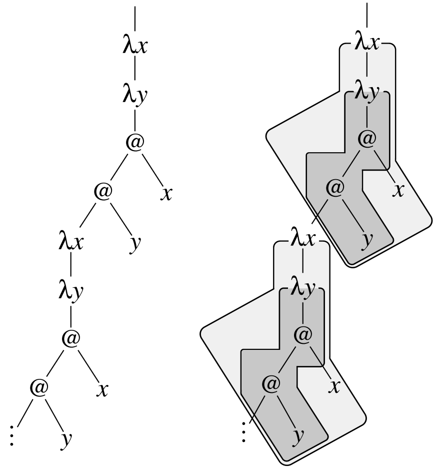


$$
\begin{aligned}
()\,M &= ()\,\lambda xy.\,M\,y\,x \\
&\to_{\lambda} (x)\,\lambda y.\,M\,y\,x \\
&\to_{\lambda} (xy)\,M\,y\,x \\
&\to_{@_0} (xy)\,M\,y \\
&\to_{@_0} (xy)\,M \\
&\to_{\mathsf{S}} (x)\,M
\end{aligned}
$$

$M = \lambda xy.\, M\, y\, x$

$\to_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

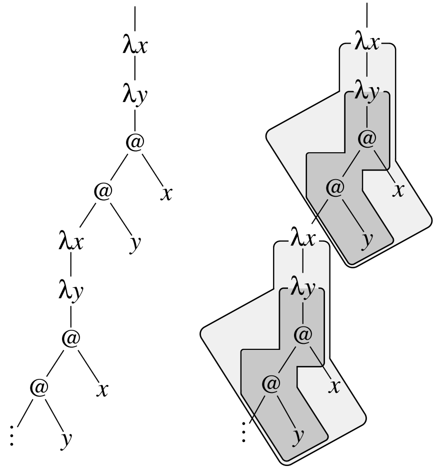


$$
\begin{aligned}
()\,M &= ()\,\lambda xy.\,M\,y\,x \\
&\to_{\lambda} (x)\,\lambda y.\,M\,y\,x \\
&\to_{\lambda} (xy)\,M\,y\,x \\
&\to_{@_0} (xy)\,M\,y \\
&\to_{@_0} (xy)\,M \\
&\to_{\mathsf{S}} (x)\,M \\
&\to_{\mathsf{S}} ()\,M
\end{aligned}
$$

$M = \lambda xy.\,M\,y\,x$

$\to_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

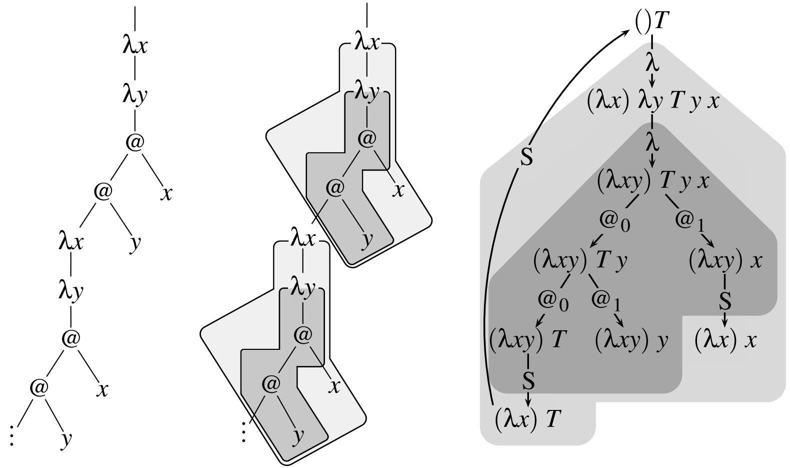


$$
\begin{aligned}
()\,M &= ()\,\lambda xy.\,M\,y\,x \\
&\rightarrow_{\lambda} (x)\,\lambda y.\,M\,y\,x \\
&\rightarrow_{\lambda} (xy)\,M\,y\,x \\
&\rightarrow_{@_0} (xy)\,M\,y \\
&\rightarrow_{@_0} (xy)\,M \\
&\rightarrow_{\mathsf{S}} (x)\,M \\
&\rightarrow_{\mathsf{S}} ()\,M \\
&\ldots
\end{aligned}
$$

$M = \lambda xy.\,M\,y\,x$

$\rightarrow_{\text{reg}^+}$-generated subterms

Strongly regular λ^∞-term

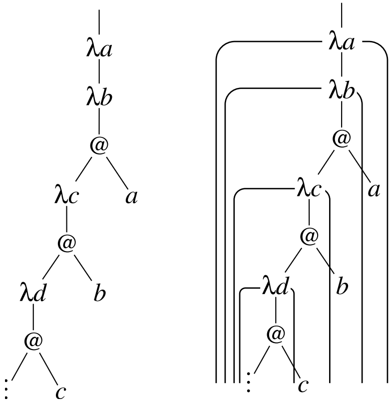


$M = \lambda xy.\, M\, y\, x$

finitely many $\rightarrow_{\mathrm{reg}^+}$-generated subterms

$\Longrightarrow$ M is strongly regular

Not strongly regular λ^∞-term

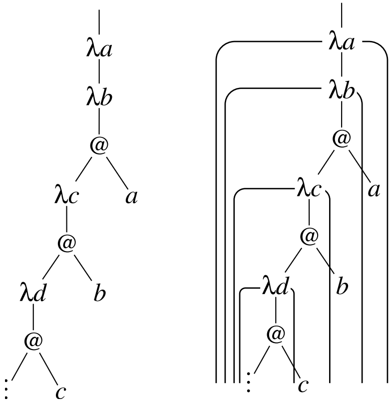


$$N \quad = \quad ()\,\lambda a.\,\lambda b.\,(\ldots)\,a$$

λ^∞-term N

$\rightarrow_{\mathrm{reg}^+}$-generated subterms

Not strongly regular λ^∞-term

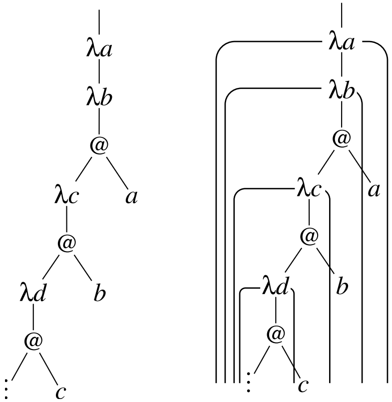


$$N \;=\; ()\,\lambda a.\,\lambda b.\,(\ldots)\,a$$
$$\to_\lambda \; (a)\,\lambda b.\,(\lambda c.\;\ldots)\,a$$

λ^∞-term N

$\to_{\text{reg}^+}$-generated subterms

Not strongly regular λ^∞-term

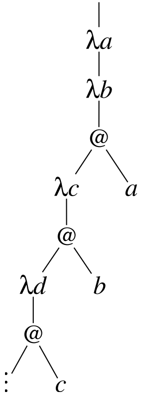


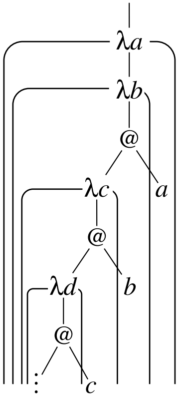
$$N \;=\; ()\,\lambda a.\,\lambda b.\,(\ldots)\,a$$
$$\rightarrow_\lambda \;(a)\,\lambda b.\,(\lambda c.\;\ldots)\,a$$
$$\rightarrow_\lambda \;(ab)\,(\lambda c.\,(\ldots)\,b)\,a$$

λ^∞-term N

$\rightarrow_{\text{reg}^+}$-generated subterms

Not strongly regular λ^∞-term



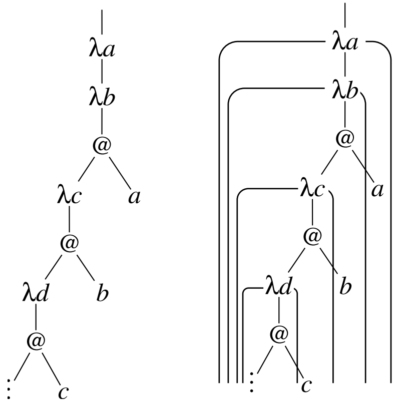


$$
\begin{aligned}
N &= \; ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_\lambda \; (a)\,\lambda b.\,(\lambda c.\;\ldots)\,a \\
&\to_\lambda \; (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} \; (ab)\,\lambda c.\,(\lambda d.\;\ldots)\,b
\end{aligned}
$$

λ^∞-term N

$\to_{\mathsf{reg}^+}$-generated subterms

Not strongly regular λ^∞-term

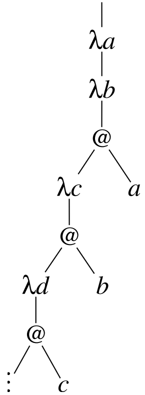


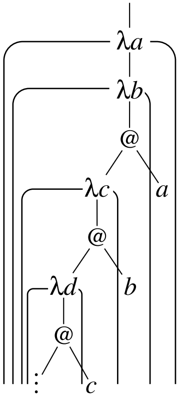
$$
\begin{aligned}
N \quad &= \quad (\,)\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_{\lambda} \quad (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_{\lambda} \quad (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} \quad (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\lambda} \quad (abc)\,(\lambda d.\,(\ldots)\,c)\,b
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}^+}$-generated subterms

Not strongly regular λ^∞-term



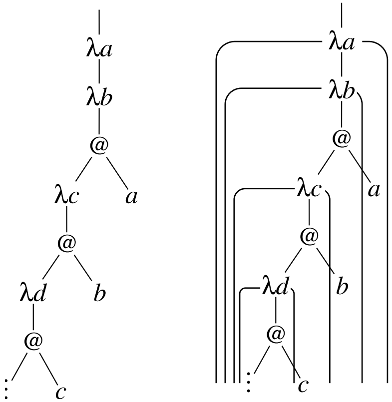


$$
\begin{aligned}
N \quad &= &&(\,)\,\lambda a.\,\lambda b.\,(\ldots)\,a\\
&\to_\lambda &&(a)\,\lambda b.\,(\lambda c.\ \ldots)\,a\\
&\to_\lambda &&(ab)\,(\lambda c.\,(\ldots)\,b)\,a\\
&\to_{@_0} &&(ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b\\
&\to_\lambda &&(abc)\,(\lambda d.\,(\ldots)\,c)\,b\\
&\to_{@_0} &&(abc)\,\lambda d.\,(\lambda e.\ \ldots)\,c
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}^+}$-generated subterms

Not strongly regular λ^∞-term

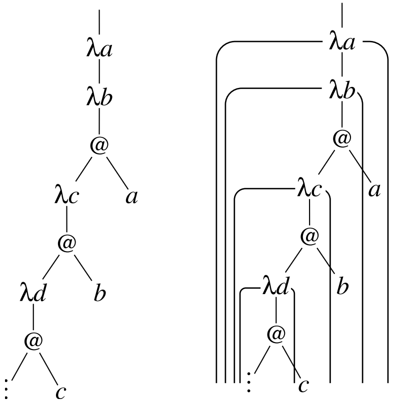


$$
\begin{aligned}
N &= ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_\lambda (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_\lambda (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_\lambda (abc)\,(\lambda d.\,(\ldots)\,c)\,b \\
&\to_{@_0} (abc)\,\lambda d.\,(\lambda e.\ \ldots)\,c \\
&\to_\lambda (abcd)\,(\lambda e.\,(\ldots)\,d)\,c
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}^+}$-generated subterms

Not strongly regular λ^∞-term

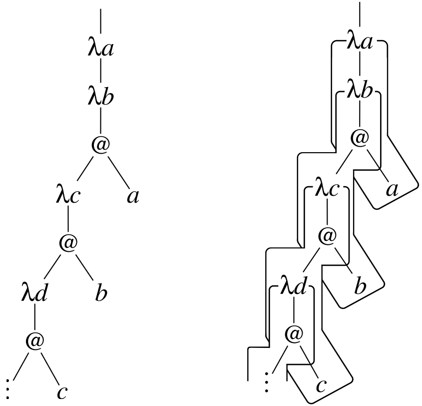


$$
\begin{array}{lll}
N & = & ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
 & \to_\lambda & (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
 & \to_\lambda & (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
 & \to_{@_0} & (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
 & \to_\lambda & (abc)\,(\lambda d.\,(\ldots)\,c)\,b \\
 & \to_{@_0} & (abc)\,\lambda d.\,(\lambda e.\ \ldots)\,c \\
 & \to_\lambda & (abcd)\,(\lambda e.\,(\ldots)\,d)\,c \\
 & \to_{@_0} & (abcd)\,\lambda e.\,(\lambda f.\ \ldots)\,d \\
 & \ldots &
\end{array}
$$

λ^∞-term N

infinitely many $\to_{\text{reg}^+}$-generated subterms
$\Longrightarrow$ N is not strongly regular

Regular λ^∞-term

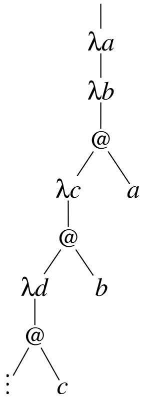


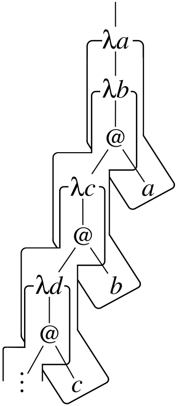
$N \;=\; ()\,\lambda a.\,\lambda b.\,(\ldots)\,a$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



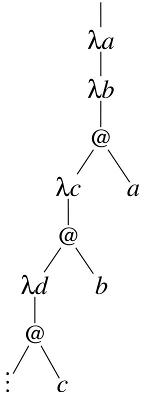


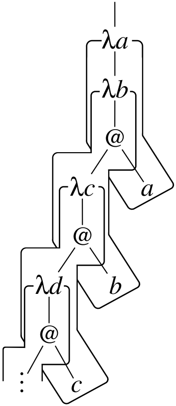
$$N \;=\; ()\,\lambda a.\,\lambda b.\,(\ldots)\,a$$
$$\to_\lambda \;(a)\,\lambda b.\,(\lambda c.\;\ldots)\,a$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



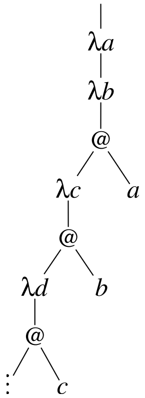


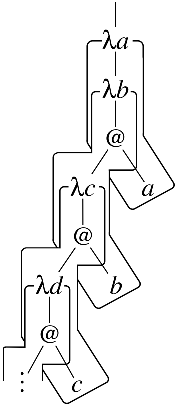
$$
\begin{aligned}
N &= && (\,)\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_\lambda && (a)\,\lambda b.\,(\lambda c.\;\ldots)\,a \\
&\to_\lambda && (ab)\,(\lambda c.\,(\ldots)\,b)\,a
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



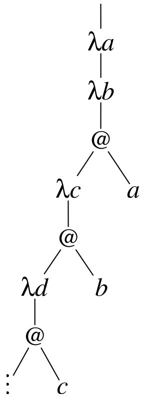


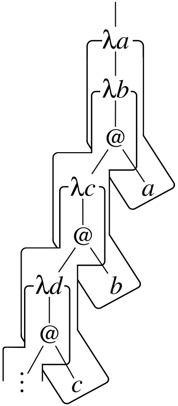
$$
\begin{aligned}
N \quad &= \quad ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_\lambda \quad (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_\lambda \quad (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} \quad (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



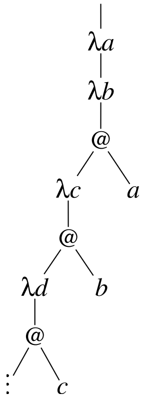


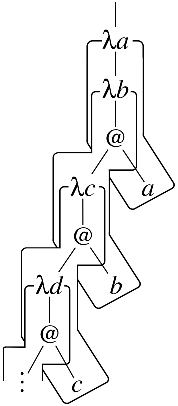
$$
\begin{aligned}
N &= && ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_\lambda && (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_\lambda && (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} && (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\text{del}} && (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



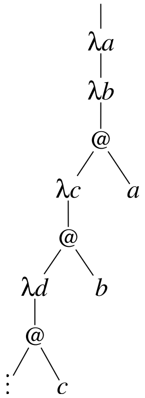


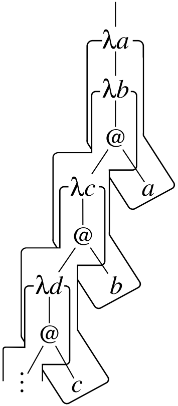
$$
\begin{array}{lll}
N & = & ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
 & \to_{\lambda} & (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
 & \to_{\lambda} & (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
 & \to_{@_0} & (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
 & \to_{\text{del}} & (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
 & \to_{\lambda} & (bc)\,(\lambda d.\,(\ldots)\,c)\,b
\end{array}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



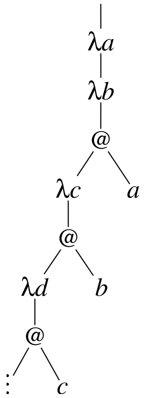


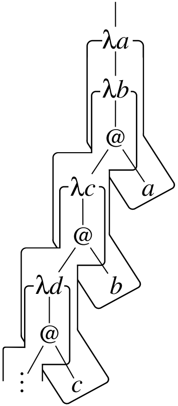
$$
\begin{aligned}
N &= && ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_{\lambda} && (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_{\lambda} && (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} && (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\mathsf{del}} && (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\lambda} && (bc)\,(\lambda d.\,(\ldots)\,c)\,b \\
&\to_{@_0} && (bc)\,\lambda d.\,(\lambda d.\ \ldots)\,c
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^{∞}-term



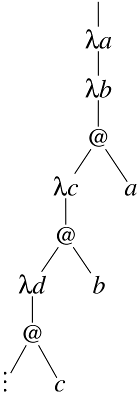


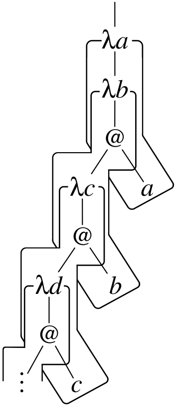
$$
\begin{array}{lll}
N & = & (\,)\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
 & \to_{\lambda} & (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
 & \to_{\lambda} & (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
 & \to_{@_0} & (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
 & \to_{\mathsf{del}} & (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
 & \to_{\lambda} & (bc)\,(\lambda d.\,(\ldots)\,c)\,b \\
 & \to_{@_0} & (bc)\,\lambda d.\,(\lambda d.\ \ldots)\,c \\
 & \to_{\mathsf{del}} & (c)\,\lambda d.\,(\lambda e.\ \ldots)\,d
\end{array}
$$

λ^{∞}-term N

$\to_{\mathsf{reg}}$-generated subterms

Regular λ^{∞}-term



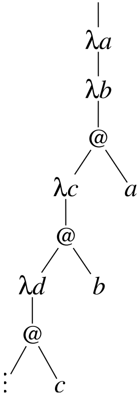


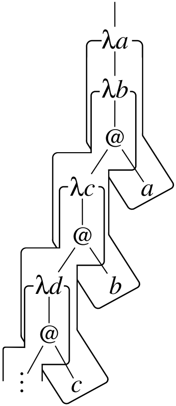
$$
\begin{aligned}
N &= ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_{\lambda} (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_{\lambda} (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\mathsf{del}} (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\lambda} (bc)\,(\lambda d.\,(\ldots)\,c)\,b \\
&\to_{@_0} (bc)\,\lambda d.\,(\lambda d.\ \ldots)\,c \\
&\to_{\mathsf{del}} (c)\,\lambda d.\,(\lambda e.\ \ldots)\,d \\
&\to_{\lambda} (cd)\,(\lambda e.\,(\ldots)\,d)\,c
\end{aligned}
$$

λ^{∞}-term N

$\to_{\mathsf{reg}}$-generated subterms

Regular λ^{∞}-term





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\begin{aligned}
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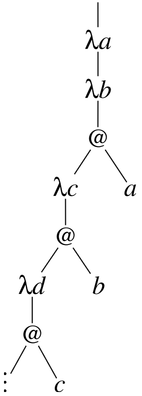
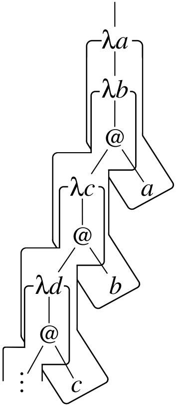
&\to_{\mathsf{del}}\; (b)\,\lambda c.\,(\lambda d.\;\ldots)\,b \\
&\to_{\lambda}\; (bc)\,(\lambda d.\,(\ldots)\,c)\,b \\
&\to_{@_0}\; (bc)\,\lambda d.\,(\lambda d.\;\ldots)\,c \\
&\to_{\mathsf{del}}\; (c)\,\lambda d.\,(\lambda e.\;\ldots)\,d \\
&\to_{\lambda}\; (cd)\,(\lambda e.\,(\ldots)\,d)\,c \\
&\to_{@_0}\; (cd)\,\lambda e.\,(\lambda f.\;\ldots)\,d
\end{aligned}
$$

λ^{∞}-term N

$\to_{\mathsf{reg}}$-generated subterms

Regular λ^∞-term



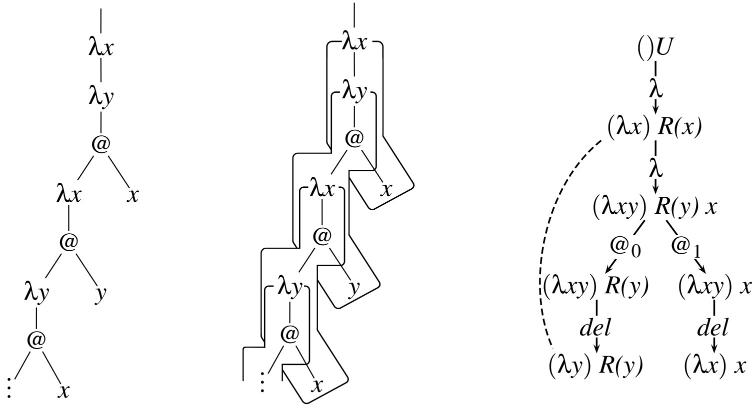


$$
\begin{aligned}
N &= ()\,\lambda a.\,\lambda b.\,(\ldots)\,a \\
&\to_{\lambda} (a)\,\lambda b.\,(\lambda c.\ \ldots)\,a \\
&\to_{\lambda} (ab)\,(\lambda c.\,(\ldots)\,b)\,a \\
&\to_{@_0} (ab)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\text{del}} (b)\,\lambda c.\,(\lambda d.\ \ldots)\,b \\
&\to_{\lambda} (bc)\,(\lambda d.\,(\ldots)\,c)\,b \\
&\to_{@_0} (bc)\,\lambda d.\,(\lambda d.\ \ldots)\,c \\
&\to_{\text{del}} (c)\,\lambda d.\,(\lambda e.\ \ldots)\,d \\
&\to_{\lambda} (cd)\,(\lambda e.\,(\ldots)\,d)\,c \\
&\to_{@_0} (cd)\,\lambda e.\,(\lambda f.\ \ldots)\,d \\
&\to_{\text{del}} (d)\,\lambda e.\,(\lambda f.\ \ldots)\,d \\
&\ldots
\end{aligned}
$$

λ^∞-term N

$\to_{\text{reg}}$-generated subterms

Regular λ^∞-term



λ^∞-term N

$\{N = \lambda xy.\, R(y)\, x,$
$\ R(z) = \lambda u.\, R(u)\, z\}$

finitely many $\rightarrow_{\text{reg}}$-generated subterms

$\Longrightarrow$ M is regular

Strongly regular ⇒ regular

Proposition

- Every strongly regular λ^∞-term is also regular.
- Finite λ-terms are both regular and strongly regular.

$\boldsymbol{\lambda}_{\text{letrec}}$-Expressibility

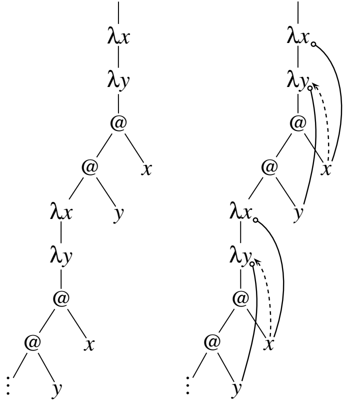
Proposition

- Every strongly regular λ^∞-term is also regular.
- Finite λ-terms are both regular and strongly regular.

Theorem (λ_{letrec}-expressibility)

An λ^∞-term is $\boldsymbol{\lambda}_{\text{letrec}}$-expressible if and only if it is strongly regular.

Binding–capturing chains



Definition (Melliés, van Oostrom)

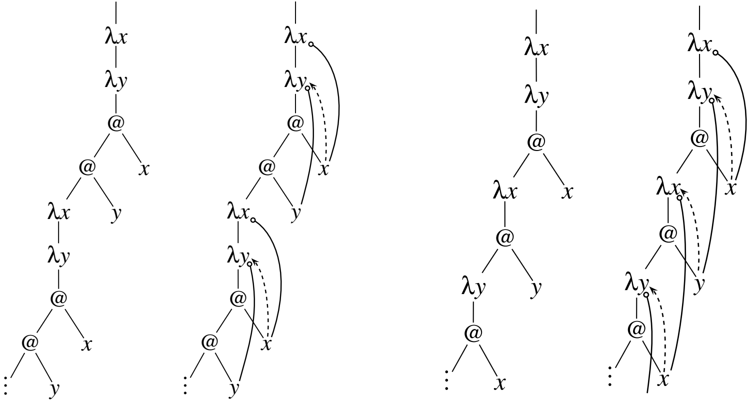
For positions p, q, r, s:

$p \multimap q$ $:\Longleftrightarrow$ binder at p binds variable occurrence at position q

$r \dashrightarrow s$ $:\Longleftrightarrow$ variable occurrence at r is captured by binding at s

Binding–capturing chains: $p_0 \multimap p_1 \dashrightarrow p_2 \multimap p_3 \dashrightarrow p_4 \multimap \ldots$

Binding–capturing chains



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Main theorem (extended)

Theorem (binding–capturing chains)

For all λ^{∞}-term M:

M *is strongly regular* $\Longleftrightarrow$ M *is regular, and*
M *has only finite binding–capturing chains.*

Main theorem (extended)

Theorem (binding–capturing chains)

For all λ^∞-term M:

M *is strongly regular* $\iff$ M *is regular, <u>and</u>*
M *has only finite binding–capturing chains.*

Theorem ($\boldsymbol{\lambda}_{\text{letrec}}$-expressibility, extended)

For all λ^∞-terms M the following are equivalent:

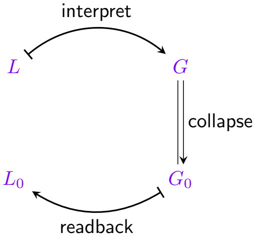
(i) M *is $\boldsymbol{\lambda}_{\text{letrec}}$-expressible.*

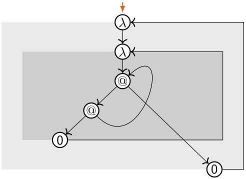
(ii) M *is strongly regular.*

(iii) M *is regular, and it only contains finite binding–capturing chains.*

Maximal sharing of functional programs

(joint work with Jan Rochel)





Motivation, questions, and results

Motivation

- desirable: increase sharing in programs
  - code that is as compact as possible
  - avoid duplication of reduction work at run-time
- useful: check equality of unfolding semantics of programs

Questions

(1): how to maximize sharing in programs?

(2): how to check for unfolding equivalence?

We restrict to λ_{letrec}, the λ-calculus with letrec

- as abstraction & syntactical core of functional languages

Results:

- efficient methods solving questions (1) and (2) for λ_{letrec}

The method

- Methods consist of the steps:
  1. interpretation of $\boldsymbol{\lambda}_{\text{letrec}}$-terms as term graphs
     - higher-order term graphs: λ-ho-term-graphs
     - first-order term graphs : λ-term-graphs
     - deterministic finite-state automata: λ-DFAs
  2. bisimilarity & bisimulation collapse of λ-term-graphs
     - implemented as: DFA-minimization of λ-DFAs
  3. readback of λ-term-graphs as $\boldsymbol{\lambda}_{\text{letrec}}$-terms
- Haskell implementation
- Complexity

Maximal sharing: example (fix)

$$\lambda f.\,\mathsf{let}\ r = f\ (f\ r)\ \mathsf{in}\ r$$

L

Maximal sharing: example (fix)

$$\lambda f.\,\mathsf{let}\ r = f\,(f\ r)\ \mathsf{in}\ r$$

L

L_0

$$\lambda f.\,\mathsf{let}\ r = f\ r\ \mathsf{in}\ r$$

Maximal sharing: the method

L

unfold $\downarrow$

M

unfold $\uparrow$

L_0

$$\lambda f.\, \mathsf{let}\ r = f\,(f\ r)\ \mathsf{in}\ r$$

$$\llbracket\cdot\rrbracket_{\lambda^\infty} \downarrow$$

$$\lambda f.\, f\,(f\,(\ldots))$$

$$\llbracket\cdot\rrbracket_{\lambda^\infty} \uparrow$$

$$\lambda f.\, \mathsf{let}\ r = f\ r\ \mathsf{in}\ r$$

Maximal sharing: the method

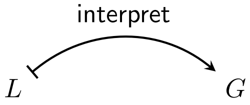
$$\lambda f.\,\mathsf{let}\ r = f\ (f\ r)\ \mathsf{in}\ r$$

L

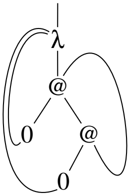
L_0

$$\lambda f.\,\mathsf{let}\ r = f\ r\ \mathsf{in}\ r$$

Maximal sharing: the method



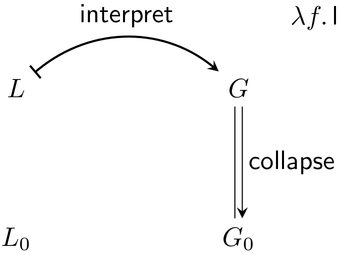
$\lambda f.\,\text{let } r = f\,(f\ r) \text{ in } r \quad \overset{[\![\cdot]\!]_{\mathcal{T}}}{\longmapsto}$



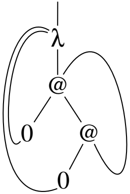
L_0

$\lambda f.\,\text{let } r = f\ r \text{ in } r$

Maximal sharing: the method

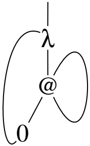


$\lambda f.\, \text{let } r = f\ (f\ \ r) \text{ in } r \quad \xmapsto{[\![\cdot]\!]_{\mathcal{T}}}$

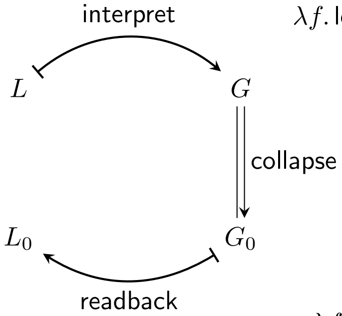


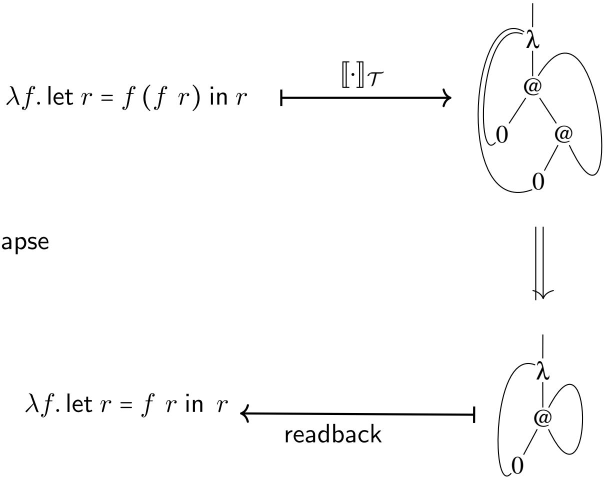
$\Downarrow$

$\lambda f.\, \text{let } r = f\ \ r \text{ in } \ r$

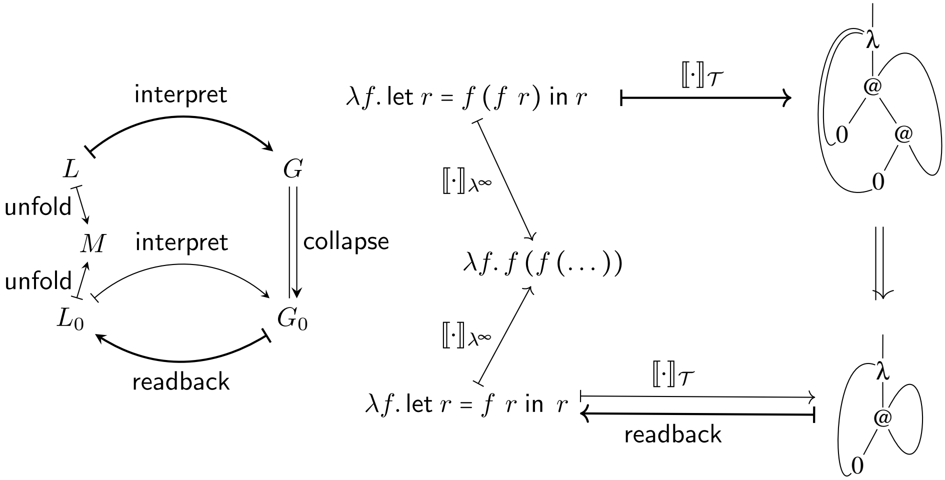


Maximal sharing: the method





Maximal sharing: the method



Maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G}$$

1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L as:
   a. higher-order term graph
      $\mathcal{G} = [\![L]\!]_{\mathcal{H}}$

Maximal sharing: the method

$$L \xmapsto{[\![\cdot]\!]_{\mathcal{H}}} \mathcal{G} \longmapsto G$$

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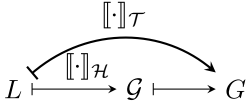
   b. first-order term graph $G = [\![L]\!]_{\mathcal{T}}$

Maximal sharing: the method



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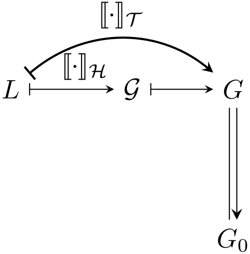
Maximal sharing: the method



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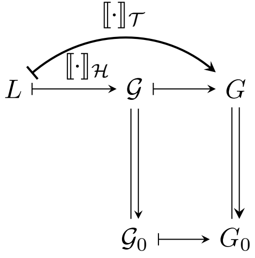
2. bisimulation collapse $\downarrow\!\!\!|$
   of f-o term graph G into G_0

Maximal sharing: the method



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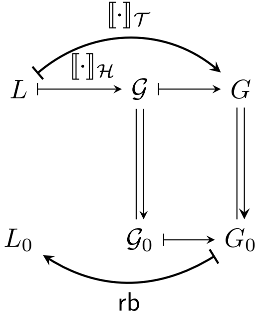

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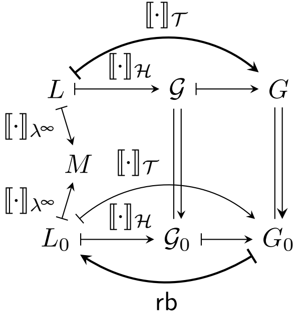
2. bisimulation collapse $\downarrow\!\!\!\!|$
   of f-o term graph G into G_0
3. readback rb
   of f-o term graph G_0
   yielding program $L_0 = \text{rb}(G_0)$.

Maximal sharing: the method



1. term graph interpretation $[\![\cdot]\!]$.
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2. bisimulation collapse $\downarrow\!\!\downarrow$
   of f-o term graph G into G_0
3. readback rb
   of f-o term graph G_0
   yielding program $L_0 = \text{rb}(G_0)$.

Unfolding equivalence: example

L_1

unfold $\downarrow$ **?**

M

unfold $\uparrow$ **?**

L_2

$\lambda f.\,\text{let } r = f\,(f\;r) \text{ in } r$

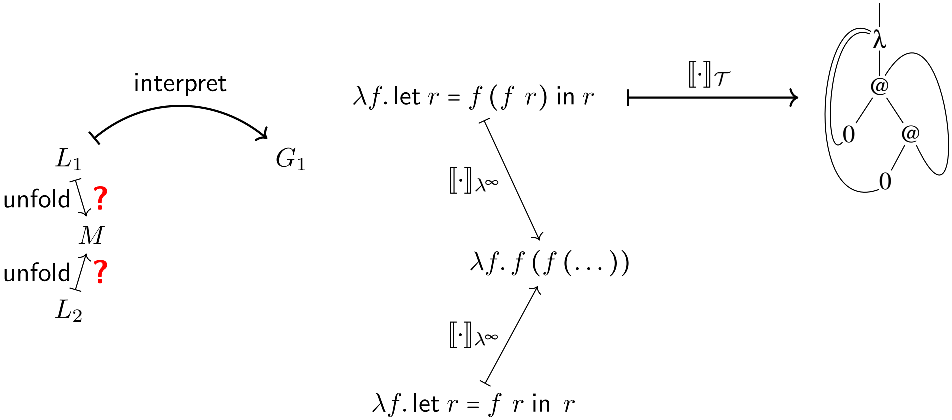
$[\![\cdot]\!]_{\lambda^\infty}$

$\lambda f.\,f\,(f\,(\ldots))$

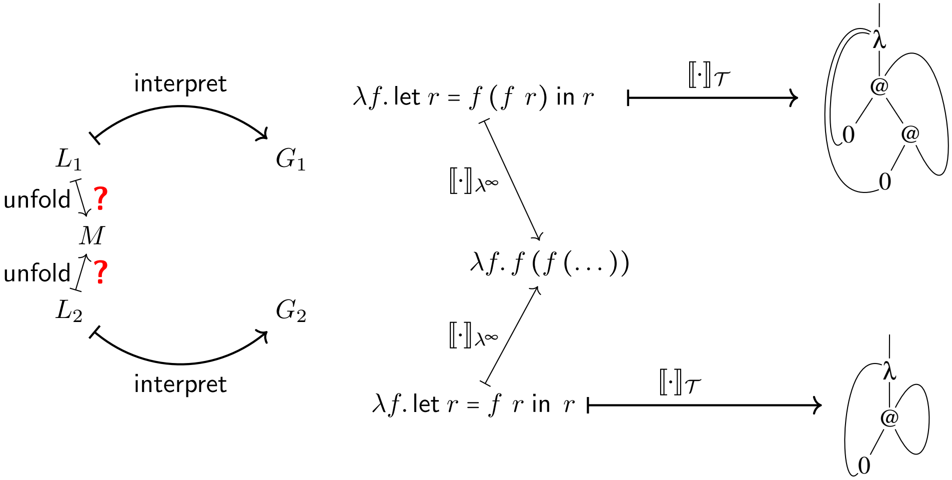
$[\![\cdot]\!]_{\lambda^\infty}$

$\lambda f.\,\text{let } r = f\;r \text{ in } r$

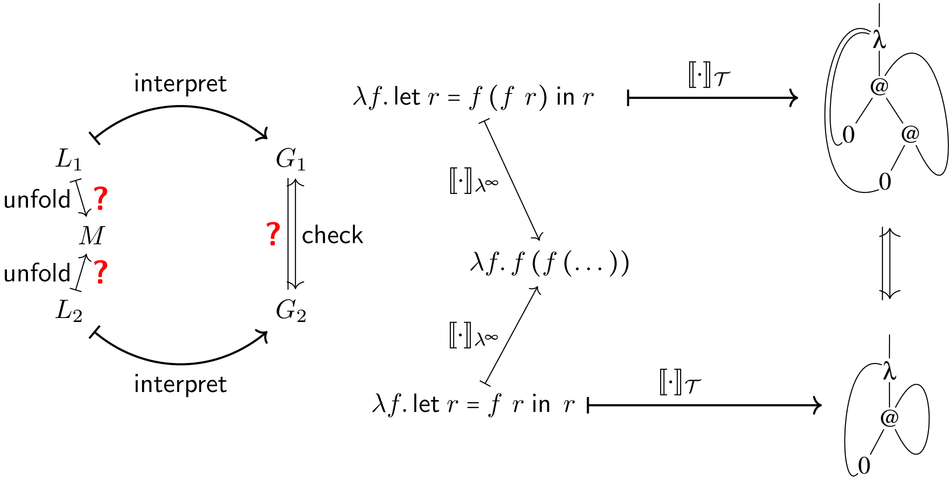
Unfolding equivalence: example



Unfolding equivalence: the method



Unfolding equivalence: the method



Unfolding equivalence: the method

$$L_1$$

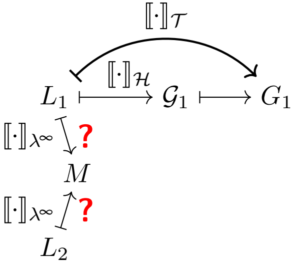
$$[\![\cdot]\!]_{\lambda^\infty} \downarrow \textbf{?}$$

$$M$$

$$[\![\cdot]\!]_{\lambda^\infty} \uparrow \textbf{?}$$

$$L_2$$

Unfolding equivalence: the method



1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L_1 and L_2 as:
   a. higher-order term graphs
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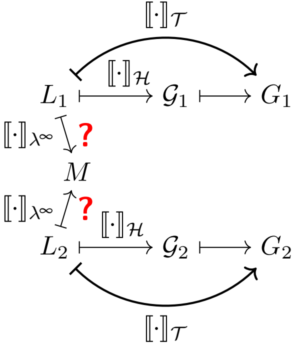
   b. first-order term graphs
      $G_1 = [\![L_1]\!]_{\mathcal{T}}$

Unfolding equivalence: the method



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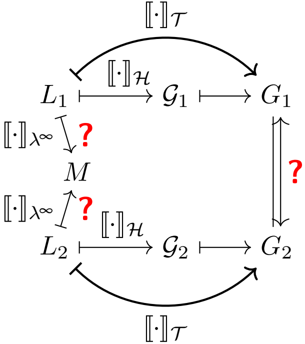
   b. first-order term graphs
      $G_1 = [\![L_1]\!]_{\mathcal{T}}$ and $G_2 = [\![L_2]\!]_{\mathcal{T}}$

Unfolding equivalence: the method



1. term graph interpretation $[\![\cdot]\!]$.
   of $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L_1 and L_2 as:
   a. higher-order term graphs
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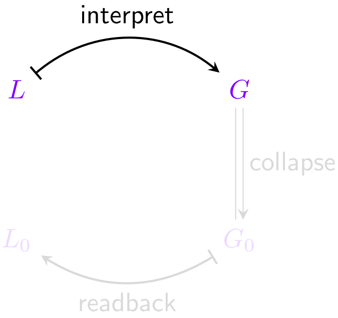
   b. first-order term graphs
      $G_1 = [\![L_1]\!]_{\mathcal{T}}$ and $G_2 = [\![L_2]\!]_{\mathcal{T}}$
2. check bisimilarity
   of f-o term graphs G_1 and G_2

Interpretation



Running example

instead of:

$\lambda f.\,\text{let } r = f\,(f\,r) \text{ in } r \quad \longmapsto_{\text{max-sharing}} \quad \lambda f.\,\text{let } r = f\,r \text{ in } r$

we use:

$\lambda x.\,\lambda f.\,\text{let } r = f\,(f\,r\,x)\,x \text{ in } r \quad \longmapsto_{\text{max-sharing}} \quad \lambda x.\,\lambda f.\,\text{let } r = f\,r\,x \text{ in } r$

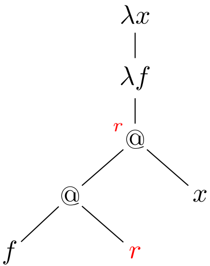
$L \quad \longmapsto_{\text{max-sharing}} \quad L_0$

Graph interpretation (example 1)

$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$

Graph interpretation (example 1)

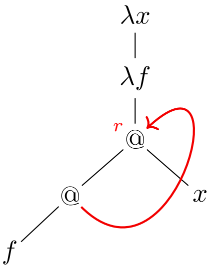
$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\; r\; x \text{ in } r$



syntax tree

Graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



syntax tree (+ recursive backlink)

Graph interpretation (example 1)

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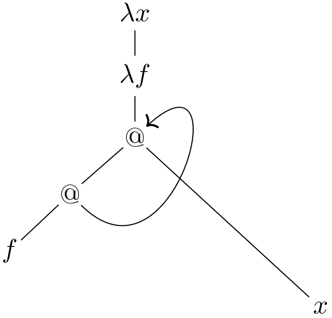

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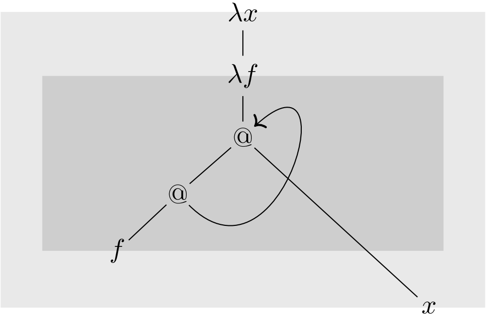

syntax tree (+ recursive backlink, + scopes)

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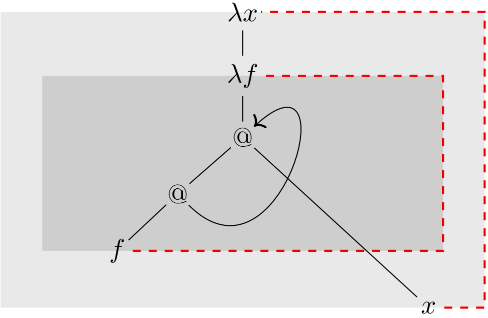

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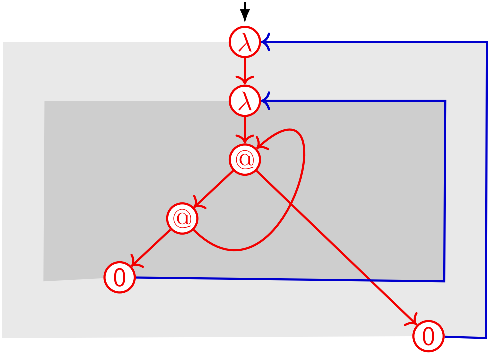

first-order term graph with binding backlinks (+ scope sets)

Graph interpretation (example 1)

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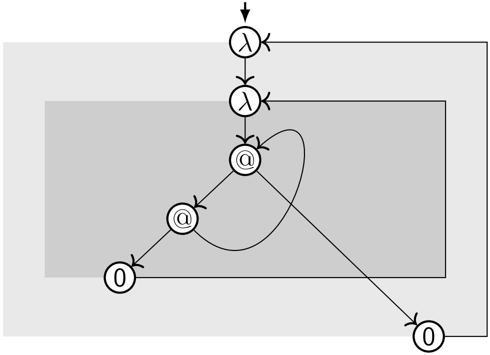

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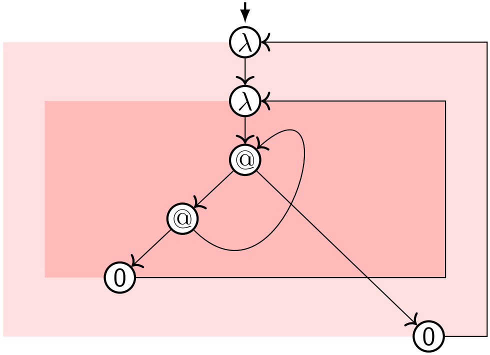

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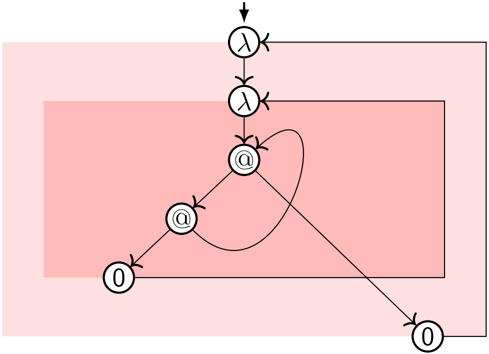

higher-order term graph (with scope sets, Blom [2003])

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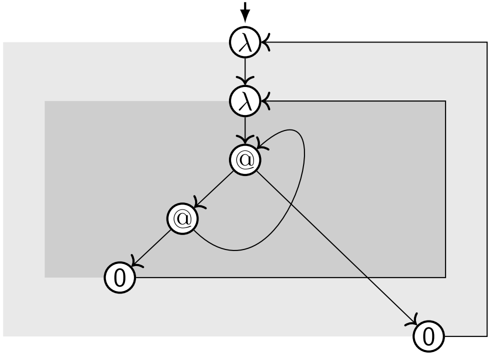

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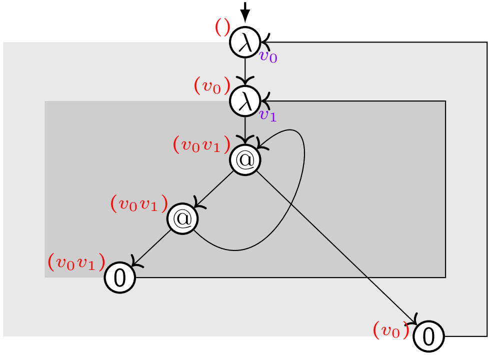

higher-order term graph (with scope sets, + abstraction-prefix function)

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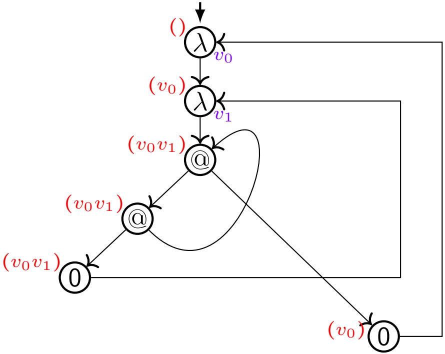
$L_0 = \lambda x. \lambda f. \text{let } r = f\, r\, x \text{ in } r$



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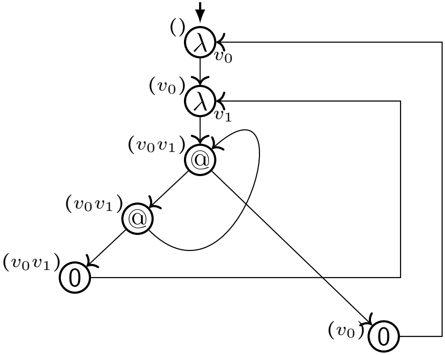

λ-higher-order-term-graph $[\![L_0]\!]_{\mathcal{H}}$

Graph interpretation (example 1)

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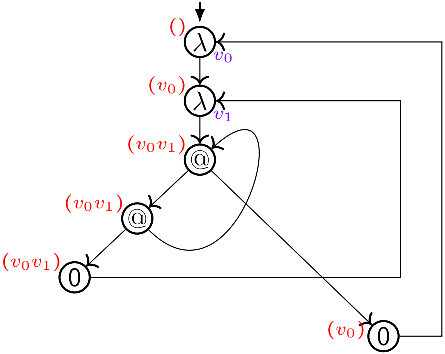

first-order term graph (+ abstraction-prefix function)

Graph interpretation (example 1)

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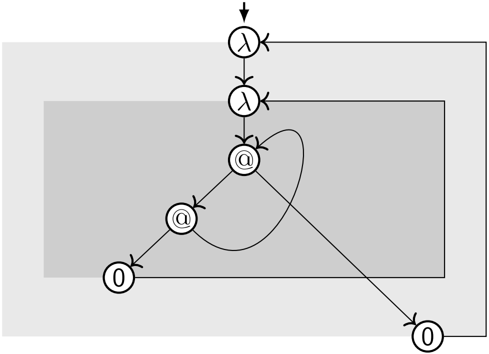

first-order term graph with binding backlinks (+ scope sets)

Graph interpretation (example 1)

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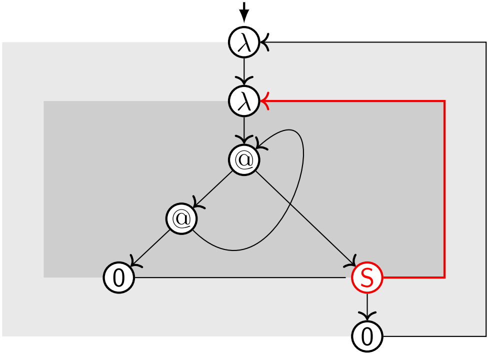
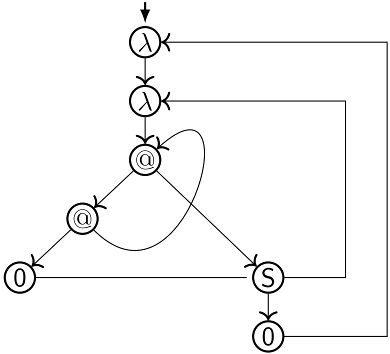


first-order term graph with scope vertices with backlinks (+ scope sets)

Graph interpretation (example 1)

$L_0 = \lambda x.\, \lambda f.\, \text{let } r = f\, r\, x \text{ in } r$



first-order term graph with scope vertices with backlinks

Graph interpretation (example 1)

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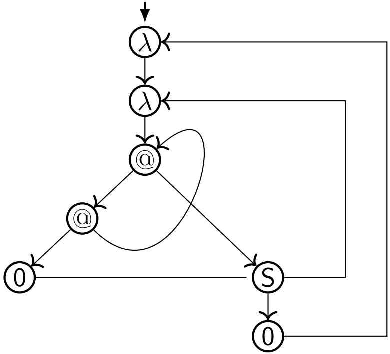

λ-term-graph $[\![L_0]\!]_{\mathcal{T}}$

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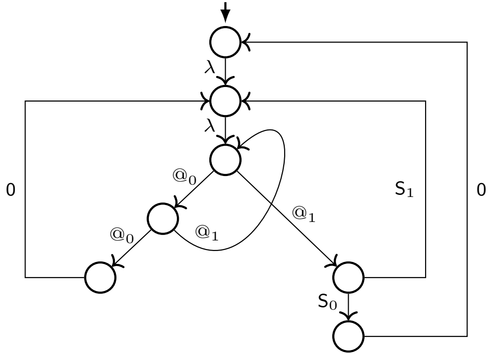

incomplete DFA

Graph interpretation (example 1)

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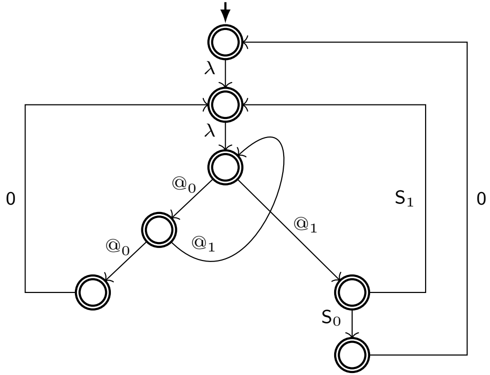

incomplete λ-DFA

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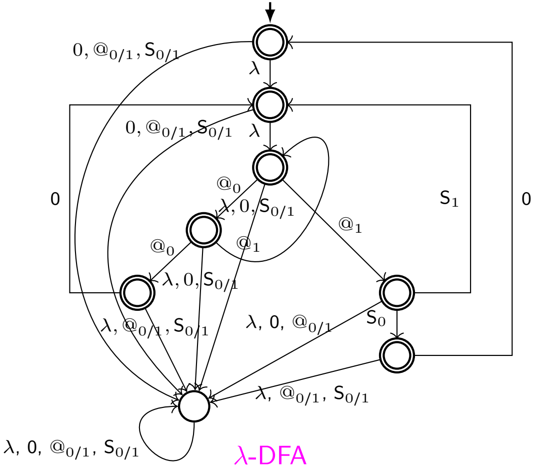

Graph interpretation (example 2)

$L = \lambda x.\, \lambda f.\, \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$

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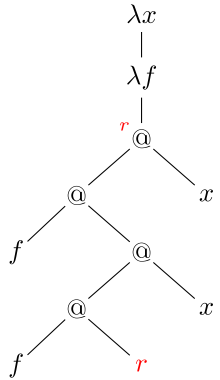
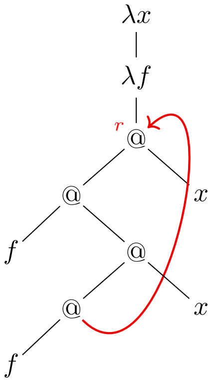


syntax tree

Graph interpretation (example 2)

$L = \lambda x. \lambda f. \text{let } r = f\,(f\,r\,x)\,x \text{ in } r$



syntax tree (+ recursive backlink)

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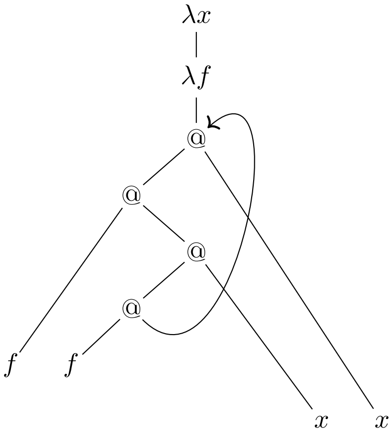

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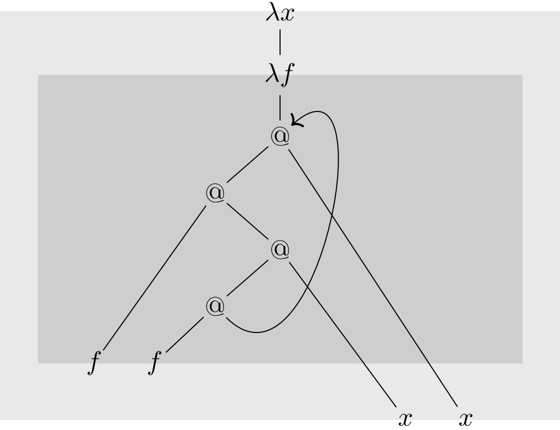

syntax tree (+ recursive backlink, + scopes)

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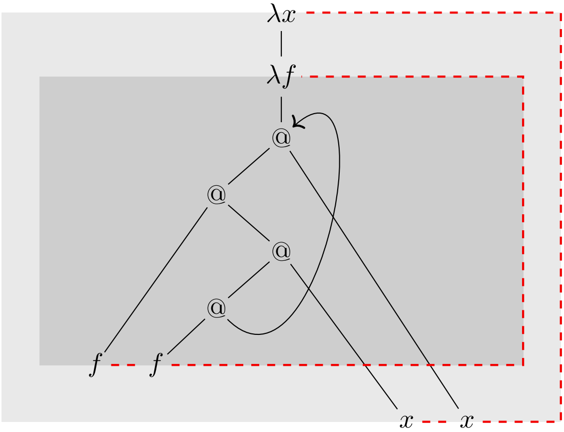

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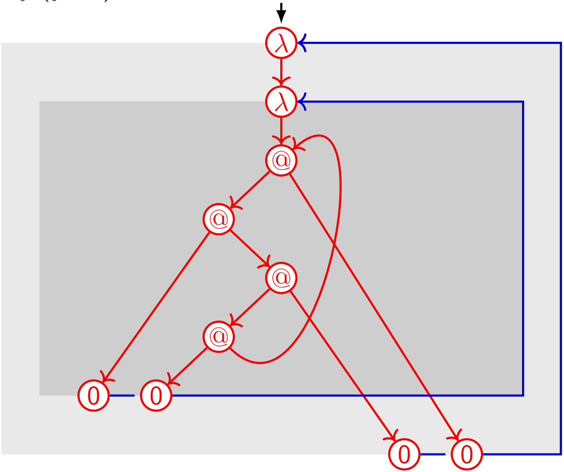

first-order term graph with binding backlinks (+ scope sets)

Graph interpretation (example 2)

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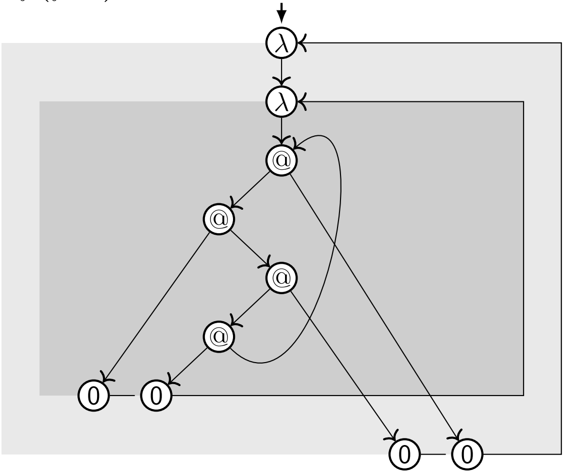

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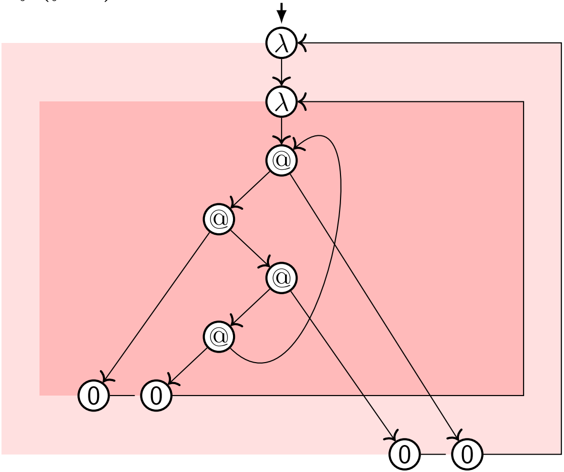

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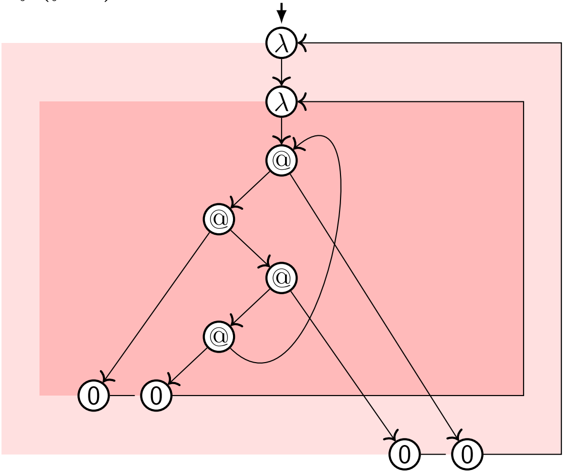

higher-order term graph (with scope sets, Blom [2003])

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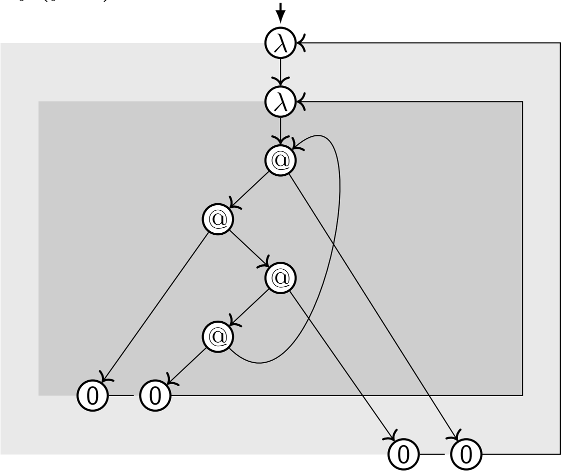

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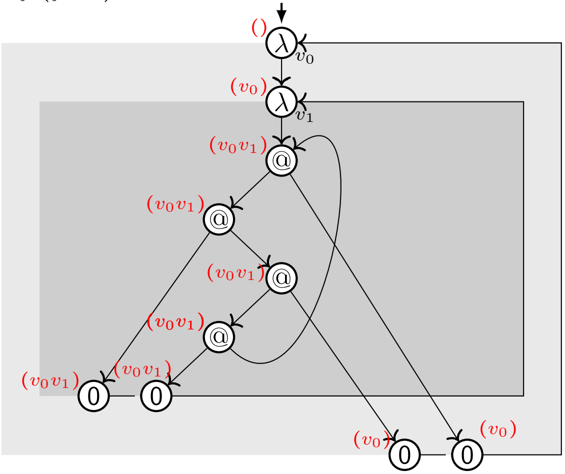

higher-order term graph (with scope sets, + abstraction-prefix function)

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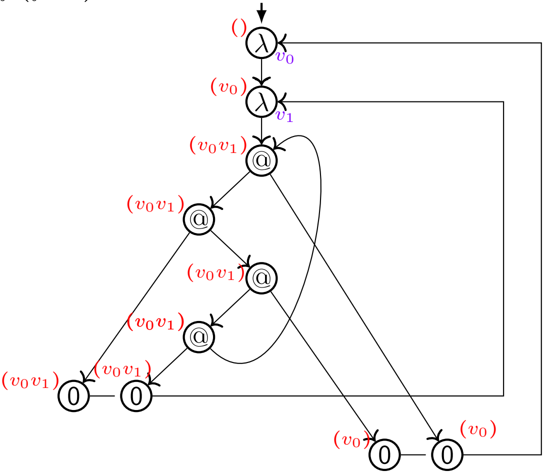

higher-order term graph (with abstraction-prefix function)

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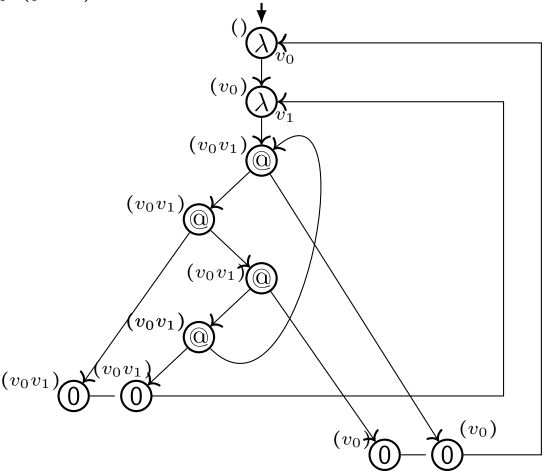

λ-higher-order-term-graph $[\![L]\!]_{\mathcal{H}}$

Graph interpretation (example 2)

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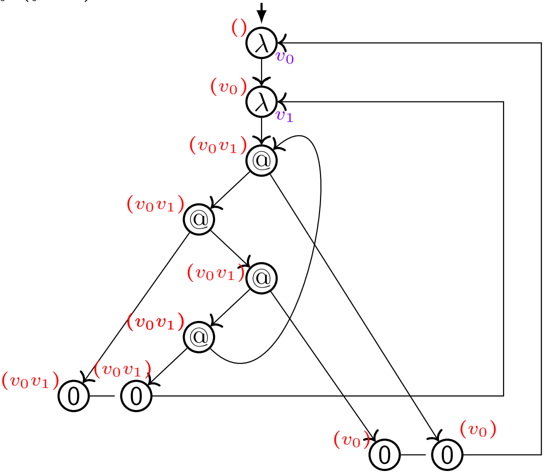

first-order term graph (+ abstraction-prefix function)

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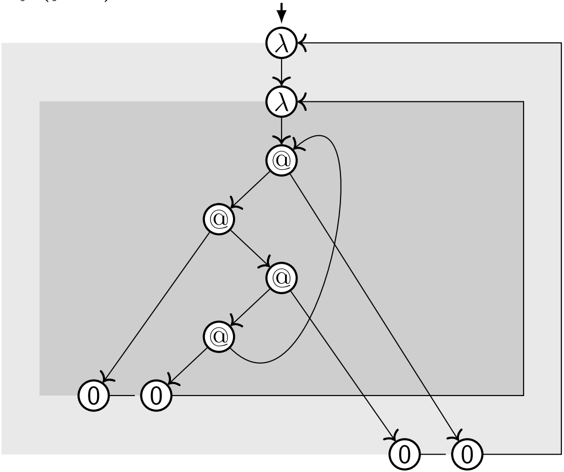

first-order term graph with binding backlinks (+ scope sets)

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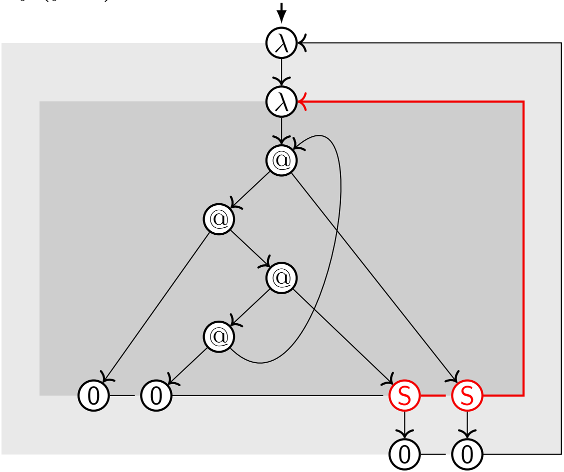

first-order term graph with scope vertices with backlinks (+ scope sets)

Graph interpretation (example 2)

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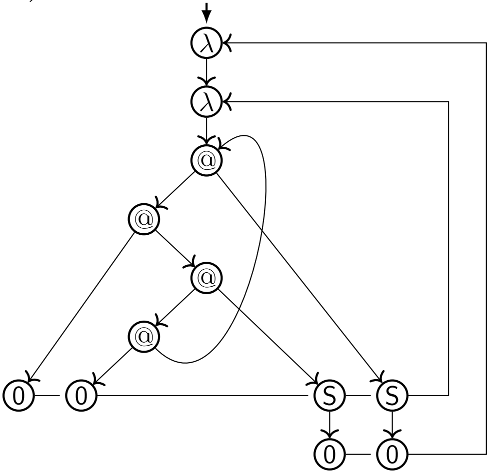

first-order term graph with scope vertices with backlinks

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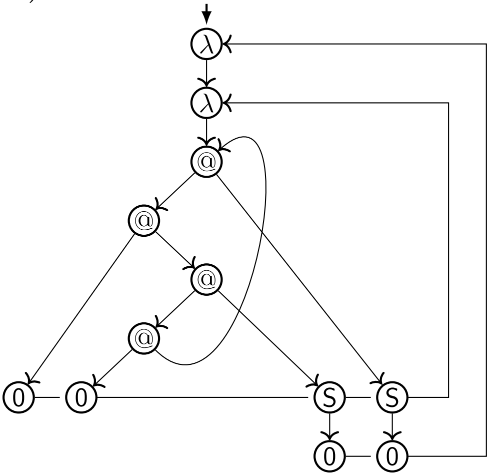

λ-term-graph $[\![L]\!]_{\mathcal{T}}$

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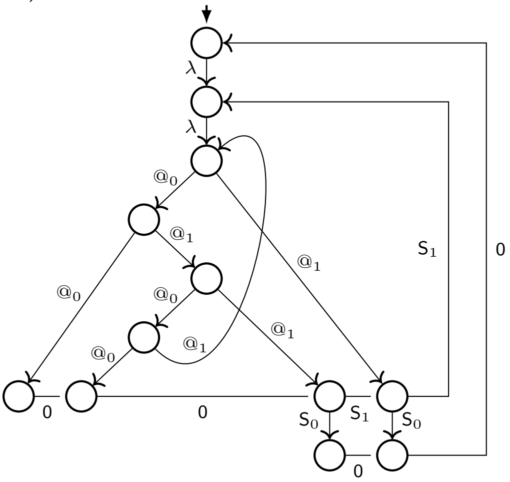

incomplete DFA

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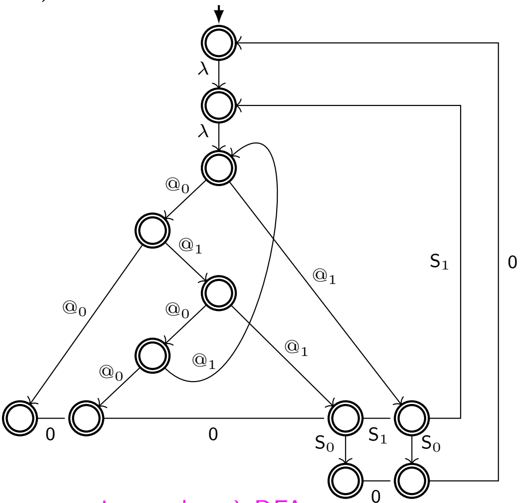

incomplete λ-DFA

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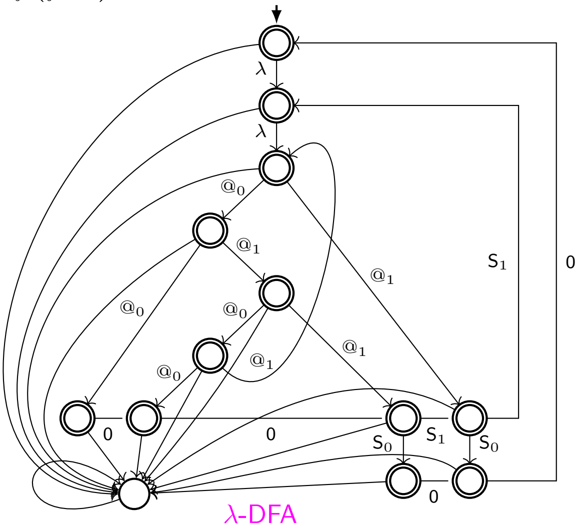
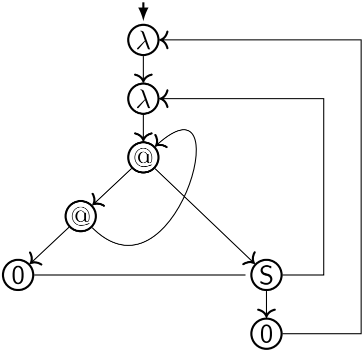
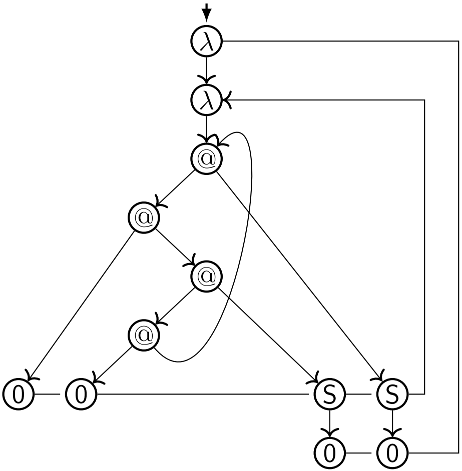


Graph interpretation (examples 1 and 2)



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

Interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

interpretation $\boldsymbol{\lambda}_{\text{letrec}}$-term $L \longmapsto \lambda$-term-graph $[\![L]\!]_{\mathcal{T}}$

- defined by induction on structure of L
- similar analysis as fully-lazy lambda-lifting
- yields eager-scope λ-term-graphs: $\sim$ minimal scopes

Theorem

For $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 it holds: Equality of infinite unfolding coincides with bisimilarity of λ-term-graph interpretations:

$$[\![L_1]\!]_{\lambda^\infty} = [\![L_2]\!]_{\lambda^\infty} \iff [\![L_1]\!]_{\mathcal{T}} \leftrightarrows [\![L_2]\!]_{\mathcal{T}}$$

Interpretation $[\![\cdot]\!]_{\mathcal{T}}$: properties (cont.)

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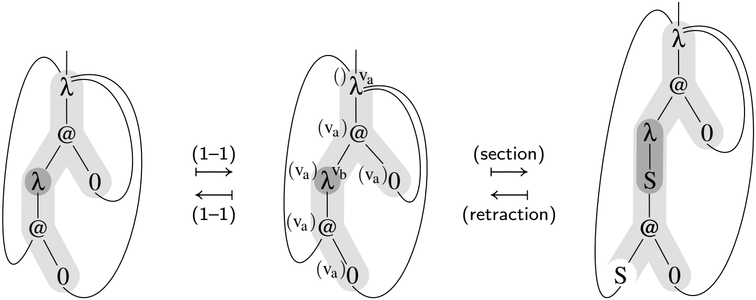
structure constraints (L'Aquila)

higher-order as first-order term graphs

let $f = \lambda x. (\lambda y. f\ x)\, x$ in f

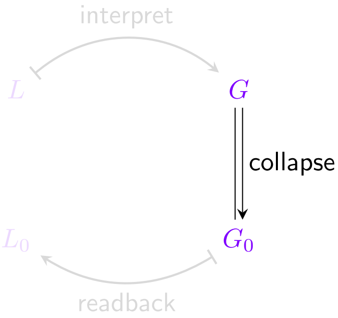


higher-order term graph [Blom '03]

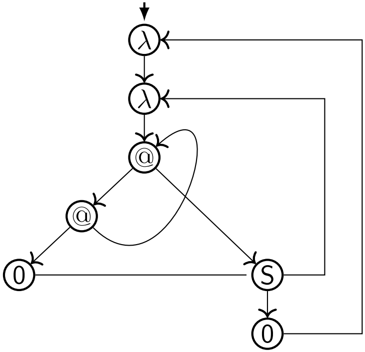
higher-order term graph (abstraction-prefix funct.)

first-order term graph

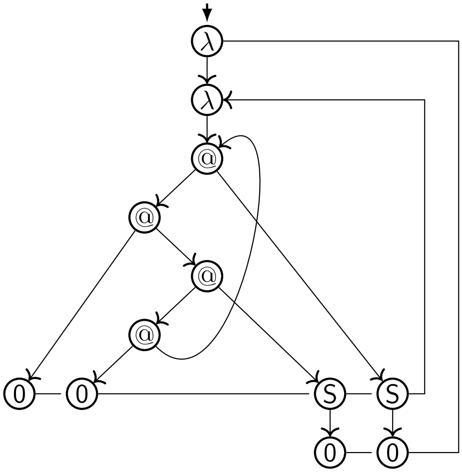
Collapse



Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$



$[\![L]\!]_{\mathcal{T}}$

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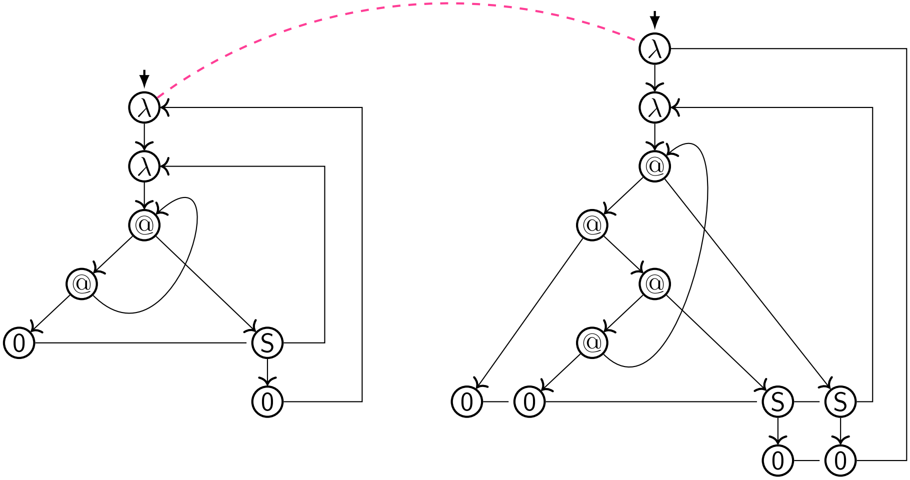
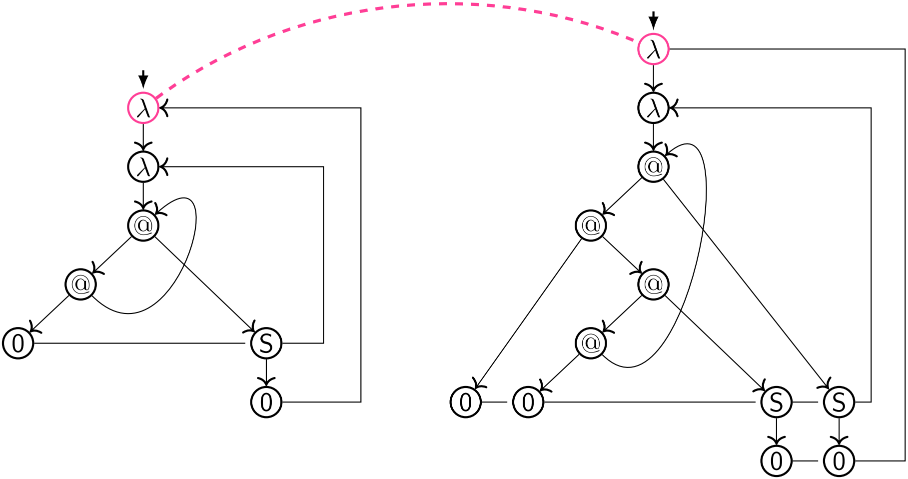


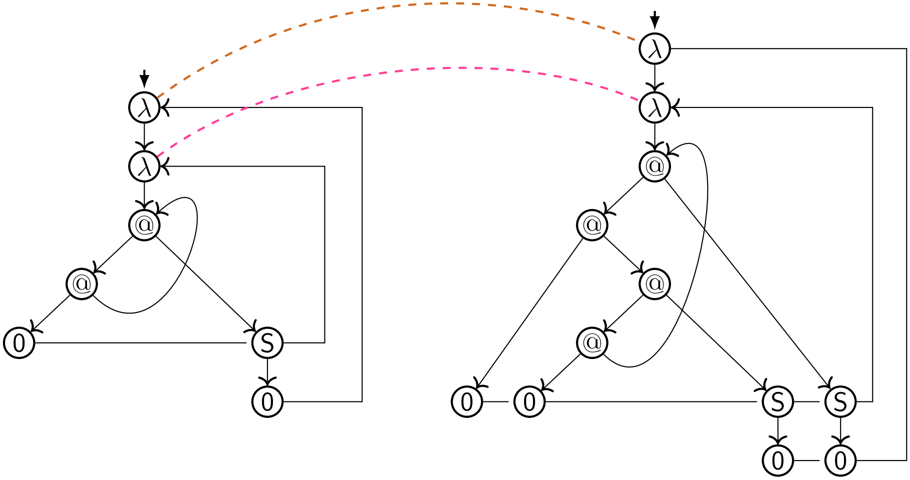
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



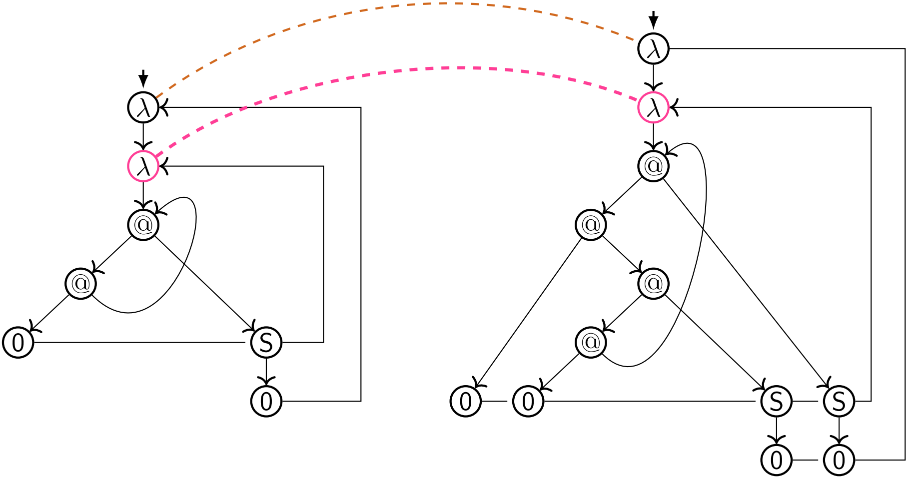
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



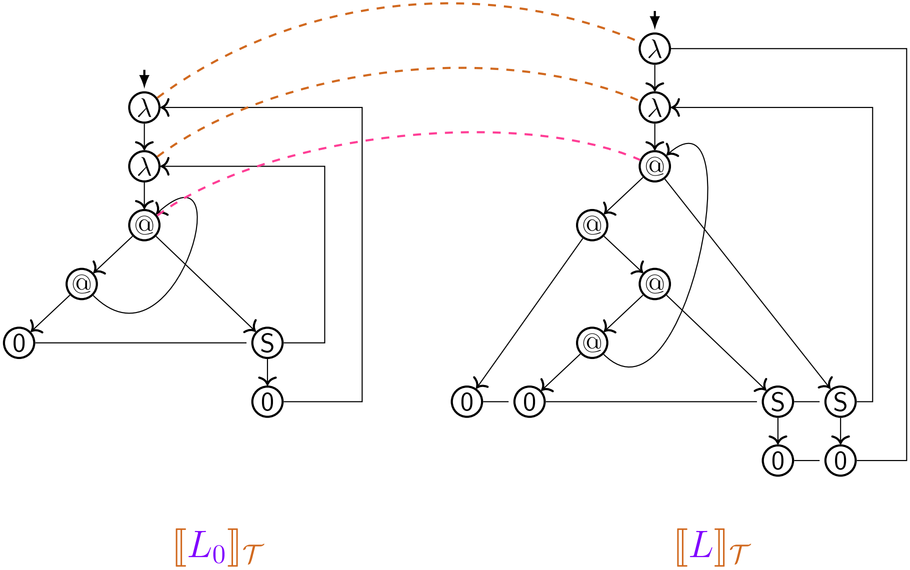
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs

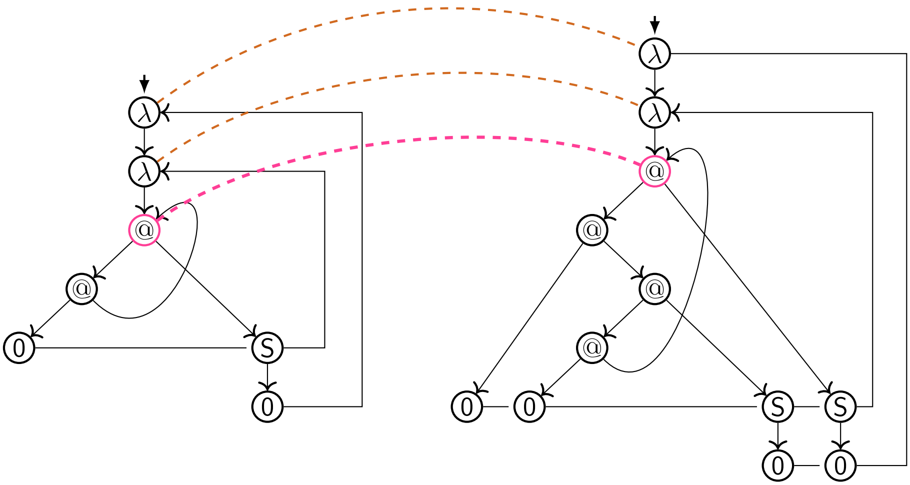


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs

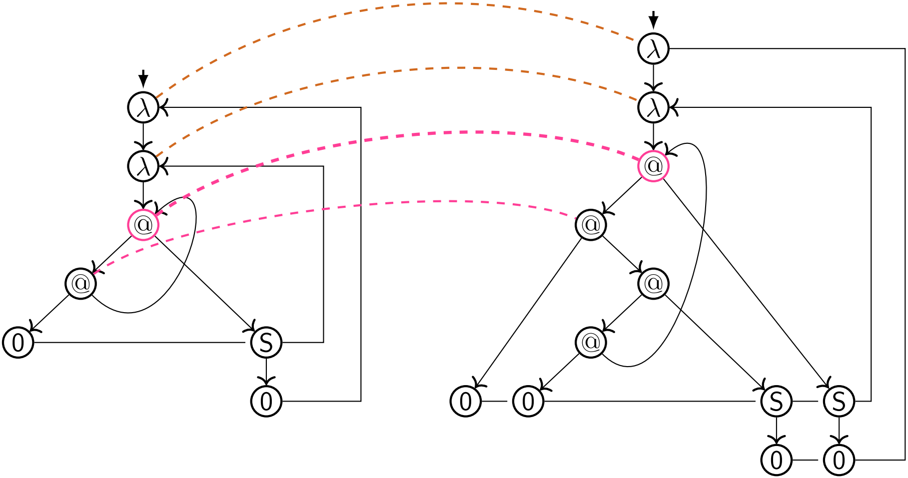


Bisimulation check between λ-term-graphs



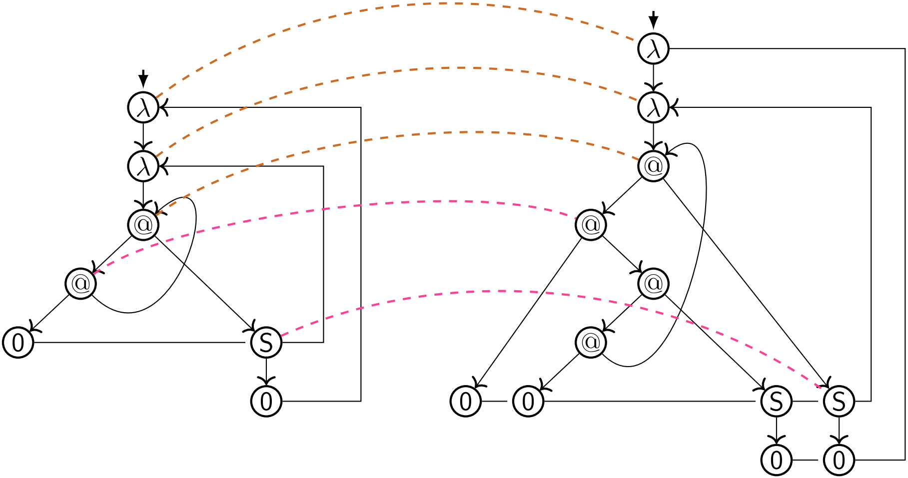
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



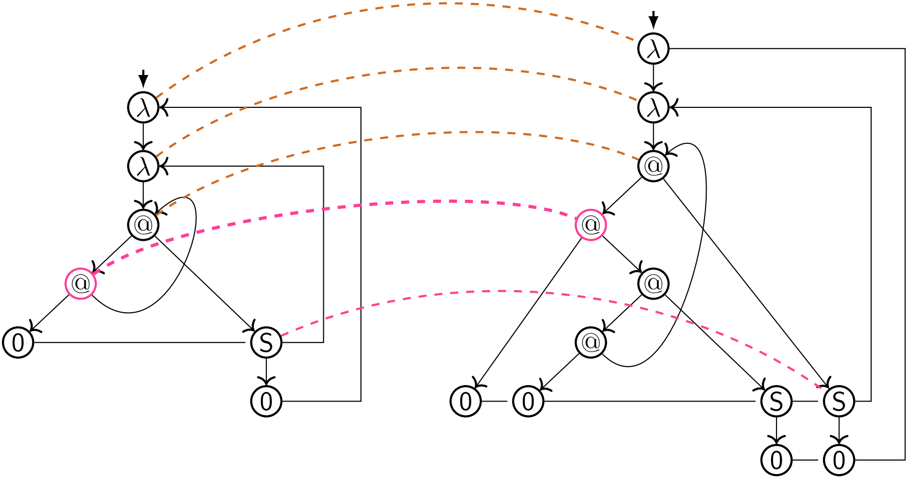
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



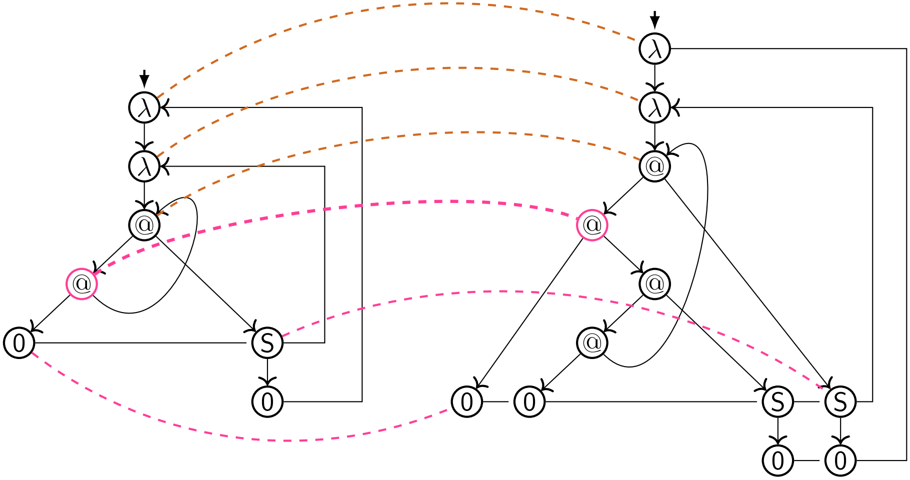
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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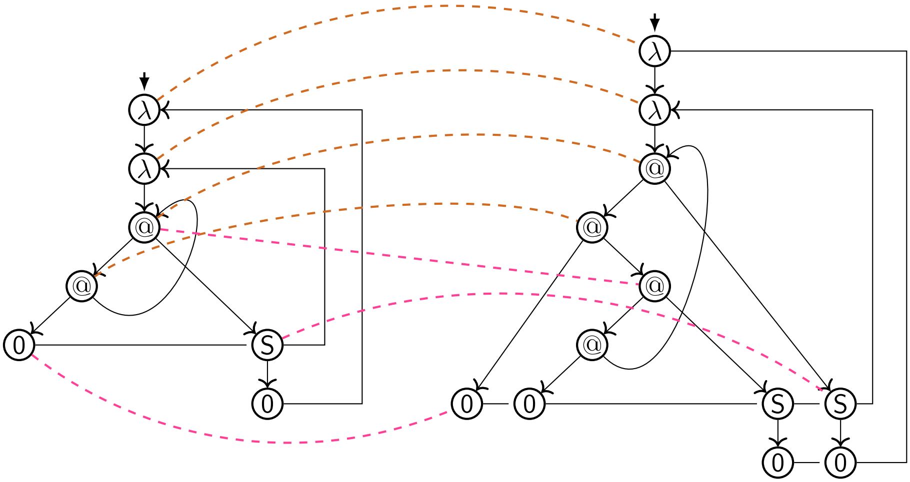

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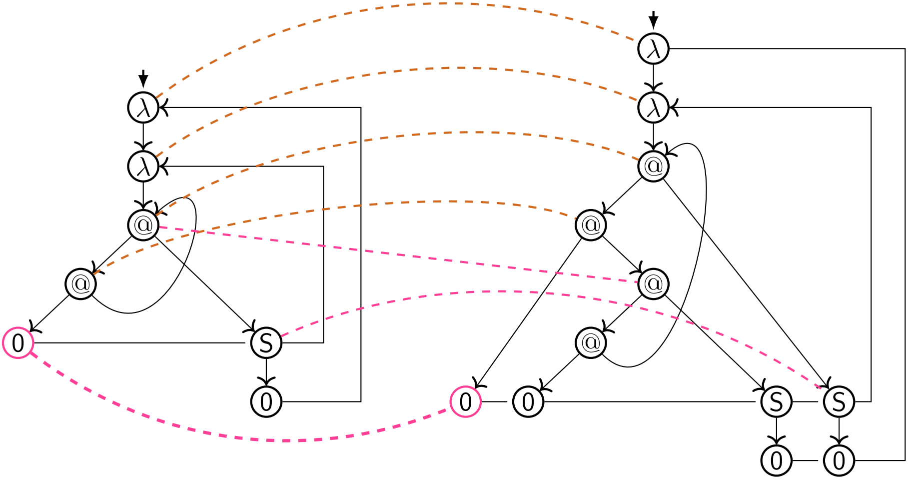

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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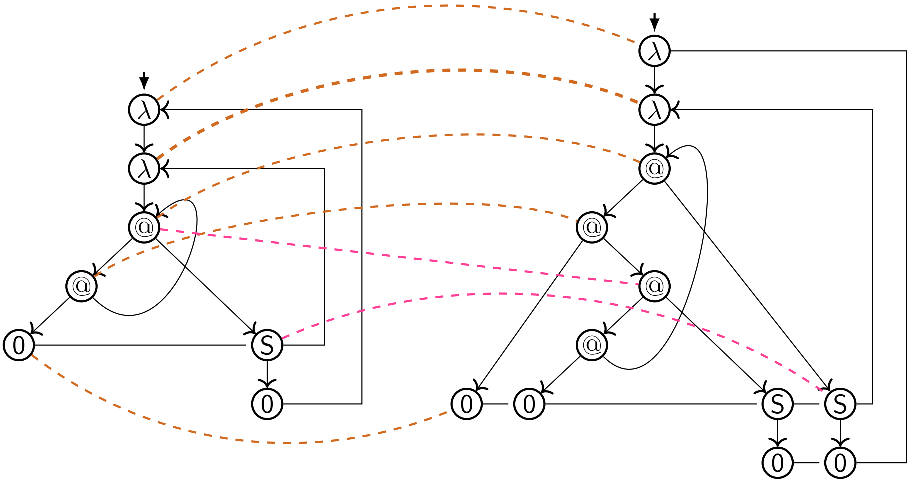

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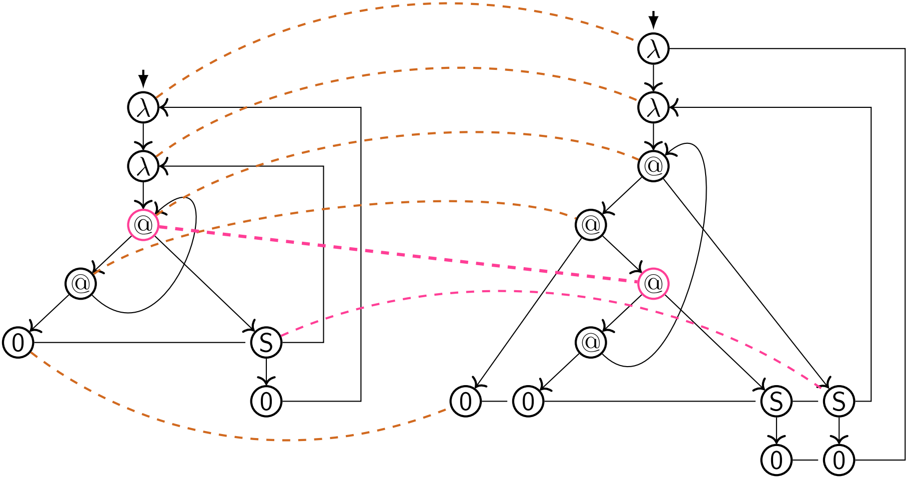

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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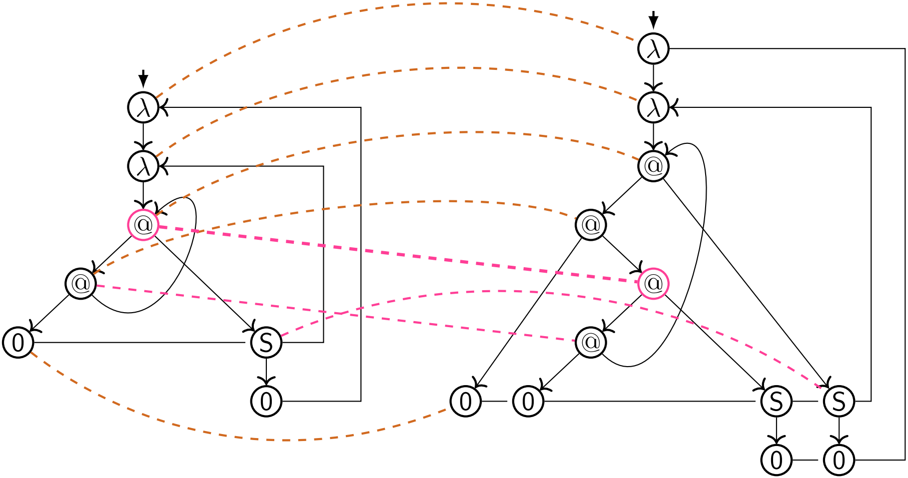

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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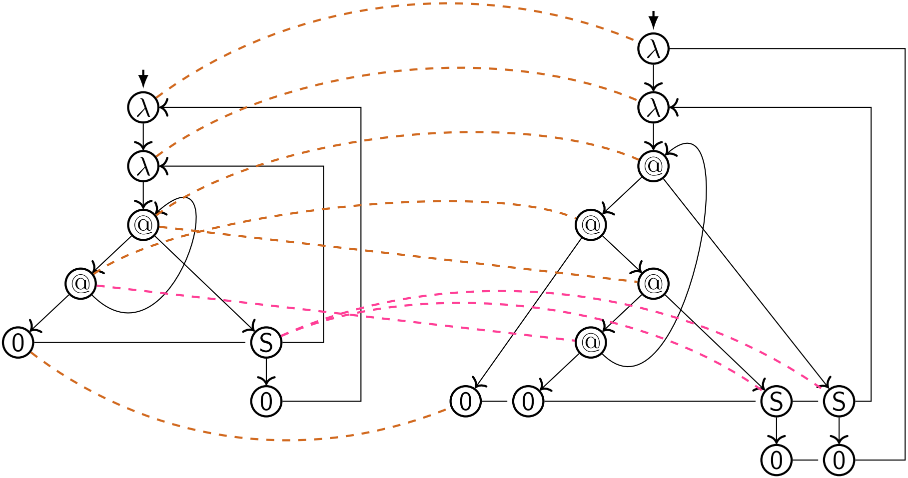

$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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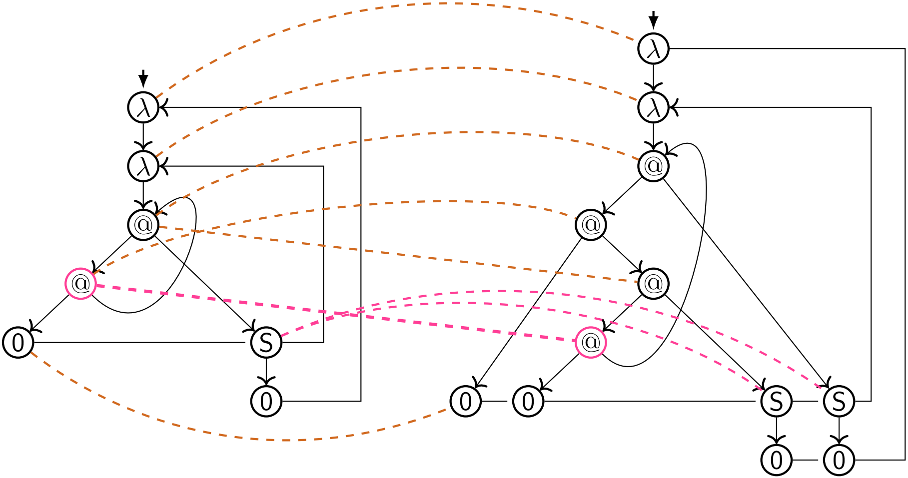
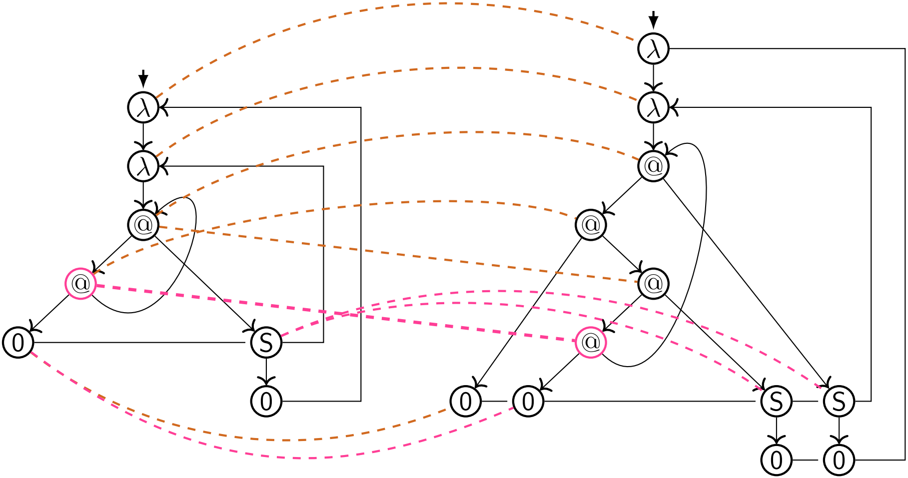


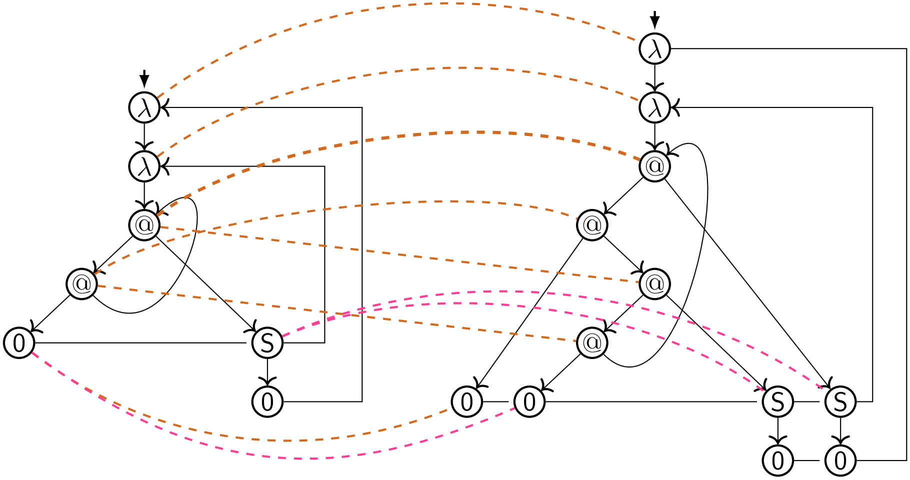
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



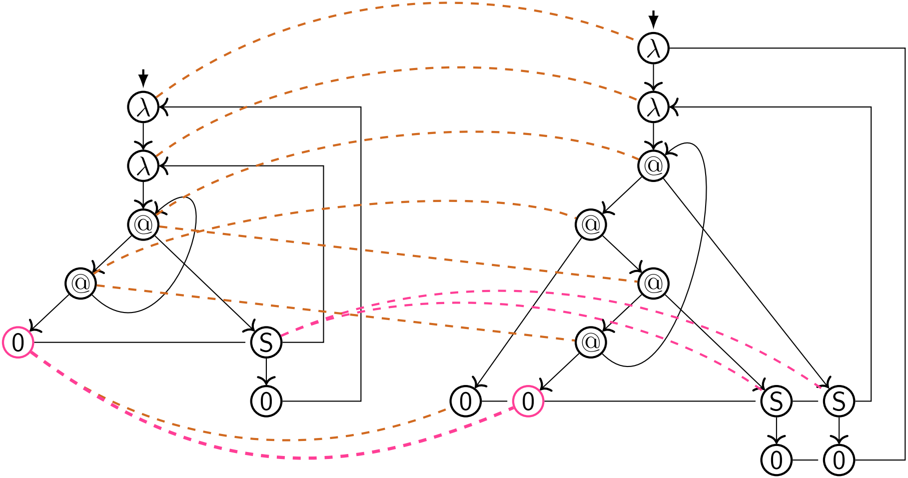
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



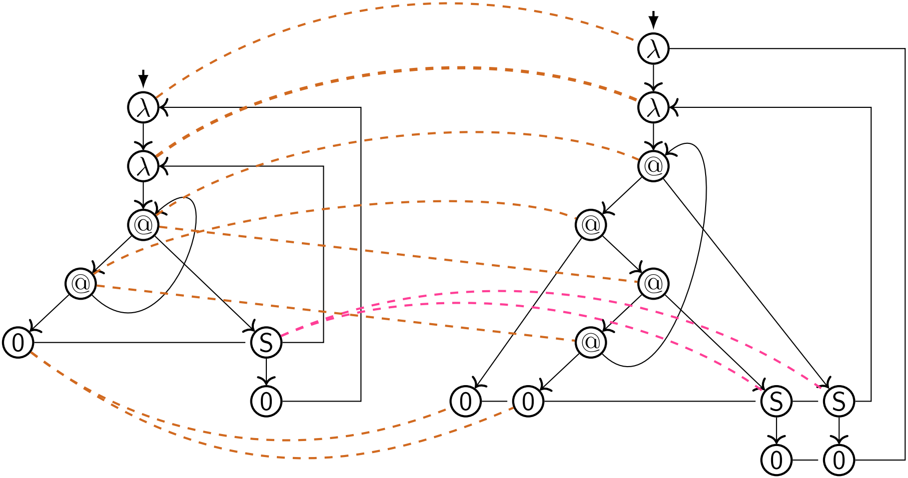
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



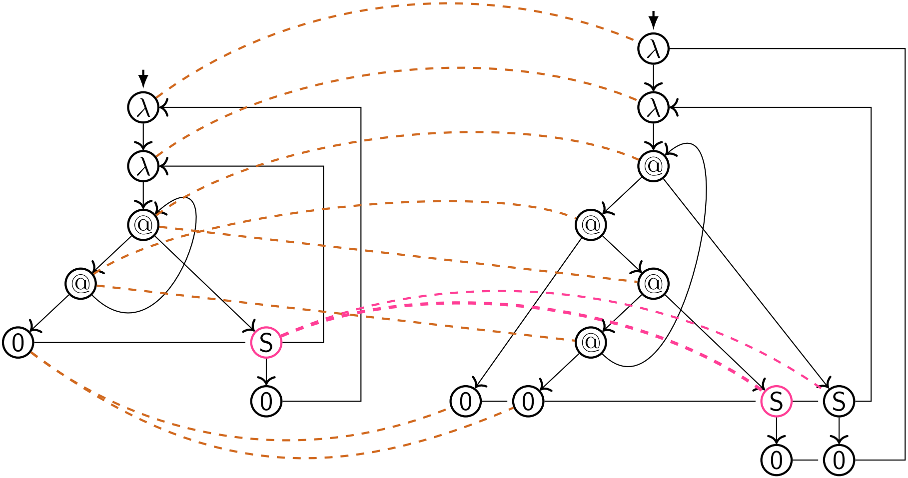
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



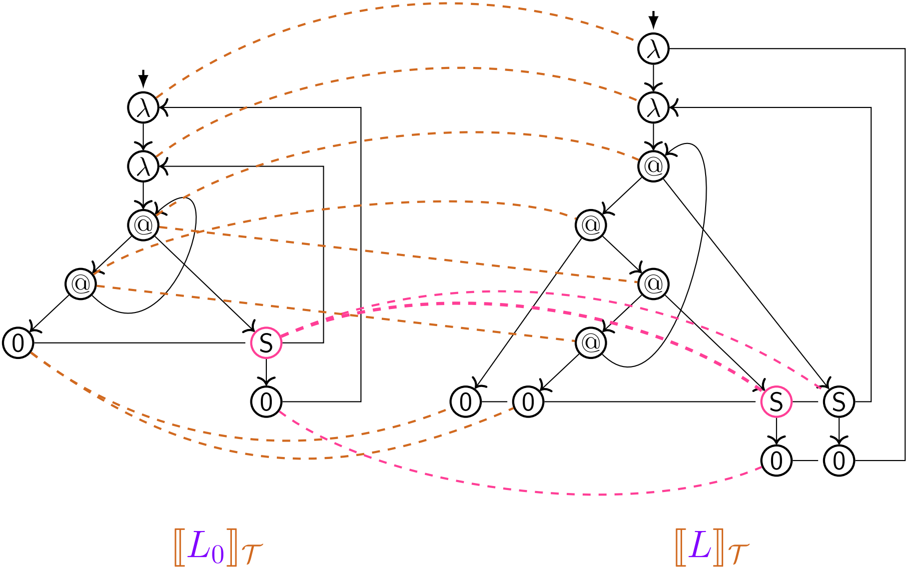
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs

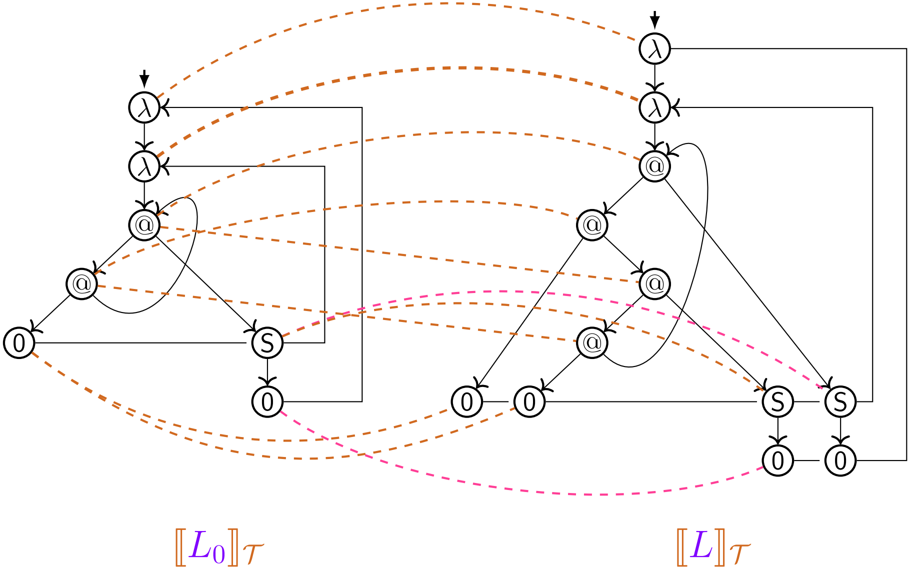


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

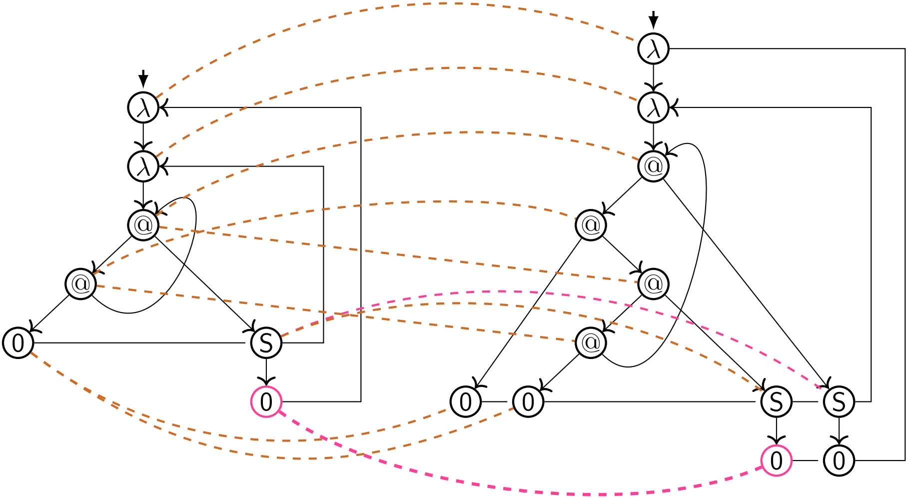
Bisimulation check between λ-term-graphs



Bisimulation check between λ-term-graphs

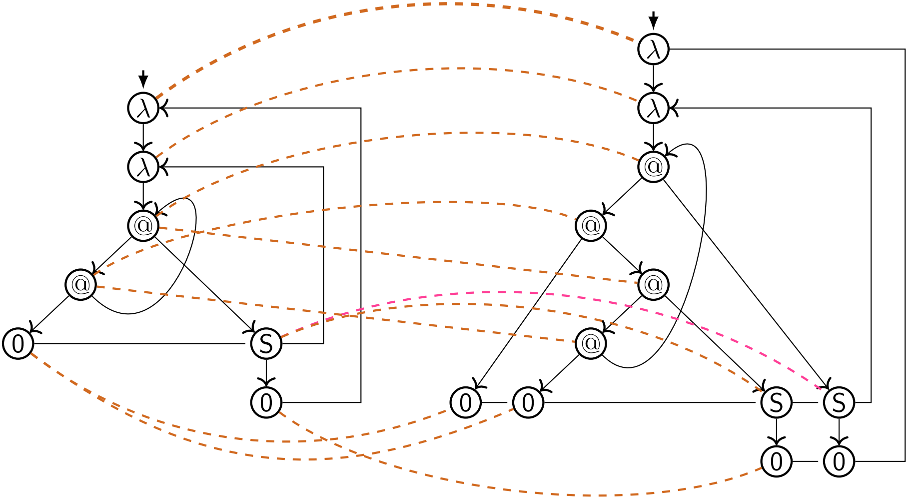


Bisimulation check between λ-term-graphs



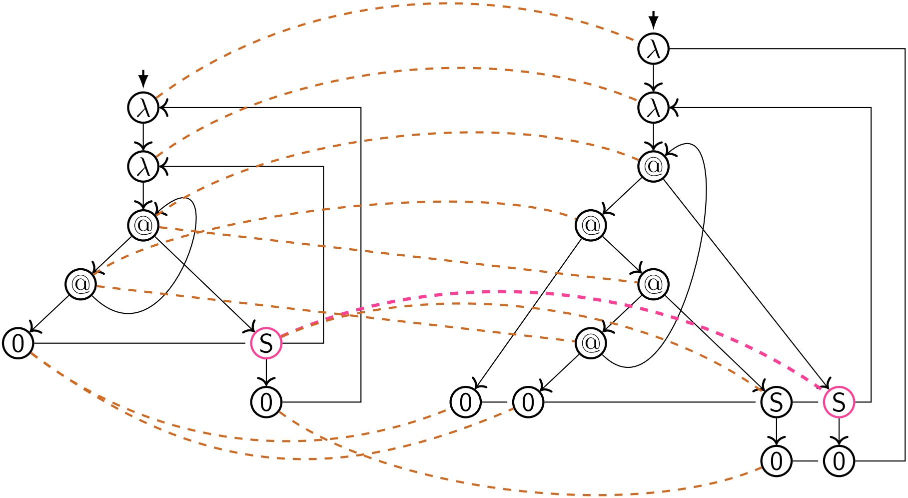
$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimulation check between λ-term-graphs



$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

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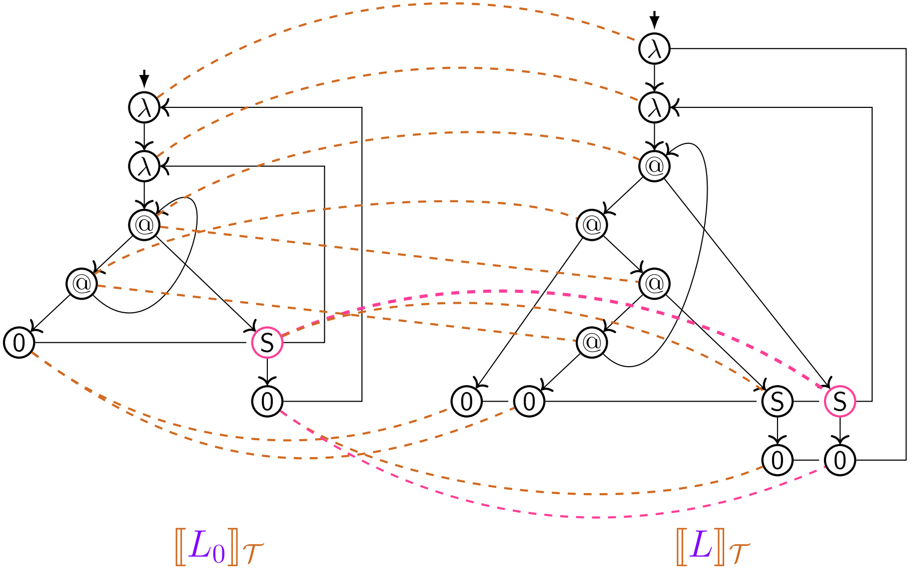
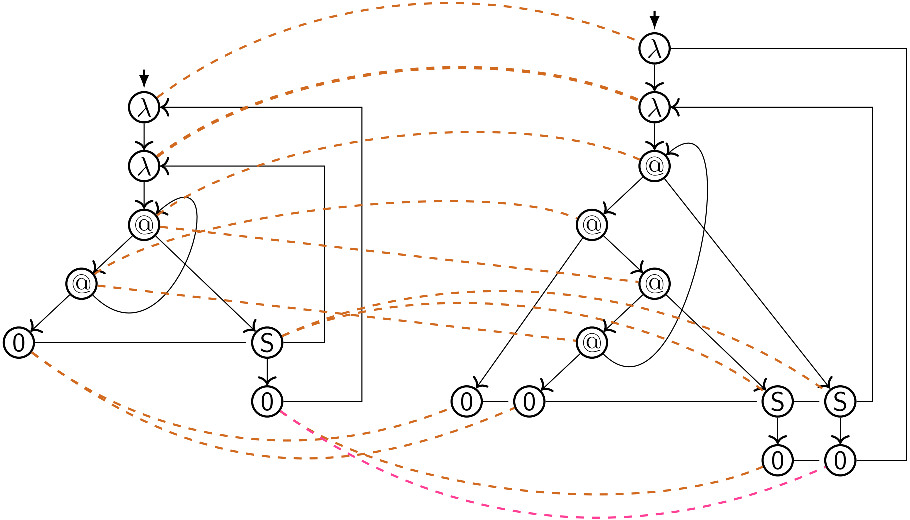
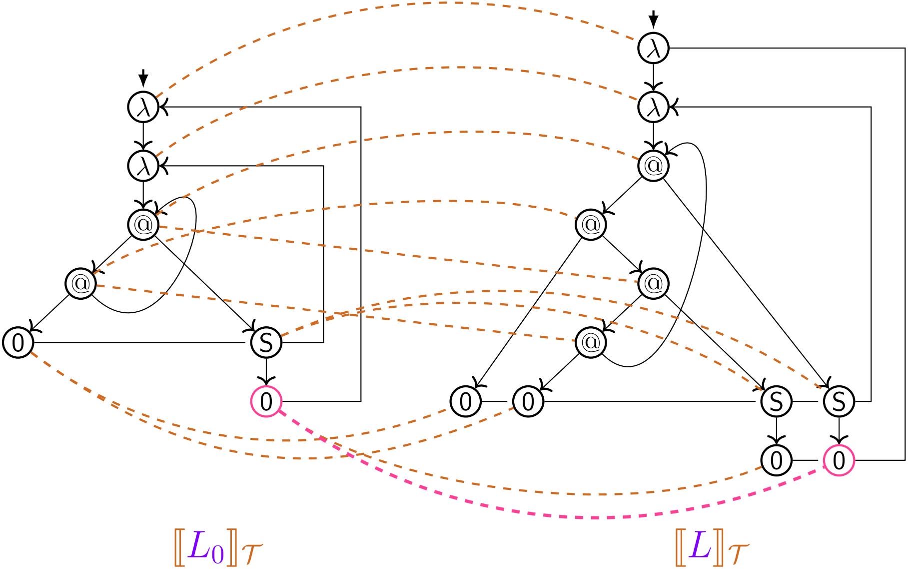


Bisimulation check between λ-term-graphs

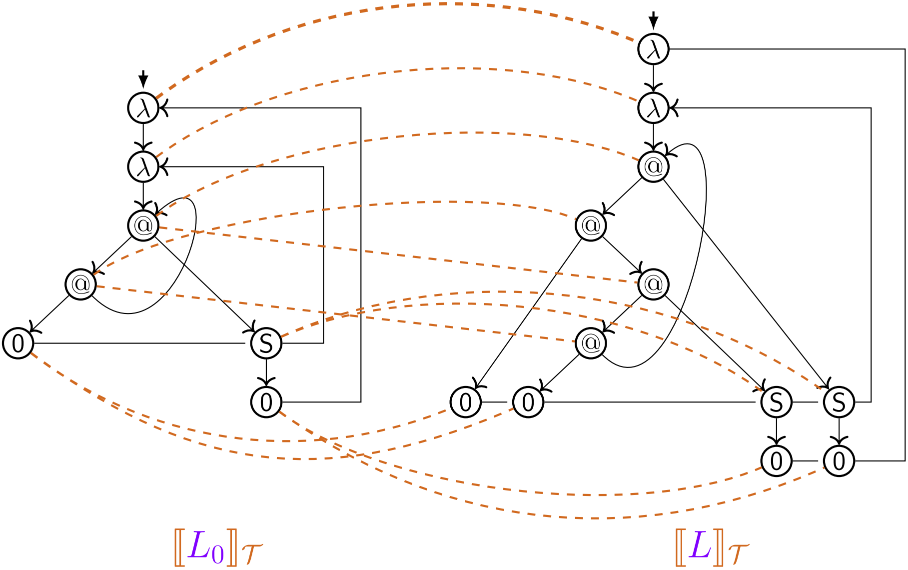


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

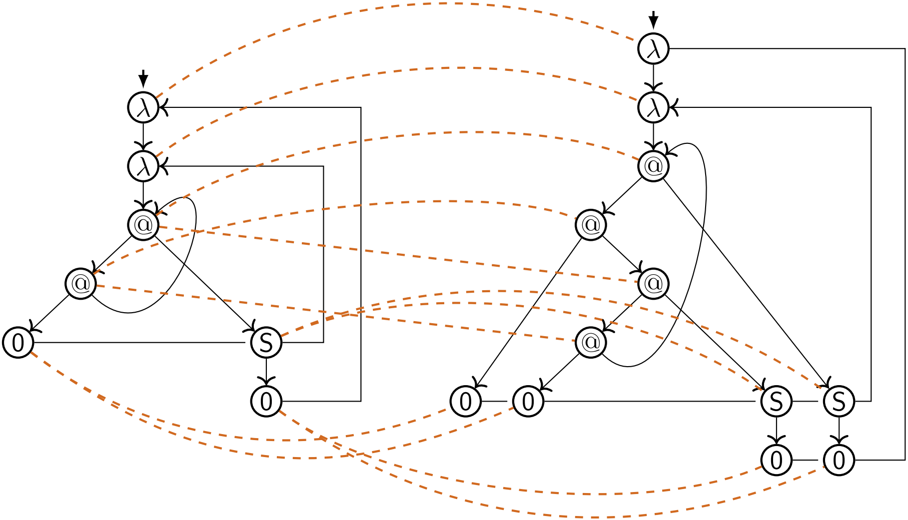
Bisimulation check between λ-term-graphs



Bisimulation check between λ-term-graphs

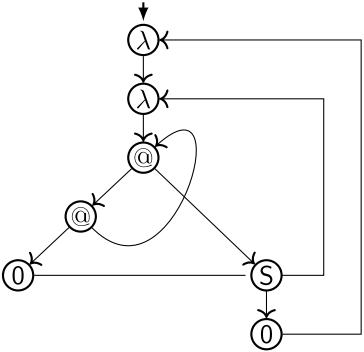


Bisimulation between λ-term-graphs

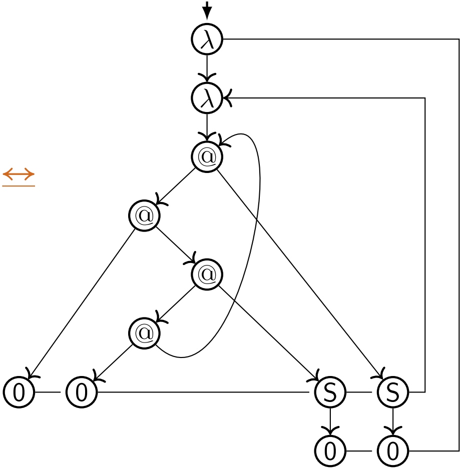


$[\![L_0]\!]_{\mathcal{T}}$ $[\![L]\!]_{\mathcal{T}}$

Bisimilarity between λ-term-graphs

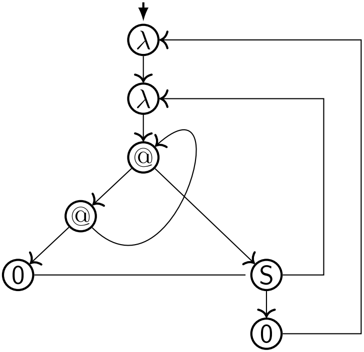


$\underline{\leftrightarrow}$

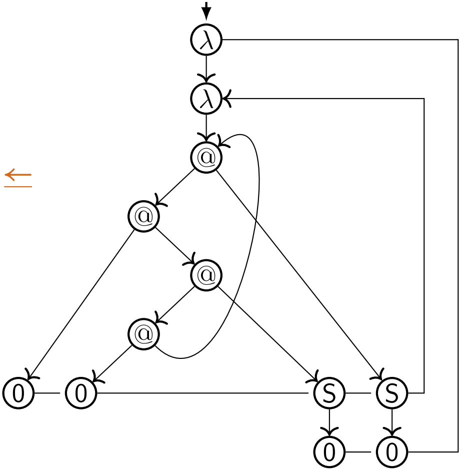


$$[\![L_0]\!]_{\mathcal{T}} \quad \underline{\leftrightarrow} \quad [\![L]\!]_{\mathcal{T}}$$

Functional bisimilarity and bisimulation collapse



$\Leftarrow$



$[\![L_0]\!]_{\mathcal{T}}$ $\Leftarrow$ $[\![L]\!]_{\mathcal{T}}$

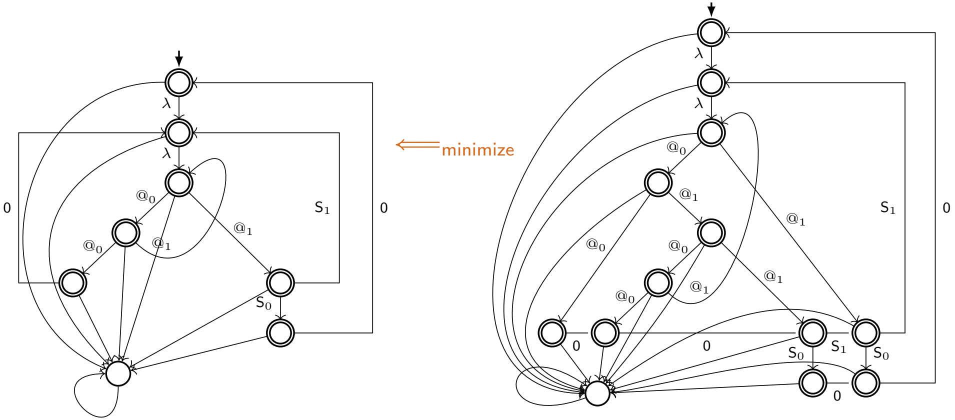
Bisimulation collapse: property

Theorem

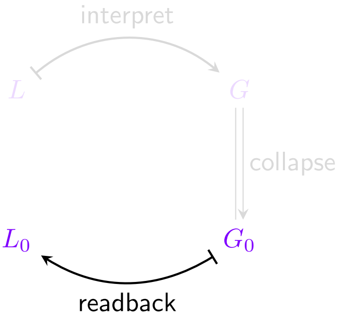
The class of eager-scope λ-term-graphs
is closed under functional bisimilarity $\rightharpoonup$.

$\Longrightarrow$ For a $\boldsymbol{\lambda}_{\mathsf{letrec}}$-term L
the bisimulation collapse of $[\![L]\!]_{\mathcal{T}}$ is again an eager-scope λ-term-graph.

λ-DFA-Minimization

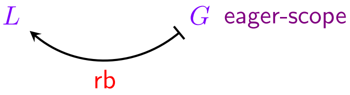


Readback



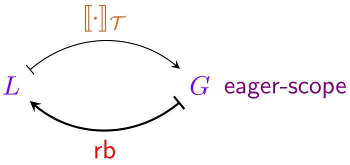
Readback

defined with property:



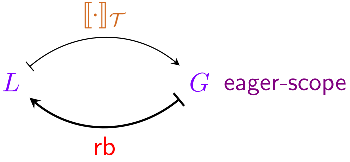
Readback

defined with property:



Readback

defined with property:



Theorem

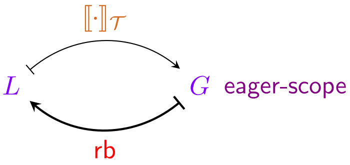
For all eager-scope λ-term-graphs G:

$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb *is a right-inverse of* $[\![\cdot]\!]_{\mathcal{T}}$ *modulo isomorphism* $\simeq$.

Readback

defined with property:



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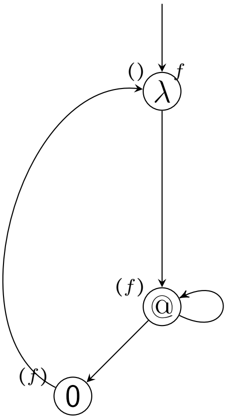
$$([\![\cdot]\!]_{\mathcal{T}} \circ \mathsf{rb})(G) \simeq G$$

The readback rb is a right-inverse of $[\![\cdot]\!]_{\mathcal{T}}$ modulo isomorphism $\simeq$.

idea:

1. construct a spanning tree T of G
2. using local rules, in a bottom-up traversal of T synthesize $L = \mathsf{rb}(G)$

Readback: example (fix)



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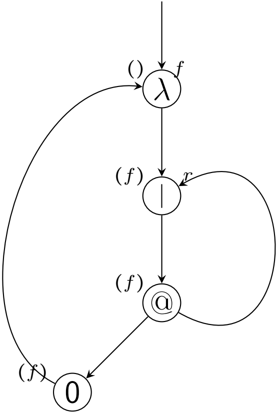

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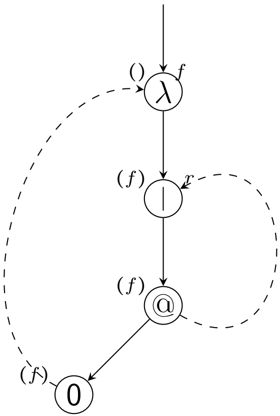

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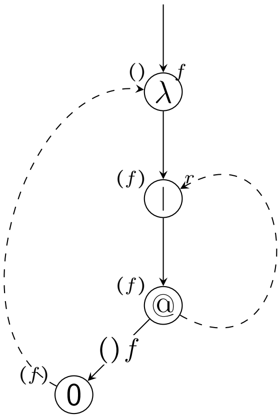

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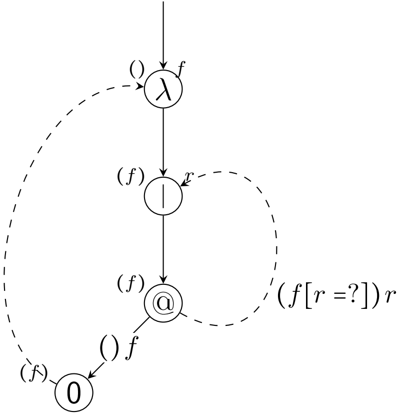

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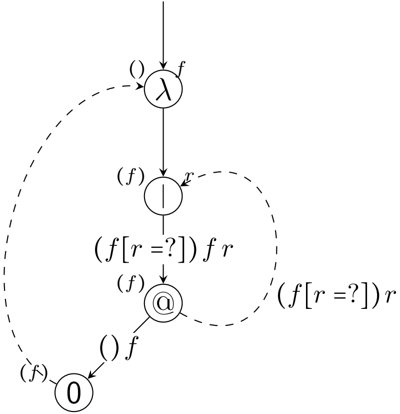

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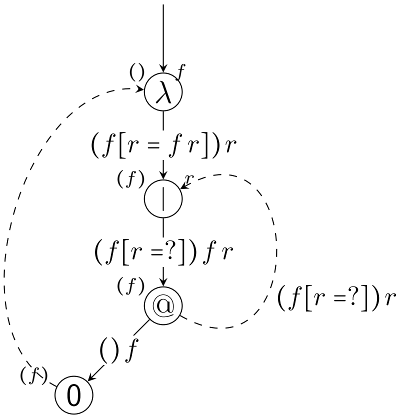

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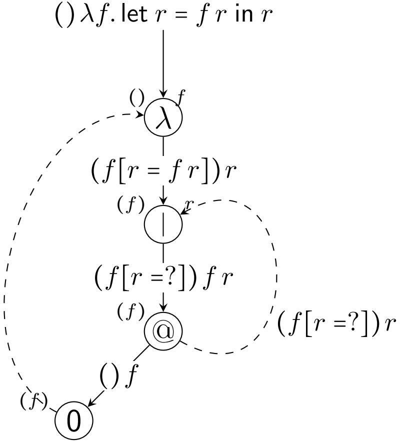

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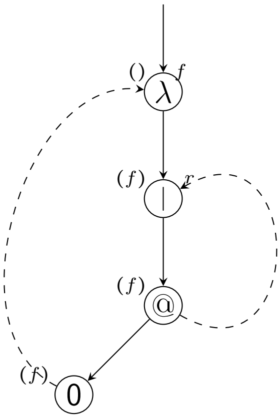

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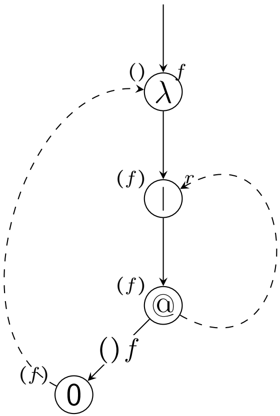

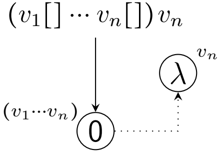

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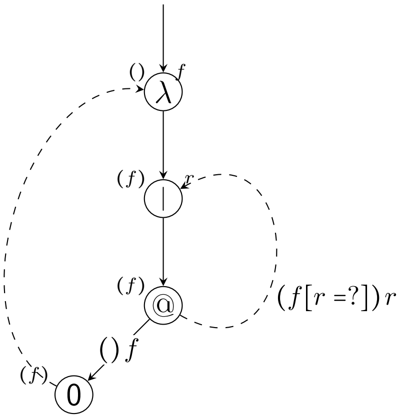

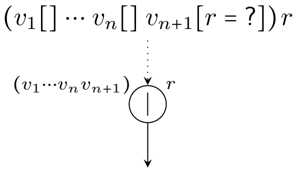

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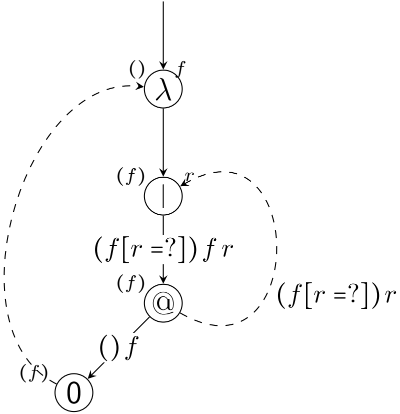

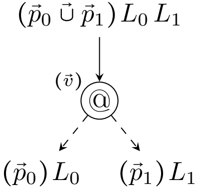

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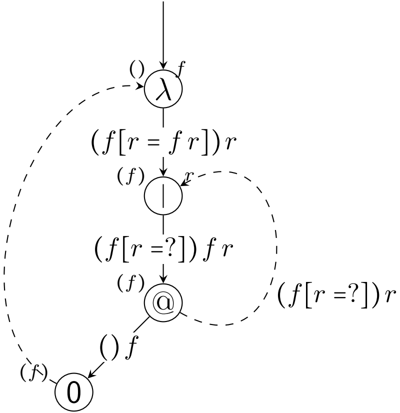

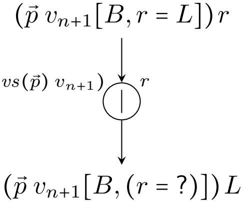

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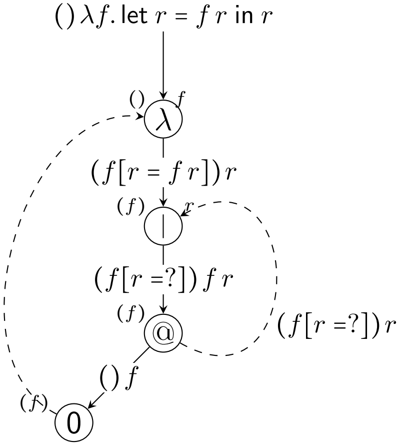
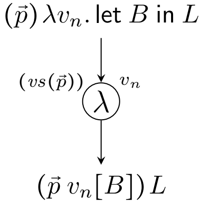
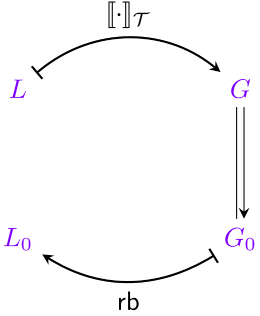




Maximal sharing: complexity

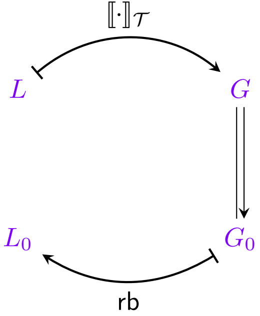


1. interpretation
   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L
   as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$

2. bisimulation collapse $|\!\downarrow$
   of f-o term graph G into G_0

3. readback rb
   of f-o term graph G_0
   yielding $\boldsymbol{\lambda}_{\text{letrec}}$-term $L_0 = \text{rb}(G_0)$.

Maximal sharing: complexity



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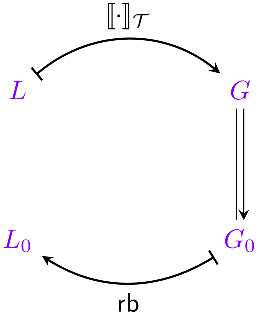
   of f-o term graph G_0
   yielding $\boldsymbol{\lambda}_{\text{letrec}}$-term $L_0 = \text{rb}(G_0)$.

Maximal sharing: complexity



1. interpretation

   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L with $|L| = n$

   as λ-term-graph $G = [\![L]\!]_{\mathcal{T}}$

   ▶ in time $O(n^2)$, size $|G| \in O(n^2)$.

2. bisimulation collapse $|\!\!\downarrow$

   of f-o term graph G into G_0

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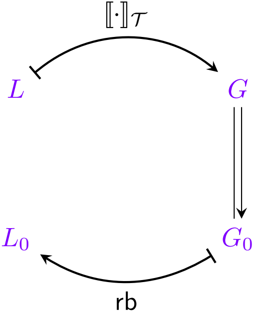

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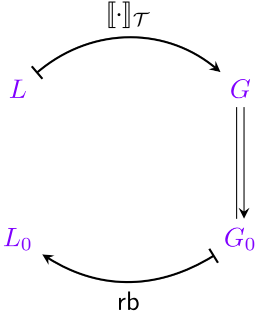

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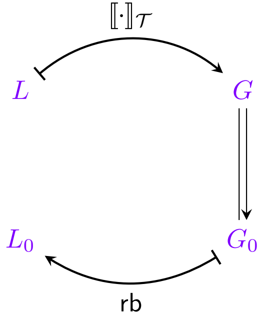

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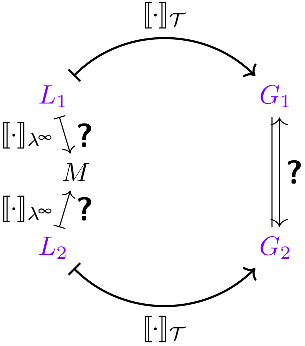
Theorem

Computing a maximally compact form $L_0 = (\text{rb} \circ |\!\!\downarrow \circ [\![\cdot]\!]_{\mathcal{T}})(L)$ *of* L *for a* $\boldsymbol{\lambda}_{\text{letrec}}$*-term* L *requires time* $O(n^2 \log n)$, *where* $|L| = n$.

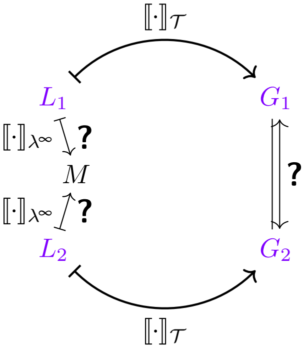
Unfolding equivalence: complexity



1. interpretation
   of $\boldsymbol{\lambda}_{\text{letrec}}$-term L_1, L_2
   as λ-term-graphs $G_1 = [\![L_1]\!]_{\mathcal{T}}$ and $G_2 = [\![L_2]\!]_{\mathcal{T}}$

2. check bisimilarity
   of λ-term-graphs G_1 and G_2

Unfolding equivalence: complexity



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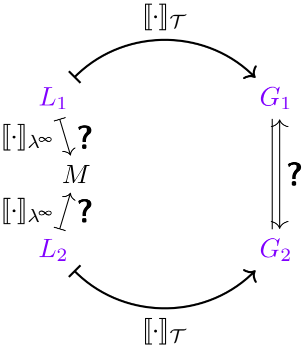

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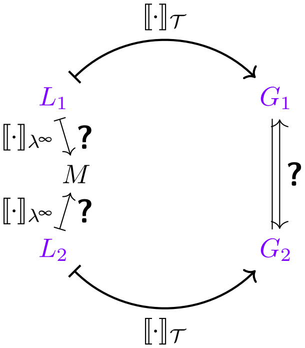

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2. check bisimilarity

   of λ-term-graphs G_1 and G_2

   - in time $O(|G_i|\,\alpha(|G_i|)) = O(n^2\,\alpha(n))$

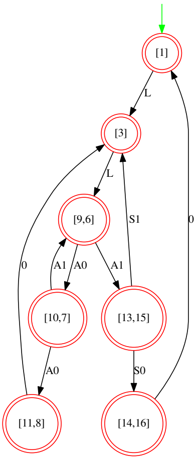
Theorem

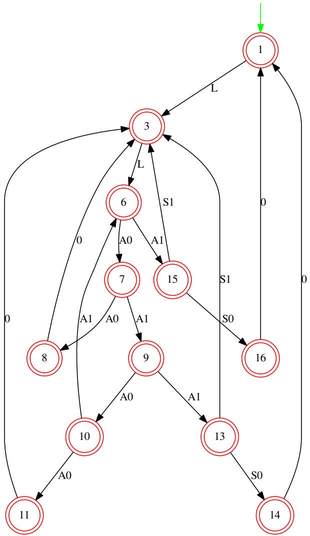
Deciding whether $\boldsymbol{\lambda}_{\text{letrec}}$-terms L_1 and L_2 are unfolding-equivalent requires almost quadratic time $O(n^2\alpha(n))$ for $n = \max\{|L_1|, |L_2|\}$.

Demo: console output

```
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014> maxsharing running.l
λ-letrec-term:
λx. λf. let r = f (f r x) x in r

derivation:
                ---------- 0                ------- 0
                (x f[r]) f    (x f[r]) r    (x) x
                ------------------------ @  ---------- S
                (x f[r]) f r                (x f[r]) x
------------ 0  ---------------------------------------- @    ------- 0
(x f[r]) f      (x f[r]) f r x                                (x) x
---------------------------------------------------------- @  ---------- S
(x f[r]) f (f r x)                                            (x f[r]) x
-------------------------------------------------------------------------- @
(x f[r]) f (f r x) x                                                          (x f[r]) r
---------------------------------------------------------------------------------------- let
(x f) let r = f (f r x) x in r
-------------------------------------------------------------------------------------------- λ
(x) λf. let r = f (f r x) x in r
---------------------------------------------------------------------------------------------- λ
() λx. λf. let r = f (f r x) x in r

writing DFA to file: running-dfa.pdf

readback of DFA:
λx. λy. let F = y (y F x) x in F

writing minimised DFA to file: running-mindfa.pdf

readback of minimised DFA:
λx. λy. let F = y F x in F
jan:~/papers/maxsharing-ICFP/talks/ICFP-2014>
```

Demo: generated λ-DFAs





Desiderata $\rightharpoonup$ results: structure-constrained term graphs

λ-calculus with letrec under unfolding semantics $[\![\cdot]\!]_{\lambda^\infty}$

Not available: term graph semantics that is studied under $\leftrightarrow$

- graph representations used by compilers were not intended for use under $\leftrightarrow$

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Desired: term graph semantics that:

- natural correspondence with terms in $\boldsymbol{\lambda}_{\text{letrec}}$
- supports compactification under $\leftrightarrow$
- efficient translation and readback

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Desired: term graph semantics that:

- natural correspondence with terms in $\boldsymbol{\lambda}_{\text{letrec}}$
- supports compactification under $\leftrightarrow$
- efficient translation and readback

Defined: int's $[\![\cdot]\!]_{\mathcal{H}}/[\![\cdot]\!]_{\mathcal{T}}$ as higher-order/first-order λ-term graphs

- closed under $\Rightarrow$ (hence under collapse)
- back-/forth correspondence with λ-calculus with letrec
  - efficient translation and readback
  - translation is inverse of readback

Desiderata ⇾ results: structure-constrained process graphs

Regular expressions under process semantics (bisimilarity $\underline{\leftrightarrow}$)

Given: process graph interpretation $[\![\cdot]\!]_P$, studied under $\underline{\leftrightarrow}$

- not closed under $\underline{\rightarrow}$, and $\underline{\leftrightarrow}$, modulo $\underline{\leftrightarrow}$ incomplete

Desiderata $\rightarrowtriangle$ results: structure-constrained process graphs

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Desired: reason with graphs that are $[\![\cdot]\!]_P$-expressible modulo $\underline{\leftrightarrow}$ (at least with 'sufficiently many')

understand incompleteness by a structural graph property

Desiderata → results: structure-constrained process graphs

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Desired: reason with graphs that are $[\![\cdot]\!]_P$-expressible modulo $\underline{\leftrightarrow}$
(at least with 'sufficiently many')

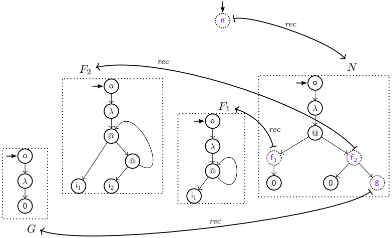
understand incompleteness by a structural graph property

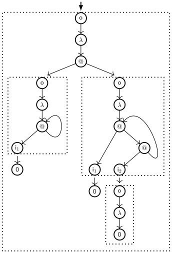
Defined: class of process graphs with LEE / (layered) LEE-witness

- closed under $\underline{\rightarrow}$ (hence under collapse)
- back-/forth correspondence with 1-return-less expr's
- contains the collapse of a process graph G
  $\iff$ G is $[\![\cdot]\!]_P^{\not{1r}\backslash\star}$-expressible modulo $\underline{\leftrightarrow}$

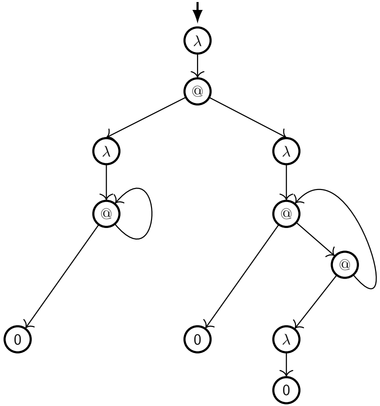
Nested Term Graphs

(joint work with Vincent van Oostrom)



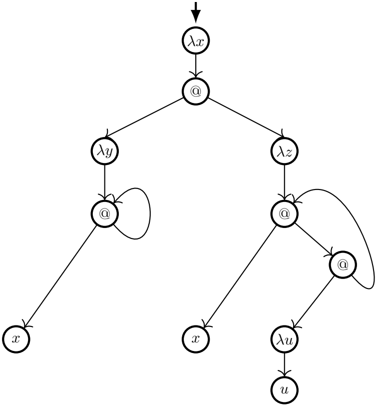


Nested scopes in λ_{letrec} terms



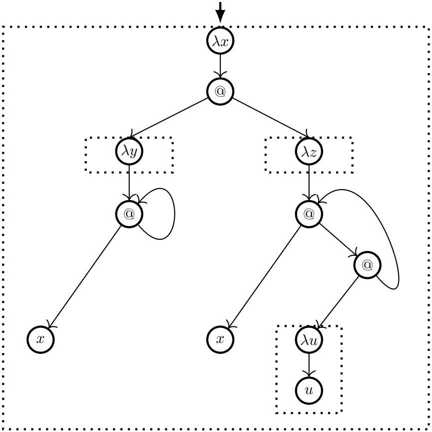
First-order term graph over $\Sigma = \{\lambda/1,\ @/2,\ 0/0\}$

Nested scopes in $\lambda_{\mathsf{letrec}}$ terms



$$\lambda x.\,(\lambda y.\,\mathsf{let}\ \alpha = x\,\alpha\ \mathsf{in}\ \alpha)\,(\lambda z.\,\mathsf{let}\ \beta = x\,(\lambda u.\,u)\,\beta\ \mathsf{in}\ \beta)$$

Nested scopes in λ_{letrec} terms



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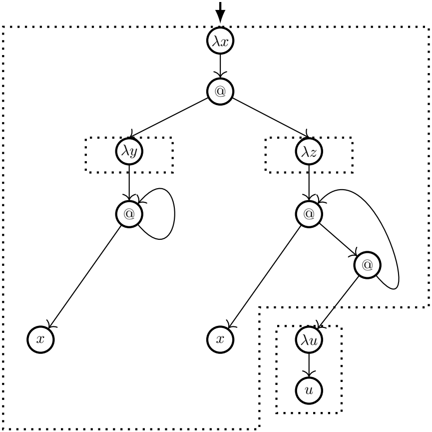

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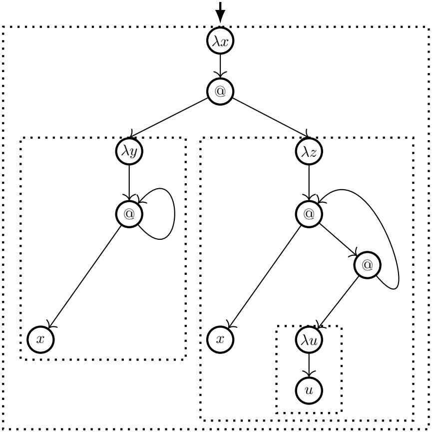
Nested scopes in λ_{letrec} terms



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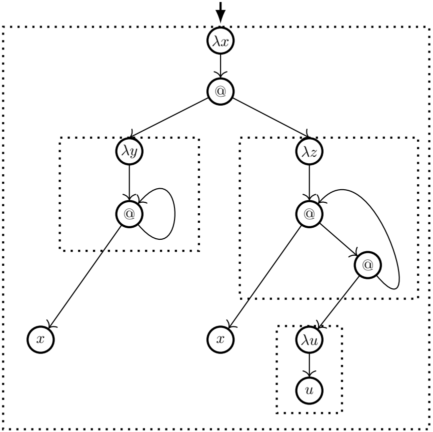
Nested scopes in $\lambda_{\mathsf{letrec}}$ terms



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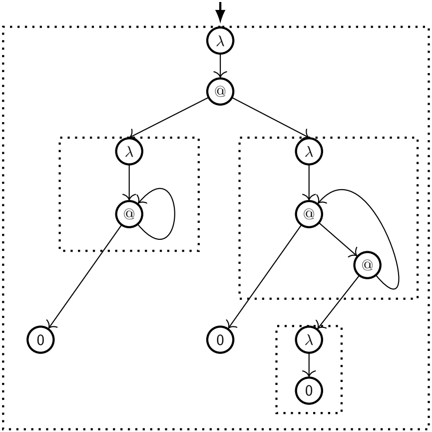
Nested scopes in λ_{letrec} terms



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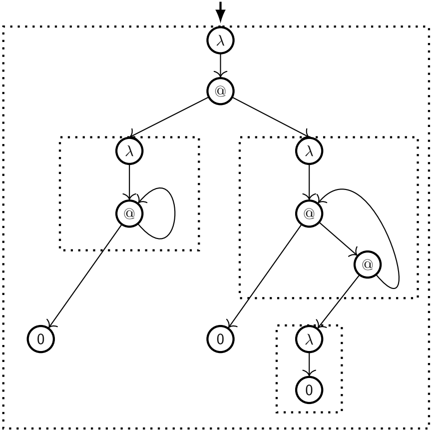
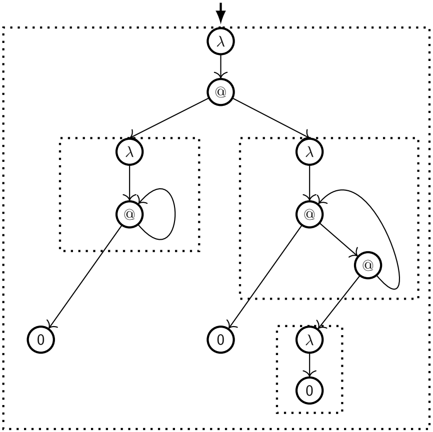
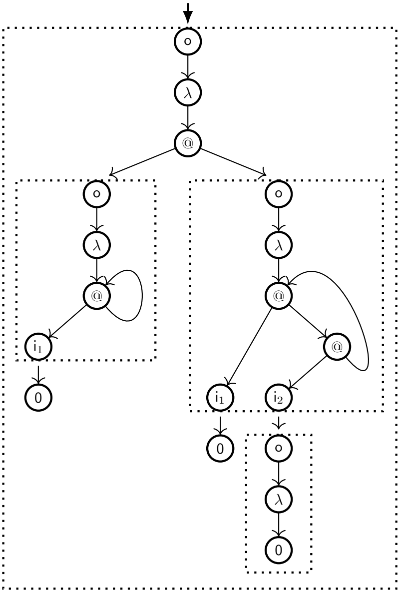
Nested scopes in λ-terms



Nested scopes → nested term graph





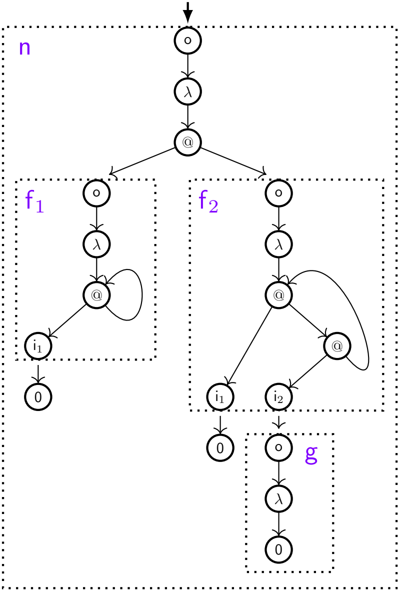
nested term graph

gletrec

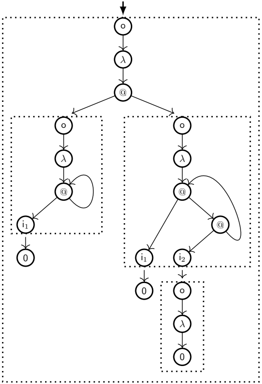
$$
\begin{aligned}
\mathsf{n}() &= \lambda x.\mathsf{f}_1(x)\mathsf{f}_2(x,\mathsf{g}())\\
\mathsf{f}_1(X_1) &= \lambda x.\mathsf{let}\,\alpha = X_1\alpha\,\mathsf{in}\,\alpha\\
\mathsf{f}_2(X_1,X_2) &= \lambda y.\mathsf{let}\,\beta = X_1(X_2\beta)\,\mathsf{in}\,\beta\\
\mathsf{g}() &= \lambda z.z
\end{aligned}
$$

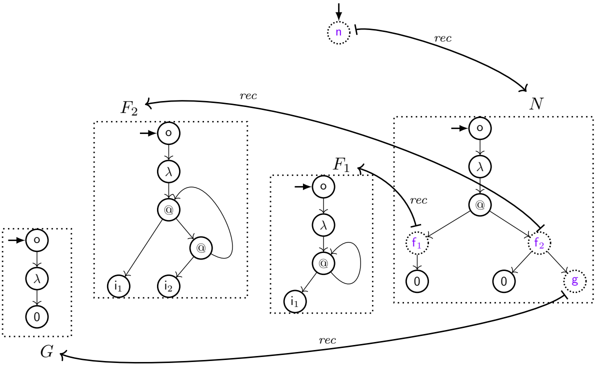
in

$$\mathsf{n}()$$



nested term graph





Signature

A *signature for nested term graphs* (*ntg-signature*) is a signature Σ that is partitioned into:

- *atomic* symbol alphabet Σ_{at}
- *nested* symbol alphabet Σ_{ne}

Additionally used:

- *interface* symbols alphabet $OI = O \cup I$
  - $O = \{\mathsf{o}\}$ with o unary
  - $I = \{\mathsf{i}_1, \mathsf{i}_2, \mathsf{i}_3, \ldots\}$ with i_j nullary

Recursive graph specification

Definition

Let Σ be an ntg-signature.
A *recursive graph specification* (a *rgs*) $\mathcal{R} = \langle rec, r \rangle$ consists of:

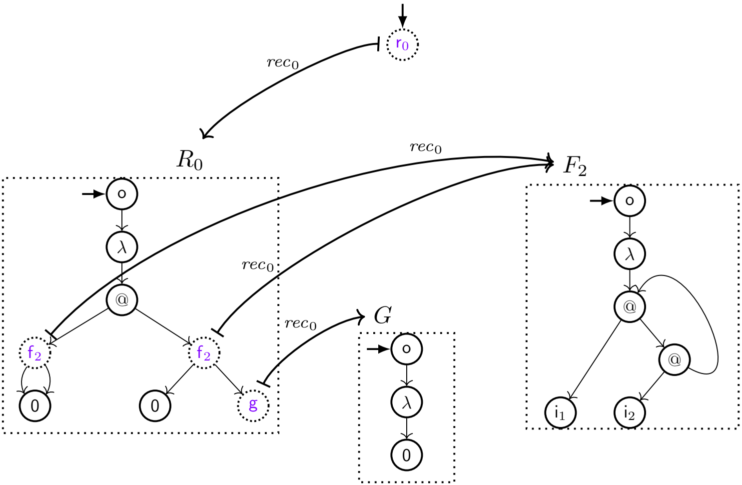
- *specification function*

$$rec : \Sigma_{\mathsf{ne}} \longrightarrow \mathsf{TG}(\Sigma \cup OI)$$
$$f_{/k} \longmapsto rec(f) = F \in \mathsf{TG}(\Sigma \cup \{\mathsf{o}, \mathsf{i}_1, \ldots, \mathsf{i}_k\})$$

  where F contains precisely one vertex labeled by o, the root, and one vertex each labeled by i_j, for $j \in \{1, \ldots, k\}$;

- nullary *root symbol* $r \in \Sigma_{\mathsf{ne}}$.

Recursive graph specification



$\Sigma_{\mathsf{at}} = \{\lambda/1,\ @/2,\ 0/0\}$, $\Sigma_{\mathsf{ne}} = \{r_0/0,\ f_2/2,\ g/0\}$, $O = \{o/1\}$,
$I = \{i_1/0, i_2/0, \ldots\}$.

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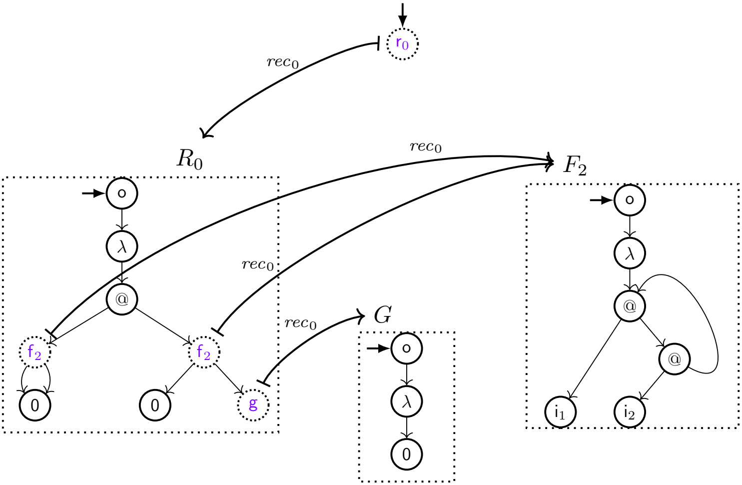
– nullary *root symbol* $r \in \Sigma_{\text{ne}}$.

rooted *dependency* ARS $\multimap$ of $\mathcal{R}$:

- objects: nested symbols in Σ_{ne}
- steps: for all $f, g \in \Sigma_{\text{ne}}$:

$$p : f \multimap g \iff g \text{ occurs in the term graph } rec(f) \text{ at position } p$$

Recursive graph specification



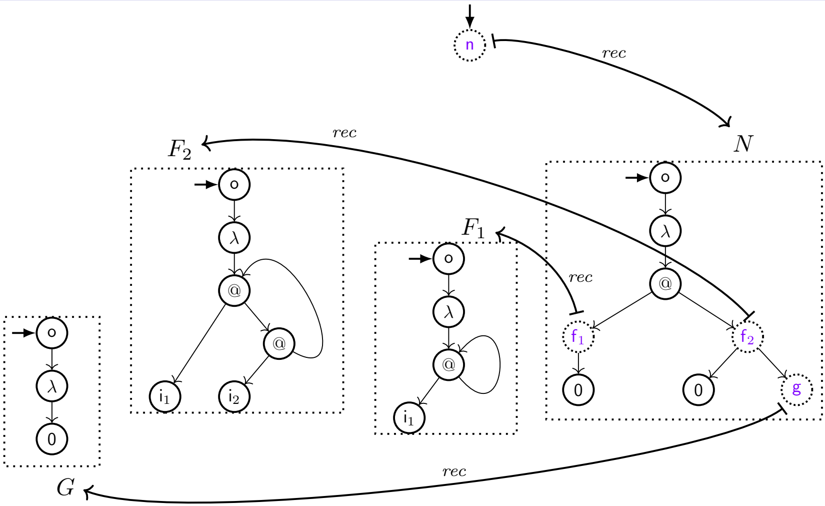
dependency ARS: $\mathsf{f}_2 \; \substack{\multimap \\ \multimap} \; \mathsf{r}_0 \; \multimapinv \; \mathsf{g}$ is a dag (but not a tree).

Nested term graph: intensional definition

Definition

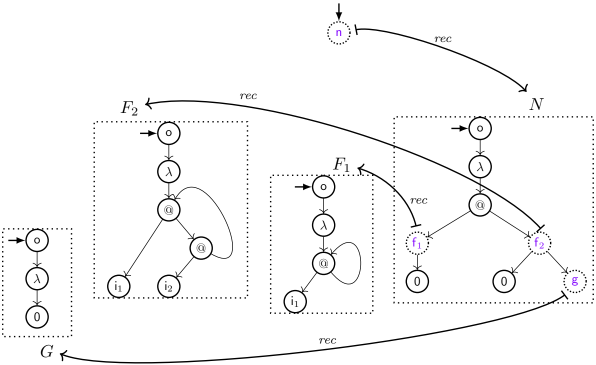
Let Σ be an ntg-signature.
A *nested term graph* over Σ is an rgs $\mathcal{N} = \langle rec, r \rangle$ such that the rooted dependency ARS $\multimapdotinv$ is a tree.

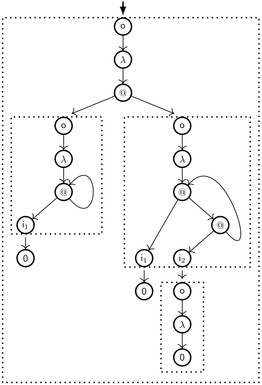
Nested term graph (intensionally)



dependency ARS: $f_1 \multimap n \; \substack{\multimap f_2 \\ \multimap g}$ is a tree.

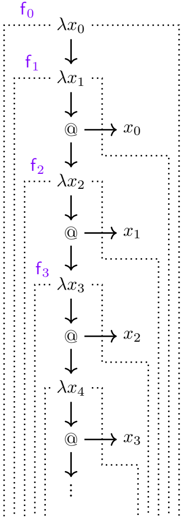
Nested term graph (intensionally)





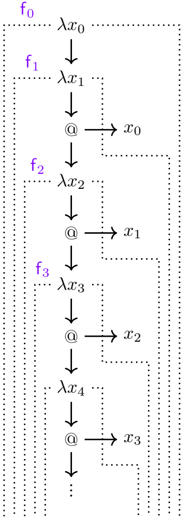
dependency ARS: $\mathsf{f}_1 \multimap \mathsf{n} \begin{array}{l} \circ\!\!- \ \mathsf{f}_2 \\ \circ\!\!- \ \mathsf{g} \end{array}$ is a tree.

Nested term graph (intensionally)

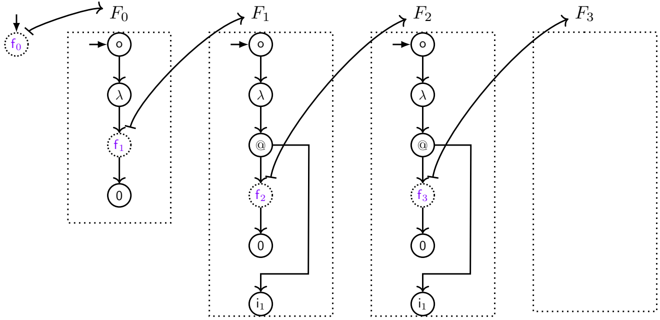


infinite λ-term
(infinitely nested scopes)

Nested term graph (intensionally)

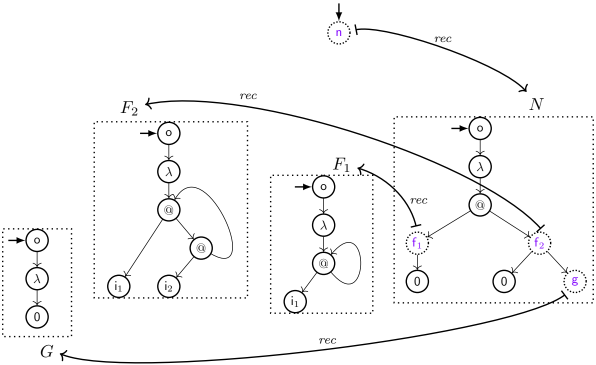


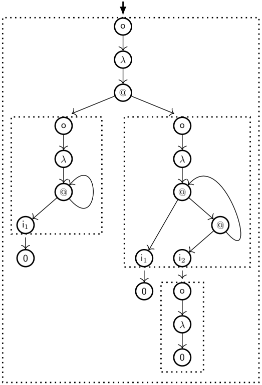
infinite λ-term
(infinitely nested scopes)



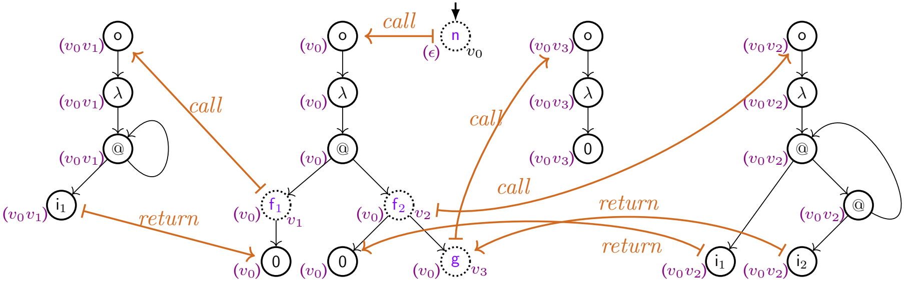
nested term graph with infinite nesting
dependency ARS: $\mathsf{f}_0 \multimap \mathsf{f}_1 \multimap \mathsf{f}_2 \multimap \mathsf{f}_3 \multimap \ldots$

Nested term graph (intensionally)





Nested term graph: extensional definition



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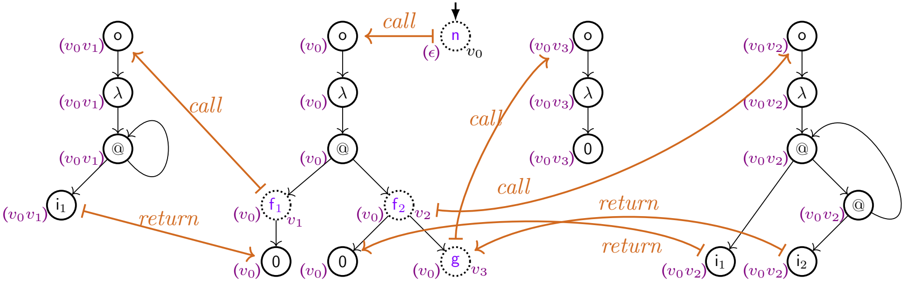

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- *call* : $V \rightharpoonup V$, (v with nested symbol) $\mapsto$ (root of graph nested into v)

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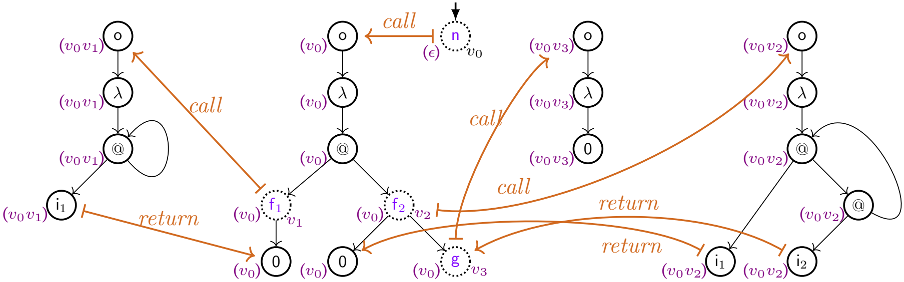

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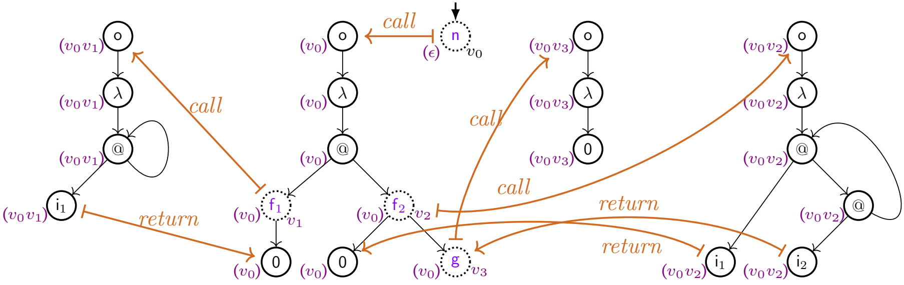

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- *anc* : $V \to V^*$ *ancestor function*: $v \mapsto$ word $anc(v) = v_1 \cdots v_n$ of the vertices in which v is nested

Nested term graphs: intensional vs. extensional definition

Proposition

- Every nested term graph has an extensional description.
- For every entg $\mathcal{G}$ there is a nested term graph for which $\mathcal{G}$ is the extensional description.

Bisimulation

Bisimulation (for intensional ntg-definition)

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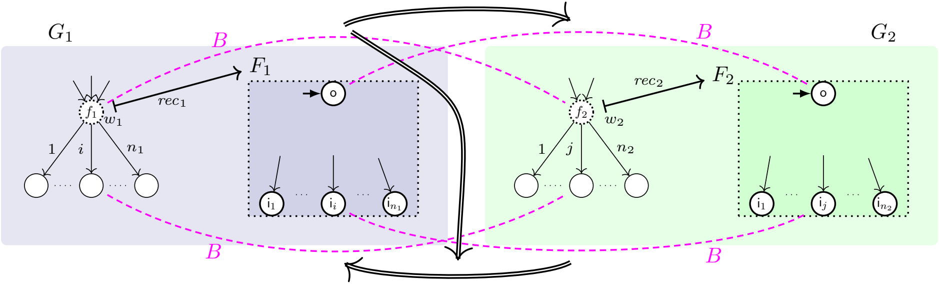

Bisimulation (for extensional ntg-definition)

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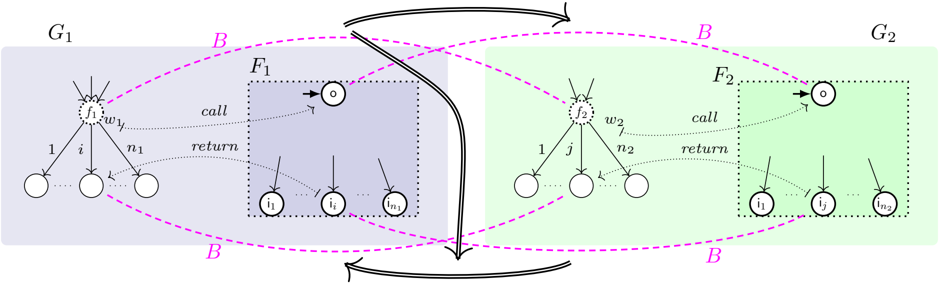
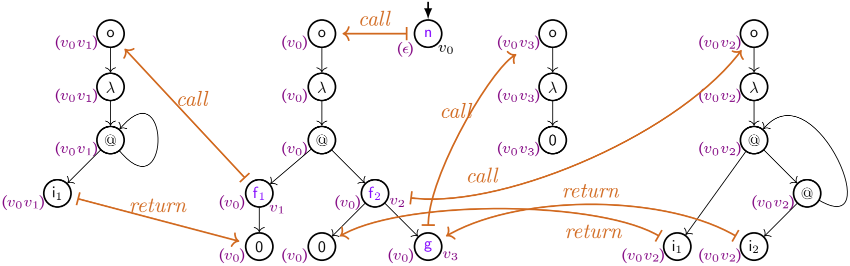


Implementation by first-order term graph (via entg)



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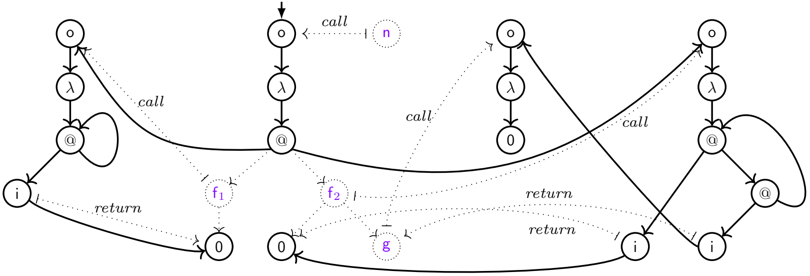

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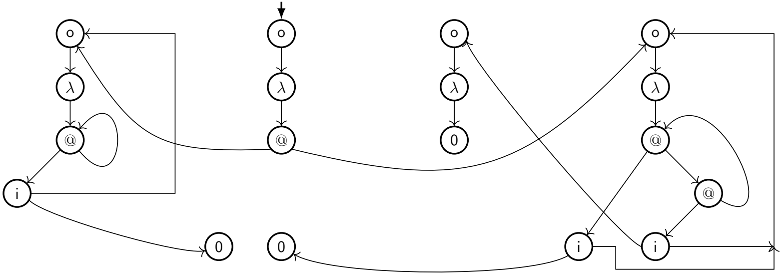

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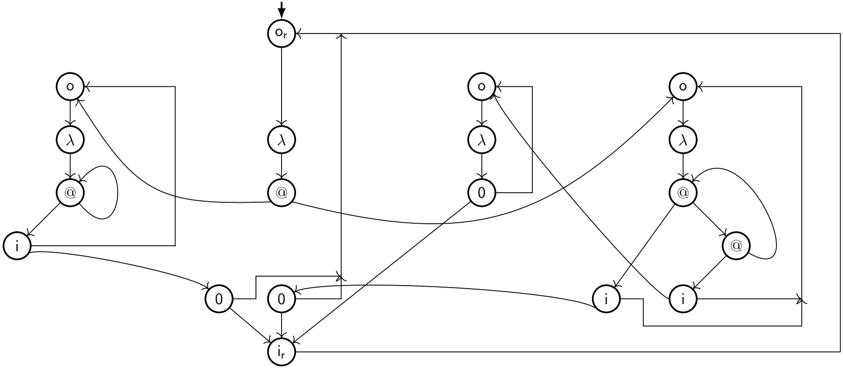

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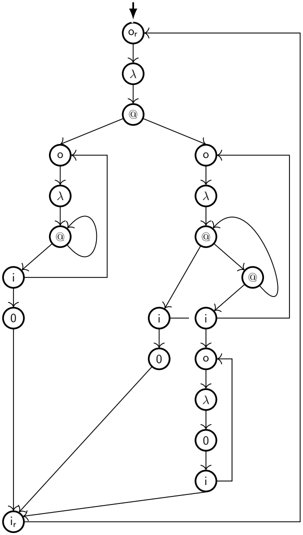

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- Expressibility of λ_{letrec} via unfolding
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  - Basic ideas for a general framework
    for graph representations of terms with nested scopes

Resources

- papers and reports
  - G: Modeling Terms by Graphs with Structure Constraints
    - TERMGRAPH 2018 post-proceedings in in EPTCS 288
  - G, Rochel: Maximal Sharing in the Lambda Calculus with Letrec
    - ICFP 2014 paper, extending report arXiv:1401.1460
  - G, Rochel: Term Graph Representations for Cyclic Lambda Terms
    - TERMGRAPH 2013 proceedings, report arXiv:1308.1034
  - G, Vincent van Oostrom: Nested Term Graphs
    - TERMGRAPH 2014 post-proceedings in EPTCS 183
- thesis Jan Rochel
  - Unfolding Semantics of the Untyped λ-Calculus with letrec
    - Ph.D. Thesis, Utrecht University, 2016
- tools by Jan Rochel
  - maxsharing on hackage.haskell.org
  - port graph rewriting